of analyticity. Since $I_M = -I$, the integral of $f(z)$ along this closed contour vanishes:

$$\oint f(z)\,dz = I + I_M = I - I = 0 \tag{5.25}$$

and we have again derived Cauchy's integral theorem.

Since every polynomial $P(z) = c_0 + c_1 z + \cdots + c_n z^n$ is entire (everywhere analytic), it follows that its integral along any closed contour must vanish

$$\oint P(z)\,dz = 0. \tag{5.26}$$

Example 5.3 (A pole) The derivative of the function $f(z) = 1/(z - z_0)$

$$f'(z) = \lim_{dz \to 0} \left(\frac{1}{z + dz - z_0} - \frac{1}{z - z_0} \right) \frac{1}{dz} = -\frac{1}{(z - z_0)^2} \tag{5.27}$$

exists everywhere except at $z = z_0$, a region that is not simply connected. □

5.3 Cauchy's Integral Formula

Let $f(z)$ be analytic in a simply connected region $\mathcal{R}$ and z_0 a point inside this region. We first will integrate the function $f(z)/(z - z_0)$ along a tiny closed counterclockwise contour around the point z_0. The contour is a circle of radius ϵ with center at z_0 with points $z = z_0 + \epsilon\, e^{i\theta}$ for $0 \le \theta \le 2\pi$, and $dz = i\epsilon\, e^{i\theta} d\theta$. Since $z - z_0 = \epsilon\, e^{i\theta}$, the contour integral in the limit $\epsilon \to 0$ is

$$\begin{aligned}
\oint_\epsilon \frac{f(z)}{z - z_0}\,dz &= \int_0^{2\pi} \frac{[f(z_0) + f'(z_0)\,(z - z_0)]}{z - z_0}\, i\epsilon\, e^{i\theta} d\theta \\
&= \int_0^{2\pi} \frac{\left[f(z_0) + f'(z_0)\,\epsilon\, e^{i\theta}\right]}{\epsilon\, e^{i\theta}}\, i\epsilon\, e^{i\theta} d\theta \\
&= \int_0^{2\pi} \left[f(z_0) + f'(z_0)\,\epsilon\, e^{i\theta}\right]\, i d\theta.
\end{aligned} \tag{5.28}$$

The θ-integral involving $f'(z_0)$ vanishes, and so we have

$$f(z_0) = \frac{1}{2\pi i} \oint_\epsilon \frac{f(z)}{z - z_0}\,dz \tag{5.29}$$

which is a miniature version of Cauchy's integral formula.

Now consider the counterclockwise contour $\mathcal{C}'$ in Fig. 5.3 which is a big counterclockwise circle, a small clockwise circle, and two parallel straight lines, all within a simply connected region $\mathcal{R}$ in which $f(z)$ is analytic. The