If we generalize the relations (3.12–3.14) between Fourier series and transforms from one to \(n\) dimensions, then we find that the Fourier series corresponding to the Fourier transform (3.94) is

\[
f(x_1, \ldots, x_n) = \left(\frac{2\pi}{L}\right)^n \sum_{j_1 = -\infty}^{\infty} \cdots \sum_{j_n = -\infty}^{\infty} e^{i(k_{j_1} x_1 + \cdots + k_{j_n} x_n)} \frac{\tilde{f}(k_{j_1}, \ldots, k_{j_n})}{(2\pi)^{n/2}}. \tag{3.95}
\]

in which \(k_{j} = 2\pi j/L\). Thus, for \(n = 3\) we have

\[
f(x) = \frac{(2\pi)^3}{V} \sum_{j_1 = -\infty}^{\infty} \sum_{j_2 = -\infty}^{\infty} \sum_{j_3 = -\infty}^{\infty} e^{i\mathbf{k}_j \cdot \mathbf{x}} \frac{\tilde{f}(\mathbf{k}_j)}{(2\pi)^{3/2}} \tag{3.96}
\]

in which \(\mathbf{k}_j = (k_{j_1}, k_{j_2}, k_{j_3})\) and \(V = L^3\) is the volume of the box.

**Example 3.8** (The Feynman Propagator) For a spinless quantum field of mass \(m\), Feynman’s propagator is the four-dimensional Fourier transform

\[
\Delta_F(x) = \int \frac{\exp(i \mathbf{k} \cdot \mathbf{x})}{k^2 + m^2 - i\epsilon} \frac{d^4k}{(2\pi)^4} \tag{3.97}
\]

where \(k \cdot x = \mathbf{k} \cdot \mathbf{x} - k_0 x^0\), all physical quantities are in **natural units** \((c = h = 1)\), and \(x^0 = ct = t\). The tiny imaginary term \(-i\epsilon\) makes \(\Delta_F(x - y)\) proportional to the mean value in the vacuum state \(|0\rangle\) of the **time-ordered product** of the fields \(\phi(x)\) and \(\phi(y)\) (section 5.34)

\[
-i \Delta_F(x - y) = \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle \tag{3.98}
\]

\[
\equiv \theta(x^0 - y^0) \langle 0 | \phi(x)\phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y)\phi(x) | 0 \rangle
\]

in which \(\theta(a) = (a + |a|)/2|a|\) is the Heaviside function (2.166).

### 3.6 Convolutions

The convolution of \(f(x)\) with \(g(x)\) is the integral

\[
f \ast g(x) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} f(x - y) g(y). \tag{3.99}
\]

The convolution product is symmetric

\[
f \ast g(x) = g \ast f(x) \tag{3.100}
\]
because setting $z = x - y$, we have
\[
f * g(x) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} f(x - y) g(y) = -\int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} f(z) g(x - z) \\
= \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} g(x - z) f(z) = g * f(x).
\] (3.101)

Convolutions may look strange at first, but they often occur in physics in the three-dimensional form
\[
F(x) = \int G(x - x') S(x') d^3x
\] (3.102)
in which $G$ is a Green’s function and $S$ is a source.

**Example 3.9 (Gauss’s Law)** The divergence of the electric field $E$ is the microscopic charge density $\rho$ divided by the electric permittivity of the vacuum $\epsilon_0 = 8.854 \times 10^{-12} \, \text{F/m}$, that is, $\nabla \cdot E = \rho/\epsilon_0$. This constraint is known as Gauss’s law. If the charges and fields are independent of time, then the electric field $E$ is the gradient of a scalar potential $E = -\nabla \phi$. These last two equations imply that $\phi$ obeys Poisson’s equation
\[
-\nabla^2 \phi = \frac{\rho}{\epsilon_0}.
\] (3.103)

We may solve this equation by using Fourier transforms as described in Sec. 3.13. If $\tilde{\phi}(k)$ and $\tilde{\rho}(k)$ respectively are the Fourier transforms of $\phi(x)$ and $\rho(x)$, then Poisson’s differential equation (3.103) gives
\[
-\nabla^2 \phi(x) = -\nabla^2 \int e^{ik \cdot x} \tilde{\phi}(k) d^3k = \int k^2 e^{ik \cdot x} \tilde{\phi}(k) d^3k
\]
\[
= \frac{\rho(x)}{\epsilon_0} = \int e^{ik \cdot x} \frac{\tilde{\rho}(k)}{\epsilon_0} d^3k
\] (3.104)
which implies the algebraic equation $\tilde{\phi}(k) = \tilde{\rho}(k)/\epsilon_0 k^2$ which is an instance of (3.163). Performing the inverse Fourier transformation, we find for the scalar potential
\[
\phi(x) = \int e^{ik \cdot x} \tilde{\phi}(k) d^3k = \int e^{ik \cdot x} \frac{\tilde{\rho}(k)}{\epsilon_0} \frac{d^3k}{k^2}
\]
\[
= \int e^{ik \cdot x} \frac{1}{k^2} \int e^{-ik \cdot x'} \frac{\rho(x')}{\epsilon_0} \frac{d^3x' d^3k}{(2\pi)^3} = \int G(x - x') \frac{\rho(x')}{\epsilon_0} d^3x'
\] (3.105)
in which
\[
G(x - x') = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} e^{ik \cdot (x - x')}.
\] (3.106)
This function \( G(\mathbf{x} - \mathbf{x}') \) is the Green’s function for the differential operator \(-\nabla^2\) in the sense that
\[
-\nabla^2 G(\mathbf{x} - \mathbf{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} = \delta^{(3)}(\mathbf{x} - \mathbf{x}'). \tag{3.107}
\]

This Green’s function ensures that expression (3.105) for \( \phi(\mathbf{x}) \) satisfies Poisson’s equation (3.103). To integrate (3.106) and compute \( G(\mathbf{x} - \mathbf{x}') \), we use spherical coordinates with the \( z \)-axis parallel to the vector \( \mathbf{x} - \mathbf{x}' \).

\[
G(\mathbf{x} - \mathbf{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} = \int_0^\infty \frac{dk}{(2\pi)^2} \int_{-1}^1 d\cos \theta \ e^{ik|\mathbf{x} - \mathbf{x}'| \cos \theta}
= \frac{1}{2\pi^2|\mathbf{x} - \mathbf{x}'|} \int_0^\infty \frac{\sin k|\mathbf{x} - \mathbf{x}'| \, dk}{k} = \frac{1}{2\pi^2|\mathbf{x} - \mathbf{x}'|} \int_0^\infty \frac{\sin k \, dk}{k}. \tag{3.108}
\]

In example 5.35 of section 5.34 on Cauchy’s principal value, we’ll show that
\[
\int_0^\infty \frac{\sin k \, dk}{k} = \frac{\pi}{2}. \tag{3.109}
\]

Using this result, we have
\[
\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} = G(\mathbf{x} - \mathbf{x}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}. \tag{3.110}
\]

Finally by substituting this formula for \( G(\mathbf{x} - \mathbf{x}') \) into Eq. (3.105), we find that the Fourier transform \( \phi(\mathbf{x}) \) of the product \( \tilde{\rho}(\mathbf{k})/k^2 \) of the functions \( \tilde{\rho}(\mathbf{k}) \) and \( 1/k^2 \) is the convolution
\[
\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \tag{3.111}
\]
of their Fourier transforms \( 1/|\mathbf{x} - \mathbf{x}'| \) and \( \rho(\mathbf{x}') \). The Fourier transform of the product of any two functions is the convolution of their Fourier transforms, as we’ll see in the next section. (George Green 1793–1841)

**Example 3.10 (The Magnetic Vector Potential)** The magnetic induction \( \mathbf{B} \) has zero divergence (as long as there are no magnetic monopoles) and so may be written as the curl \( \mathbf{B} = \nabla \times \mathbf{A} \) of a vector potential \( \mathbf{A} \). For time-independent currents, Ampère’s law is \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \) in which \( \mu_0 = 1/(\epsilon_0 c^2) = 4\pi \times 10^{-7} \) N A\(^{-2}\) is the permeability of the vacuum. It follows
that in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the magnetostatic vector potential $\mathbf{A}$ satisfies the equation

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \quad (3.112)$$

Applying the Fourier-transform technique (3.103–3.111), we find that the Fourier transforms of $\mathbf{A}$ and $\mathbf{J}$ satisfy the algebraic equation

$$\tilde{\mathbf{A}}(\mathbf{k}) = \mu_0 \frac{\tilde{\mathbf{J}}(\mathbf{k})}{k^2}. \quad (3.113)$$

which is an instance of (3.163). Performing the inverse Fourier transform, we see that $\mathbf{A}$ is the convolution

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \quad (3.114)$$

If in the solution (3.111) of Poisson’s equation, $\rho(\mathbf{x})$ is translated by $\mathbf{a}$, then so is $\phi(\mathbf{x})$. That is, if $\rho'(\mathbf{x}) = \rho(\mathbf{x} + \mathbf{a})$ then $\phi'(\mathbf{x}) = \phi(\mathbf{x} + \mathbf{a})$. Similarly, if the current $\mathbf{J}(\mathbf{x})$ in (3.114) is translated by $\mathbf{a}$, then so is the potential $\mathbf{A}(\mathbf{x})$. **Convolutions respect translational invariance.** That’s one reason why they occur so often in the formulas of physics.

### 3.7 The Fourier Transform of a Convolution

The Fourier transform of the convolution $f * g$ is the product of the Fourier transforms $\tilde{f}$ and $\tilde{g}$:

$$\tilde{f * g}(k) = \tilde{f}(k) \tilde{g}(k). \quad (3.115)$$

To see why, we form the Fourier transform $\tilde{f * g}(k)$ of the convolution $f * g(x)$

$$\tilde{f * g}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} f * g(x)$$

$$= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} f(x - y) g(y). \quad (3.116)$$

Now we write $f(x - y)$ and $g(y)$ in terms of their Fourier transforms $\tilde{f}(p)$ and $\tilde{g}(q)$

$$\tilde{f * g}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \tilde{f}(p) e^{ip(x-y)} \int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}} \tilde{g}(q) e^{iqy} \quad (3.117)$$
and use the representation (3.36) of Dirac's delta function twice to get
\[
\tilde{f} * \tilde{g}(k) = \int_{-\infty}^{\infty} \frac{dy}{2\pi} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \delta(p - k) \tilde{f}(p) \tilde{g}(q) e^{i(q-p)y} \\
= \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \delta(p - k) \delta(q - p) \tilde{f}(p) \tilde{g}(q) \\
= \int_{-\infty}^{\infty} dp \delta(p - k) \tilde{f}(p) \tilde{g}(p) = \tilde{f}(k) \tilde{g}(k)
\]
which is (3.115). Examples 3.9 and 3.10 were illustrations of this result.

\section*{3.8 Fourier Transforms and Green’s Functions}
A Green’s function $G(x)$ for a differential operator $P$ turns into a delta function when acted upon by $P$, that is, $PG(x) = \delta(x)$. If the differential operator is a polynomial $P(\partial) \equiv P(\partial_1, \ldots, \partial_n)$ in the derivatives $\partial_1, \ldots, \partial_n$ with constant coefficients, then a suitable Green’s function $G(x)$ will satisfy
\[
P(\partial)G(x) = \delta^{(n)}(x).
\]
(3.119)
Expressing both $G(x)$ and $\delta^{(n)}(x)$ as Fourier transforms, we get
\[
P(\partial)G(x) = \int d^n k P(ik) e^{ik \cdot x} \tilde{G}(k) = \delta^{(n)}(x) = \int \frac{d^n k}{(2\pi)^n} e^{ik \cdot x}
\]
which gives us the algebraic equation
\[
\tilde{G}(k) = \frac{1}{(2\pi)^n P(ik)}.
\]
(3.121)
Thus the Green’s function $G_P$ for the differential operator $P(\partial)$ is
\[
G_P(x) = \int \frac{d^n k}{(2\pi)^n} \frac{e^{ik \cdot x}}{P(ik)}.
\]
(3.122)
\textbf{Example 3.11 (Green and Yukawa)} In 1935, Hideki Yukawa (1907–1981) proposed the partial differential equation
\[
P_Y(\partial)G_Y(x) \equiv (-\Delta + m^2)G_Y(x) = (-\nabla^2 + m^2)G_Y(x) = \delta(x).
\]
(3.123)
Our (3.122) gives as the Green’s function for $P_Y(\partial)$ the Yukawa potential
\[
G_Y(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{e^{ik \cdot x}}{P_Y(ik)} = \int \frac{d^3 k}{(2\pi)^3} \frac{e^{ik \cdot x}}{k^2 + m^2} = \frac{e^{-mr}}{4\pi r}
\]
an integration done in example 5.21. \qed