If we generalize the relations (3.12–3.14) between Fourier series and transforms from one to n dimensions, then we find that the Fourier series corresponding to the Fourier transform (3.94) is

$$f(x_1, \ldots, x_n) = \left(\frac{2\pi}{L}\right)^n \sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_n=-\infty}^{\infty} e^{i(k_{j_1}x_1+\cdots+k_{j_n}x_n)} \, \frac{\tilde{f}(k_{j_1}, \ldots, k_{j_n})}{(2\pi)^{n/2}} \tag{3.95}$$

in which $k_{j_\ell} = 2\pi j_\ell / L$. Thus, for $n = 3$ we have

$$f(\boldsymbol{x}) = \frac{(2\pi)^3}{V} \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} \sum_{j_3=-\infty}^{\infty} e^{i\boldsymbol{k_j}\cdot\boldsymbol{x}} \, \frac{\tilde{f}(\boldsymbol{k_j})}{(2\pi)^{3/2}} \tag{3.96}$$

in which $\boldsymbol{k_j} = (k_{j_1}, k_{j_2}, k_{j_3})$ and $V = L^3$ is the volume of the box.

Example 3.8 (The Feynman Propagator) For a spinless quantum field of mass m, Feynman's propagator is the four-dimensional Fourier transform

$$\triangle_F(x) = \int \frac{\exp(ik \cdot x)}{k^2 + m^2 - i\epsilon} \, \frac{d^4k}{(2\pi)^4} \tag{3.97}$$

where $k \cdot x = \boldsymbol{k} \cdot \boldsymbol{x} - k^0 x^0$, all physical quantities are in **natural units** ($c = \hbar = 1$), and $x^0 = ct = t$. The tiny imaginary term $-i\epsilon$ makes $\triangle_F(x - y)$ proportional to the mean value in the vacuum state $|0\rangle$ of the **time-ordered product** of the fields $\phi(x)$ and $\phi(y)$ (section 5.34)

$$\begin{aligned} -i \, \triangle_F \, (x - y) &= \langle 0 | \mathcal{T} \left[\phi(x)\phi(y) \right] |0\rangle \\ &\equiv \theta(x^0 - y^0) \langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle \end{aligned} \tag{3.98}$$

in which $\theta(a) = (a + |a|)/2|a|$ is the Heaviside function (2.166). □

3.6 Convolutions

The convolution of $f(x)$ with $g(x)$ is the integral

$$f * g(x) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \, f(x - y) \, g(y). \tag{3.99}$$

The convolution product is symmetric

$$f * g(x) = g * f(x) \tag{3.100}$$

because setting $z = x - y$, we have

$$\begin{aligned} f * g(x) &= \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}}\, f(x-y)\, g(y) = -\int_{\infty}^{-\infty} \frac{dz}{\sqrt{2\pi}}\, f(z)\, g(x-z) \\ &= \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, g(x-z)\, f(z) = g * f(x). \end{aligned} \tag{3.101}$$

Convolutions may look strange at first, but they often occur in physics in the three-dimensional form

$$F(\boldsymbol{x}) = \int G(\boldsymbol{x} - \boldsymbol{x}')\, S(\boldsymbol{x}')\, d^3\boldsymbol{x} \tag{3.102}$$

in which G is a Green's function and S is a source.

Example 3.9 (Gauss's Law) The divergence of the electric field $\boldsymbol{E}$ is the microscopic charge density ρ divided by the electric permittivity of the vacuum $\epsilon_0 = 8.854 \times 10^{-12}$ F/m, that is, $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$. This constraint is known as Gauss's law. If the charges and fields are independent of time, then the electric field $\boldsymbol{E}$ is the gradient of a scalar potential $\boldsymbol{E} = -\nabla\phi$. These last two equations imply that ϕ obeys Poisson's equation

$$-\nabla^2 \phi = \frac{\rho}{\epsilon_0}. \tag{3.103}$$

We may solve this equation by using Fourier transforms as described in Sec. 3.13. If $\tilde{\phi}(\boldsymbol{k})$ and $\tilde{\rho}(\boldsymbol{k})$ respectively are the Fourier transforms of $\phi(\boldsymbol{x})$ and $\rho(\boldsymbol{x})$, then Poisson's differential equation (3.103) gives

$$\begin{aligned} -\nabla^2 \phi(\boldsymbol{x}) &= -\nabla^2 \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \tilde{\phi}(\boldsymbol{k})\, d^3\boldsymbol{k} = \int \boldsymbol{k}^2\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \tilde{\phi}(\boldsymbol{k})\, d^3\boldsymbol{k} \\ &= \frac{\rho(\boldsymbol{x})}{\epsilon_0} = \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \frac{\tilde{\rho}(\boldsymbol{k})}{\epsilon_0}\, d^3\boldsymbol{k} \end{aligned} \tag{3.104}$$

which implies the algebraic equation $\tilde{\phi}(\boldsymbol{k}) = \tilde{\rho}(\boldsymbol{k})/\epsilon_0 \boldsymbol{k}^2$ which is an instance of (3.163). Performing the inverse Fourier transformation, we find for the scalar potential

$$\begin{aligned} \phi(\boldsymbol{x}) &= \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \tilde{\phi}(\boldsymbol{k})\, d^3\boldsymbol{k} = \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \frac{\tilde{\rho}(\boldsymbol{k})}{\epsilon_0\, \boldsymbol{k}^2}\, d^3\boldsymbol{k} \\ &= \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \frac{1}{\boldsymbol{k}^2} \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}'}\, \frac{\rho(\boldsymbol{x}')}{\epsilon_0}\, \frac{d^3\boldsymbol{x}'\, d^3\boldsymbol{k}}{(2\pi)^3} = \int G(\boldsymbol{x} - \boldsymbol{x}')\, \frac{\rho(\boldsymbol{x}')}{\epsilon_0}\, d^3\boldsymbol{x}' \end{aligned} \tag{3.105}$$

in which

$$G(\boldsymbol{x} - \boldsymbol{x}') = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, \frac{1}{\boldsymbol{k}^2}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}. \tag{3.106}$$

This function $G(\boldsymbol{x}-\boldsymbol{x}')$ is the Green's function for the differential operator $-\nabla^2$ in the sense that

$$-\nabla^2 G(\boldsymbol{x}-\boldsymbol{x}') = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'). \tag{3.107}$$

This Green's function ensures that expression (3.105) for $\phi(\boldsymbol{x})$ satisfies Poisson's equation (3.103). To integrate (3.106) and compute $G(\boldsymbol{x}-\boldsymbol{x}')$, we use spherical coordinates with the z-axis parallel to the vector $\boldsymbol{x}-\boldsymbol{x}'$

$$\begin{aligned} G(\boldsymbol{x}-\boldsymbol{x}') &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{1}{\boldsymbol{k}^2}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} = \int_0^\infty \frac{dk}{(2\pi)^2}\int_{-1}^{1} d\cos\theta\, e^{ik|\boldsymbol{x}-\boldsymbol{x}'|\cos\theta} \\ &= \int_0^\infty \frac{dk}{(2\pi)^2}\,\frac{e^{ik|\boldsymbol{x}-\boldsymbol{x}'|}-e^{-ik|\boldsymbol{x}-\boldsymbol{x}'|}}{ik|\boldsymbol{x}-\boldsymbol{x}'|} \\ &= \frac{1}{2\pi^2|\boldsymbol{x}-\boldsymbol{x}'|}\int_0^\infty \frac{\sin k|\boldsymbol{x}-\boldsymbol{x}'|\,dk}{k} = \frac{1}{2\pi^2|\boldsymbol{x}-\boldsymbol{x}'|}\int_0^\infty \frac{\sin k\,dk}{k}. \end{aligned} \tag{3.108}$$

In example 5.35 of section 5.34 on Cauchy's principal value, we'll show that

$$\int_0^\infty \frac{\sin k}{k}\, dk = \frac{\pi}{2}. \tag{3.109}$$

Using this result, we have

$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{1}{\boldsymbol{k}^2}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} = G(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{3.110}$$

Finally by substituting this formula for $G(\boldsymbol{x}-\boldsymbol{x}')$ into Eq. (3.105), we find that the Fourier transform $\phi(x)$ of the product $\tilde{\rho}(\boldsymbol{k})/\boldsymbol{k}^2$ of the functions $\tilde{\rho}(\boldsymbol{k})$ and $1/\boldsymbol{k}^2$ is the convolution

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3\boldsymbol{x}' \tag{3.111}$$

of their Fourier transforms $1/|\boldsymbol{x}-\boldsymbol{x}'|$ and $\rho(\boldsymbol{x}')$. The Fourier transform of the product of any two functions is the convolution of their Fourier transforms, as we'll see in the next section. (George Green 1793–1841) □

Example 3.10 (The Magnetic Vector Potential) The magnetic induction $\boldsymbol{B}$ has zero divergence (as long as there are no magnetic monopoles) and so may be written as the curl $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ of a vector potential $\boldsymbol{A}$. For time-independent currents, Ampère's law is $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J}$ in which $\mu_0 = 1/(\epsilon_0 c^2) = 4\pi\times 10^{-7}$ N A^{-2} is the permeability of the vacuum. It follows that in the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, the magnetostatic vector potential $\boldsymbol{A}$ satisfies the equation

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = -\nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}. \tag{3.112}$$

Applying the Fourier-transform technique (3.103–3.111), we find that the Fourier transforms of $\boldsymbol{A}$ and $\boldsymbol{J}$ satisfy the algebraic equation

$$\tilde{\boldsymbol{A}}(\boldsymbol{k}) = \mu_0 \frac{\tilde{\boldsymbol{J}}(\boldsymbol{k})}{\boldsymbol{k}^2} \tag{3.113}$$

which is an instance of (3.163). Performing the inverse Fourier transform, we see that $\boldsymbol{A}$ is the convolution

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int d^3\boldsymbol{x}' \, \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{3.114}$$

If in the solution (3.111) of Poisson's equation, $\rho(\boldsymbol{x})$ is translated by $\boldsymbol{a}$, then so is $\phi(\boldsymbol{x})$. That is, if $\rho'(\boldsymbol{x}) = \rho(\boldsymbol{x}+\boldsymbol{a})$ then $\phi'(\boldsymbol{x}) = \phi(\boldsymbol{x}+\boldsymbol{a})$. Similarly, if the current $\boldsymbol{J}(\boldsymbol{x})$ in (3.114) is translated by $\boldsymbol{a}$, then so is the potential $\boldsymbol{A}(\boldsymbol{x})$. **Convolutions respect translational invariance.** That's one reason why they occur so often in the formulas of physics. □

3.7 The Fourier Transform of a Convolution

The Fourier transform of the convolution $f * g$ is the product of the Fourier transforms $\tilde{f}$ and $\tilde{g}$:

$$\widetilde{f * g}(k) = \tilde{f}(k)\, \tilde{g}(k). \tag{3.115}$$

To see why, we form the Fourier transform $\widetilde{f * g}(k)$ of the convolution $f * g(x)$

$$\begin{aligned} \widetilde{f * g}(k) &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-ikx}\, f * g(x) \\ &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-ikx} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}}\, f(x-y)\, g(y). \end{aligned} \tag{3.116}$$

Now we write $f(x-y)$ and $g(y)$ in terms of their Fourier transforms $\tilde{f}(p)$ and $\tilde{g}(q)$

$$\widetilde{f * g}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-ikx} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}}\, \tilde{f}(p)\, e^{ip(x-y)} \int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}}\, \tilde{g}(q)\, e^{iqy} \tag{3.117}$$

and use the representation (3.36) of Dirac's delta function twice to get

$$\widetilde{f * g}(k) = \int_{-\infty}^{\infty} \frac{dy}{2\pi} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \, \delta(p-k) \, \tilde{f}(p) \, \tilde{g}(q) \, e^{i(q-p)y}$$
$$= \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \, \delta(p-k) \, \delta(q-p) \, \tilde{f}(p) \, \tilde{g}(q)$$
$$= \int_{-\infty}^{\infty} dp \, \delta(p-k) \, \tilde{f}(p) \, \tilde{g}(p) = \tilde{f}(k) \, \tilde{g}(k) \tag{3.118}$$

which is (3.115). Examples 3.9 and 3.10 were illustrations of this result.

3.8 Fourier Transforms and Green's Functions

A Green's function $G(x)$ for a differential operator P turns into a delta function when acted upon by P, that is, $PG(x) = \delta(x)$. If the differential operator is a polynomial $P(\partial) \equiv P(\partial_1, \ldots, \partial_n)$ in the derivatives $\partial_1, \ldots, \partial_n$ with constant coefficients, then a suitable Green's function $G(x) \equiv G(x_1, \ldots, x_n)$ will satisfy

$$P(\partial) G(x) = \delta^{(n)}(x). \tag{3.119}$$

Expressing both $G(x)$ and $\delta^{(n)}(x)$ as Fourier transforms, we get

$$P(\partial) G(x) = \int d^n k \, P(ik) \, e^{ik \cdot x} \, \tilde{G}(k) = \delta^{(n)}(x) = \int \frac{d^n k}{(2\pi)^n} \, e^{ik \cdot x} \tag{3.120}$$

which gives us the algebraic equation

$$\tilde{G}(k) = \frac{1}{(2\pi)^n \, P(ik)}. \tag{3.121}$$

Thus the Green's function G_P for the differential operator $P(\partial)$ is

$$G_P(x) = \int \frac{d^n k}{(2\pi)^n} \, \frac{e^{ik \cdot x}}{P(ik)}. \tag{3.122}$$

Example 3.11 (Green and Yukawa) In 1935, Hideki Yukawa (1907–1981) proposed the partial differential equation

$$P_Y(\partial) G_Y(\boldsymbol{x}) \equiv (-\triangle + m^2) G_Y(\boldsymbol{x}) = (-\nabla^2 + m^2) G_Y(\boldsymbol{x}) = \delta(\boldsymbol{x}). \tag{3.123}$$

Our (3.122) gives as the Green's function for $P_Y(\partial)$ the Yukawa potential

$$G_Y(x) = \int \frac{d^3 k}{(2\pi)^3} \, \frac{e^{i\boldsymbol{k} \cdot \boldsymbol{x}}}{P_Y(ik)} = \int \frac{d^3 k}{(2\pi)^3} \, \frac{e^{i\boldsymbol{k} \cdot \boldsymbol{x}}}{\boldsymbol{k}^2 + m^2} = \frac{e^{-mr}}{4\pi r} \tag{3.124}$$

an integration done in example 5.21. □