Example 4.12 (Planck's Distribution) Max Planck (1858–1947) showed that the electromagnetic energy in a closed cavity of volume V at a temperature T in the frequency interval $d\nu$ about ν is

$$dU(\beta,\nu,V) = \frac{8\pi hV}{c^3}\,\frac{\nu^3}{e^{\beta h\nu}-1}\,d\nu \tag{4.94}$$

in which $\beta = 1/(kT)$, $k = 1.3806503\times 10^{-23}$ J/K is **Boltzmann's constant**, and $h = 6.626068\times 10^{-34}$ Js is **Planck's constant**. The total energy then is the integral

$$U(\beta,V) = \frac{8\pi hV}{c^3}\int_0^\infty \frac{\nu^3}{e^{\beta h\nu}-1}\,d\nu \tag{4.95}$$

which we may do by letting $x = \beta h\nu$ and using the geometric series (4.31)

$$\begin{aligned} U(\beta,V) &= \frac{8\pi(kT)^4V}{(hc)^3}\int_0^\infty \frac{x^3}{e^x-1}\,dx \\ &= \frac{8\pi(kT)^4V}{(hc)^3}\int_0^\infty \frac{x^3e^{-x}}{1-e^{-x}}\,dx \\ &= \frac{8\pi(kT)^4V}{(hc)^3}\int_0^\infty x^3e^{-x}\sum_{n=0}^\infty e^{-nx}\,dx. \end{aligned} \tag{4.96}$$

The geometric series is absolutely and uniformly convergent for $x > 0$, and we may interchange the limits of summation and integration. After another change of variables, the Gamma-function formula (5.102) gives

$$\begin{aligned} U(\beta,V) &= \frac{8\pi(kT)^4V}{(hc)^3}\sum_{n=0}^\infty \int_0^\infty x^3e^{-(n+1)x}\,dx \\ &= \frac{8\pi(kT)^4V}{(hc)^3}\sum_{n=0}^\infty \frac{1}{(n+1)^4}\int_0^\infty y^3e^{-y}\,dy \\ &= \frac{8\pi(kT)^4V}{(hc)^3}\,3!\,\zeta(4) = \frac{8\pi^5(kT)^4V}{15(hc)^3}. \end{aligned} \tag{4.97}$$

It follows that the power radiated by a "**black body**" is proportional to the fourth power of its temperature and to its area A

$$P = \sigma\,A\,T^4 \tag{4.98}$$

in which

$$\sigma = \frac{2\pi^5k^4}{15\,h^3c^2} = 5.670400(40)\times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}} \tag{4.99}$$

is **Stefan's constant**.

The number of photons in the black-body distribution (4.94) at inverse temperature β in the volume V is

$$\begin{aligned} N(\beta, V) &= \frac{8\pi V}{c^3} \int_0^\infty \frac{\nu^2}{e^{\beta h\nu} - 1}\, d\nu = \frac{8\pi V}{(c\beta h)^3} \int_0^\infty \frac{x^2}{e^x - 1}\, dx \\ &= \frac{8\pi V}{(c\beta h)^3} \int_0^\infty \frac{x^2 e^{-x}}{1 - e^{-x}}\, dx = \frac{8\pi V}{(c\beta h)^3} \int_0^\infty x^2 e^{-x} \sum_{n=0}^\infty e^{-nx}\, dx \\ &= \frac{8\pi V}{(c\beta h)^3} \sum_{n=0}^\infty \int_0^\infty x^2 e^{-(n+1)x}\, dx = \frac{8\pi V}{(c\beta h)^3} \sum_{n=0}^\infty \frac{1}{(n+1)^3} \int_0^\infty y^2 e^{-y}\, dy \\ &= \frac{8\pi V}{(c\beta h)^3} \zeta(3) 2! = \frac{8\pi (kT)^3 V}{(ch)^3} \zeta(3) 2!. \end{aligned} \tag{4.100}$$

The mean energy $\langle E \rangle$ of a photon in the black-body distribution (4.94) is the energy $U(\beta, V)$ divided by the number of photons $N(\beta, V)$

$$\langle E \rangle = \langle h\nu \rangle = \frac{3!\, \zeta(4)}{2!\, \zeta(3)}\, kT = \frac{\pi^4}{30\, \zeta(3)}\, kT \tag{4.101}$$

or $\langle E \rangle \approx 2.70118\, kT$ since Apéry's constant $\zeta(3)$ is $1.2020569032\ldots$ (Roger Apéry, 1916–1994). □

Example 4.13 (The Lerch Transcendent) The **Lerch transcendent** is the series

$$\Phi(z, s, \alpha) = \sum_{n=0}^\infty \frac{z^n}{(n+\alpha)^s}. \tag{4.102}$$

It converges when $|z| < 1$ and $\mathrm{Re}\, s > 0$ and $\mathrm{Re}\, \alpha > 0$. □

4.11 Bernoulli Numbers and Polynomials

The **Bernoulli numbers** B_n are defined by the infinite series

$$\frac{x}{e^x - 1} = \sum_{n=0}^\infty \frac{x^n}{n!} \left[\frac{d^n}{dx^n} \frac{x}{e^x - 1} \right] \Bigg|_{x=0} = \sum_{n=0}^\infty B_n \frac{x^n}{n!} \tag{4.103}$$

for the **generating function** $x/(e^x - 1)$. They are the successive derivatives

$$B_n = \frac{d^n}{dx^n} \frac{x}{e^x - 1} \Bigg|_{x=0}. \tag{4.104}$$