Fourier and Laplace Transforms

One often needs to relate a function’s Fourier series to its Fourier transform. So let’s compare the Fourier series (3.1) for the function \( f(x) \) on the interval \([-L/2, L/2]\) with its Fourier transform (3.9) in the limit of large \( L \)

\[
f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i2\pi nx/L} = \sum_{n=-\infty}^{\infty} f_n e^{ik_n x} = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \frac{dk}{\sqrt{2\pi}} \tag{3.12}
\]

in which \( k_n = 2\pi n/L = 2\pi y/L \). Now \( f_n = \hat{f}(y) \), and so by the definition (3.6) of \( \hat{f}(k) \), we have \( f_n = \hat{f}(Lk/2\pi) = \sqrt{2\pi/L} \hat{f}(k) \). Thus, to get the Fourier series from the Fourier transform, we multiply the series by \( 2\pi/L \) and use the Fourier transform at \( k_n \) divided by \( \sqrt{2\pi} \)

\[
f(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} f_n e^{ik_n x} = \frac{2\pi}{L} \sum_{n=-\infty}^{\infty} \hat{f}(k_n) e^{ik_n x}. \tag{3.13}
\]

Going the other way, we have

\[
f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \frac{dk}{\sqrt{2\pi}} = L/2\pi \int_{-\infty}^{\infty} \frac{\hat{f}(Lk/2\pi)}{\sqrt{L}} e^{ikx} dk. \tag{3.14}
\]

**Example 3.1** (The Fourier Transform of a Gaussian Is a Gaussian)

The Fourier transform of the gaussian \( f(x) = \exp(-m^2 x^2) \) is

\[
\hat{f}(k) = \int_{-\infty}^{\infty} dx \sqrt{2\pi} e^{-ikx} e^{-m^2 x^2}. \tag{3.15}
\]

We complete the square in the exponent:

\[
\hat{f}(k) = e^{-k^2/4m^2} \int_{-\infty}^{\infty} dx \sqrt{2\pi} e^{-m^2 (x+ik/2m)^2}. \tag{3.16}
\]

As we shall see in Sec. 5.14 when we study analytic functions, we may shift \( x \) to \( x - ik/2m^2 \), so the term \( ik/2m^2 \) in the exponential has no effect on the value of the \( x \)-integral.

\[
\hat{f}(k) = e^{-k^2/4m^2} \int_{-\infty}^{\infty} dx \sqrt{2\pi} e^{-m^2 x^2} = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} dx \sqrt{2\pi} e^{-m^2 x^2} = \frac{1}{\sqrt{2} m} e^{-k^2/4m^2}. \tag{3.17}
\]

Thus, the Fourier transform of a gaussian is another gaussian

\[
\hat{f}(k) = \int_{-\infty}^{\infty} dx \sqrt{2\pi} e^{-ikx} e^{-m^2 x^2} = \frac{1}{\sqrt{2} m} e^{-k^2/4m^2}. \tag{3.18}
\]

But the two gaussians are very different: if the gaussian \( f(x) = \exp(-m^2 x^2) \) decreases slowly as \( x \to \infty \) because \( m \) is small (or quickly because \( m \) is big), then its gaussian Fourier transform \( \hat{f}(k) = \exp(-k^2/4m^2)/m\sqrt{2} \) decreases quickly as \( k \to \infty \) because \( m \) is small (or slowly because \( m \) is big).