One often needs to relate a function's Fourier series to its Fourier transform. So let's compare the Fourier series (3.1) for the function $f(x)$ on the interval $[-L/2, L/2]$ with its Fourier transform (3.9) in the limit of large L

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \, \frac{e^{i2\pi nx/L}}{\sqrt{L}} = \sum_{n=-\infty}^{\infty} f_n \, \frac{e^{ik_n x}}{\sqrt{L}} = \int_{-\infty}^{\infty} \tilde{f}(k) \, e^{ikx} \, \frac{dk}{\sqrt{2\pi}} \tag{3.12}$$

in which $k_n = 2\pi n/L = 2\pi y/L$. Now $f_n = \hat{f}(y)$, and so by the definition (3.6) of $\tilde{f}(k)$, we have $f_n = \hat{f}(Lk/2\pi) = \sqrt{2\pi/L}\,\tilde{f}(k)$. Thus, to get the Fourier series from the Fourier transform, we multiply the series by $2\pi/L$ and use the Fourier transform at k_n divided by $\sqrt{2\pi}$

$$f(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} f_n \, e^{ik_n x} = \frac{2\pi}{L} \sum_{n=-\infty}^{\infty} \frac{\tilde{f}(k_n)}{\sqrt{2\pi}} \, e^{ik_n x}. \tag{3.13}$$

Going the other way, we have

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k) \, e^{ikx} \, \frac{dk}{\sqrt{2\pi}} = \frac{L}{2\pi} \int_{-\infty}^{\infty} \frac{f_{[Lk/2\pi]}}{\sqrt{L}} \, e^{ikx} dk. \tag{3.14}$$

Example 3.1 (The Fourier Transform of a Gaussian Is a Gaussian) The Fourier transform of the gaussian $f(x) = \exp(-m^2\, x^2)$ is

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \, e^{-ikx} \, e^{-m^2\, x^2}. \tag{3.15}$$

We complete the square in the exponent:

$$\tilde{f}(k) = e^{-k^2/4m^2} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \, e^{-m^2\,(x+ik/2m^2)^2}. \tag{3.16}$$

As we shall see in Sec. 5.14 when we study analytic functions, we may shift x to $x - ik/2m^2$, so the term $ik/2m^2$ in the exponential has no effect on the value of the x-integral.

$$\tilde{f}(k) = e^{-k^2/4m^2} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \, e^{-m^2\, x^2} = \frac{1}{\sqrt{2}\, m} \, e^{-k^2/4m^2}. \tag{3.17}$$

Thus, the Fourier transform of a gaussian is another gaussian

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \, e^{-ikx} \, e^{-m^2\, x^2} = \frac{1}{\sqrt{2}\, m} \, e^{-k^2/4m^2}. \tag{3.18}$$

But the two gaussians are very different: if the gaussian $f(x) = \exp(-m^2 x^2)$ decreases slowly as $x \to \infty$ because m is small (or quickly because m is big), then its gaussian Fourier transform $\tilde{f}(k) = \exp(-k^2/4m^2)/m\sqrt{2}$ decreases quickly as $k \to \infty$ because m is small (or slowly because m is big).