One may extend the definition (4.36) of \( n \)-factorial from positive integers to complex numbers by means of the integral formula

\[
z! \equiv \int_0^\infty e^{-t} t^z \, dt \tag{4.53}
\]

for \( \text{Re } z > -1 \). In particular

\[
0! = \int_0^\infty e^{-t} \, dt = 1 \tag{4.54}
\]

which explains the definition (4.37). The factorial function \((z - 1)!\) in turn defines the **gamma function** for \( \text{Re } z > 0 \) as

\[
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt = (z - 1)! \tag{4.55}
\]

as may be seen from (4.53). By differentiating this formula and integrating it by parts, we see that the gamma function satisfies the key identity

\[
\Gamma(z + 1) = \int_0^\infty \left(-\frac{d}{dt}e^{-t}\right) t^z \, dt = \int_0^\infty e^{-t}\left(\frac{d}{dt}t^z\right) \, dt = \int_0^\infty e^{-t} z t^{z-1} \, dt = z \Gamma(z). \tag{4.56}
\]

Since \( \Gamma(1) = 0! = 1 \), we may use this identity (4.56) to extend the definition (5.102) of the gamma function in unit steps into the left half-plane

\[
\Gamma(z) = \frac{1}{z} \Gamma(z+1) = \frac{1}{z} \frac{1}{z+1} \Gamma(z+2) = \frac{1}{z} \frac{1}{z+1} \frac{1}{z+2} \Gamma(z+3) = \ldots \tag{4.57}
\]

as long as we avoid the negative integers and zero. This extension leads to Euler’s definition

\[
\Gamma(z) = \lim_{n \to \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)(z+2) \cdots (z+n)} n^z \tag{4.58}
\]

and to Weierstrass’s (exercise 4.6)

\[
\Gamma(z) = \frac{1}{z} e^{-\gamma z} \left[ \prod_{n=1}^\infty \left(1 + \frac{z}{n}\right) e^{-z/n} \right]^{-1} \tag{4.59}
\]

(Karl Theodor Wilhelm Weierstrass, 1815–1897), and is an example of analytic continuation (section 5.12).

One may show (exercise 4.8) that another formula for \( \Gamma(z) \) is

\[
\Gamma(z) = 2 \int_0^\infty e^{-t^2} t^{2z-1} \, dt \tag{4.60}
\]
for $\text{Re } z > 0$ and that

$$\Gamma(n + \frac{1}{2}) = \frac{(2n)!}{n! 2^n \sqrt{\pi}}$$

(4.61)

which implies (exercise 4.11) that

$$\Gamma \left( n + \frac{1}{2} \right) = \frac{(2n - 1)!!}{2^n} \sqrt{\pi}.$$  (4.62)

**Example 4.7** (Bessel Function of nonintegral index) We can use the gamma-function formula (4.55) for $n!$ to extend the definition (4.49) of the Bessel function of the first kind $J_n(\rho)$ to nonintegral values $\nu$ of the index $n$. Replacing $n$ by $\nu$ and $(m + n)!$ by $\Gamma(m + \nu + 1)$, we get

$$J_\nu(\rho) = \left( \frac{\rho}{2} \right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left( \frac{\rho}{2} \right)^{2m}$$  (4.63)

which makes sense even for complex values of $\nu$. □

**Example 4.8** (Spherical Bessel Function) The spherical Bessel function is defined as

$$j_\ell(\rho) \equiv \sqrt{\frac{\pi}{2\rho}} J_{\ell+1/2}(\rho).$$  (4.64)

For small values of its argument $|\rho| \ll 1$, the first term in the series (4.63) dominates and so (exercise 4.7)

$$j_\ell(\rho) \approx \frac{\sqrt{\pi}}{2} \left( \frac{\rho}{2} \right)^\ell \frac{1}{\Gamma(\ell + 3/2)} = \frac{\ell! (2\rho)^\ell}{(2\ell + 1)!} = \frac{\rho^\ell}{(2\ell + 1)!!}$$  (4.65)

as one may show by repeatedly using the key identity $\Gamma(z + 1) = z \Gamma(z)$. □

### 4.6 Taylor Series

If the function $f(x)$ is a real-valued function of a real variable $x$ with a continuous $N$th derivative, then Taylor’s expansion for it is

$$f(x + a) = f(x) + af'(x) + \frac{a^2}{2} f''(x) + \cdots + \frac{a^{N-1}}{(N-1)!} f^{(N-1)}(x) + E_N$$

$$= \sum_{n=0}^{N-1} \frac{a^n}{n!} f^{(n)}(x) + E_N$$  (4.66)