Moreover if $f^{(k+1)}$ is piecewise continuous, then

$$\begin{aligned} f_n &= \int_{-\pi}^{\pi} \left\{ \frac{d}{dx} \left[f^{(k)}(x) \, \frac{e^{-inx}}{-(in)^{k+1}} \right] - f^{(k+1)}(x) \, \frac{e^{-inx}}{-(in)^{k+1}} \right\} dx \\ &= \int_{-\pi}^{\pi} f^{(k+1)}(x) \, \frac{e^{-inx}}{(in)^{k+1}} \, dx. \end{aligned} \tag{2.55}$$

Since $f^{(k+1)}(x)$ is piecewise continuous on the closed interval $[-\pi, \pi]$, it is bounded there in absolute value by, let us say, M. So the Fourier coefficients of a C^k periodic function with $f^{(k+1)}$ piecewise continuous are bounded by

$$|f_n| \le \frac{1}{n^{k+1}} \int_{-\pi}^{\pi} |f^{(k+1)}(x)| \, dx \le \frac{2\pi M}{n^{k+1}}. \tag{2.56}$$

We often can carry this derivation one step further. In most simple examples, the piecewise continuous periodic function $f^{(k+1)}(x)$ actually is piecewise continuously differentiable between its successive jumps at x_j. In this case, the derivative $f^{(k+2)}(x)$ is a piecewise continuous function plus a sum of a finite number of delta functions with finite coefficients. Thus we can integrate once more by parts. If for instance the function $f^{(k+1)}(x)$ jumps J times between $-\pi$ and π by $\Delta f_j^{(k+1)}$, then its Fourier coefficients are

$$\begin{aligned} f_n &= \int_{-\pi}^{\pi} f^{(k+2)}(x) \, \frac{e^{-inx}}{(in)^{k+2}} \, dx \\ &= \sum_{j=1}^{J} \int_{x_j}^{x_{j+1}} f_s^{(k+2)}(x) \, \frac{e^{-inx}}{(in)^{k+2}} \, dx + \sum_{j=1}^{J} \Delta f_j^{(k+1)} \, \frac{e^{-inx_j}}{(in)^{k+2}} \end{aligned} \tag{2.57}$$

in which the subscript s means that we've separated out the delta functions. The Fourier coefficients then are bounded by

$$|f_n| \le \frac{2\pi M}{n^{k+2}} \tag{2.58}$$

in which M is related to the maximum absolute values of $f_s^{(k+2)}(x)$ and of the $\Delta f_j^{(k+1)}$. The Fourier series of periodic C^k functions converge very rapidly if k is big.

Example 2.6 (Fourier Series of a C^0 Function) The function defined by

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ x & 0 \le x < \pi/2 \\ \pi - x & \pi/2 \le x \le \pi \end{cases} \tag{2.59}$$

is continuous on the interval $[-\pi, \pi]$ and its first derivative is piecewise