So $B_0 = 1$ and $B_1 = -1/2$. The remaining odd Bernoulli numbers vanish

$$B_{2n+1} = 0 \quad \text{for } n > 0 \tag{4.105}$$

and the remaining even ones are given by Euler's zeta function (4.92) formula

$$B_{2n} = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \, \zeta(2n) \quad \text{for } n > 0. \tag{4.106}$$

The Bernoulli numbers occur in the power series for many transcendental functions, for instance

$$\coth x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} \, x^{2k-1} \quad \text{for } x^2 < \pi^2. \tag{4.107}$$

Bernoulli's polynomials $B_n(y)$ are defined by the series

$$\frac{x e^{xy}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(y) \, \frac{x^n}{n!} \tag{4.108}$$

for the **generating function** $xe^{xy}/(e^x - 1)$.

Some authors (Whittaker and Watson, 1927, p. 125–127) define Bernoulli's numbers instead by

$$B_n = \frac{2(2n)!}{(2\pi)^{2n}} \, \zeta(2n) = 4n \int_0^{\infty} \frac{t^{2n-1} \, dt}{e^{2\pi t} - 1} \tag{4.109}$$

a result due to Carda.

4.12 Asymptotic Series

A series

$$s_n(x) = \sum_{k=0}^{n} \frac{a_k}{x^k} \tag{4.110}$$

is an **asymptotic** expansion for a real function $f(x)$ if the **remainder** R_n

$$R_n(x) = f(x) - s_n(x) \tag{4.111}$$

satisfies the condition

$$\lim_{x \to \infty} x^n R_n(x) = 0 \tag{4.112}$$

for fixed n. In this case, one writes

$$f(x) \approx \sum_{k=0}^{\infty} \frac{a_k}{x^k} \tag{4.113}$$

where the wavy equal sign indicates equality in the sense of (4.112). Some authors add the condition:

$$\lim_{n\to\infty} x^n R_n(x) = \infty \tag{4.114}$$

for fixed x.

Example 4.14 (The Asymptotic Series for E_1) Let's develop an asymptotic expansion for the function

$$E_1(x) = \int_x^\infty e^{-y}\,\frac{dy}{y} \tag{4.115}$$

which is related to the exponential-integral function

$$Ei(x) = \int_{-\infty}^x e^{y}\,\frac{dy}{y} \tag{4.116}$$

by the tricky formula $E_1(x) = -Ei(-x)$. Since

$$\frac{e^{-y}}{y} = -\frac{d}{dy}\left(\frac{e^{-y}}{y}\right) - \frac{e^{-y}}{y^2} \tag{4.117}$$

we may integrate by parts, getting

$$E_1(x) = \frac{e^{-x}}{x} - \int_x^\infty e^{-y}\,\frac{dy}{y^2}. \tag{4.118}$$

Integrating by parts again, we find

$$E_1(x) = \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\int_x^\infty e^{-y}\,\frac{dy}{y^3}. \tag{4.119}$$

Eventually, we develop the series

$$E_1(x) = e^{-x}\left(\frac{0!}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} - \dots\right) \tag{4.120}$$

with remainder

$$R_n(x) = (-1)^n\, n! \int_x^\infty e^{-y}\,\frac{dy}{y^{n+1}}. \tag{4.121}$$

Setting $y = u + x$, we have

$$R_n(x) = (-1)^n\,\frac{n!\,e^{-x}}{x^{n+1}} \int_0^\infty e^{-u}\,\frac{du}{\left(1+\frac{u}{x}\right)^{n+1}} \tag{4.122}$$