Moreover if \( f^{(k+1)} \) is piecewise continuous, then

\[
f_n = \int_{-\pi}^{\pi} \left\{ \frac{d}{dx} \left[ f^{(k)}(x) \frac{e^{-inx}}{-(in)^{k+1}} \right] - f^{(k+1)}(x) \frac{e^{-inx}}{-(in)^{k+1}} \right\} \, dx \\
= \int_{-\pi}^{\pi} f^{(k+1)}(x) \frac{e^{-inx}}{(in)^{k+1}} \, dx.
\]

Equation (2.55)

Since \( f^{(k+1)}(x) \) is piecewise continuous on the closed interval \([ -\pi, \pi ]\), it is bounded there in absolute value by, let us say, \( M \). So the Fourier coefficients of a \( C^k \) periodic function with \( f^{(k+1)} \) piecewise continuous are bounded by

\[
|f_n| \leq \frac{1}{n^{k+1}} \int_{-\pi}^{\pi} |f^{(k+1)}(x)| \, dx \leq \frac{2\pi M}{n^{k+1}}.
\]

Equation (2.56)

We often can carry this derivation one step further. In most simple examples, the piecewise continuous periodic function \( f^{(k+1)}(x) \) actually is piecewise continuously differentiable between its successive jumps at \( x_j \). In this case, the derivative \( f^{(k+2)}(x) \) is a piecewise continuous function plus a sum of a finite number of delta functions with finite coefficients. Thus we can integrate once more by parts. If for instance the function \( f^{(k+1)}(x) \) jumps \( J \) times between \( -\pi \) and \( \pi \) by \( \Delta f^{(k+1)}_j \), then its Fourier coefficients are

\[
f_n = \int_{-\pi}^{\pi} f^{(k+2)}(x) \frac{e^{-inx}}{(in)^{k+2}} \, dx \\
= \sum_{j=1}^{J} \int_{x_{j-1}}^{x_j} f^{(k+2)}(x) \frac{e^{-inx}}{(in)^{k+2}} \, dx + \sum_{j=1}^{J} \Delta f^{(k+1)}_j \frac{e^{-inx_j}}{(in)^{k+2}}
\]

Equation (2.57)

in which the subscript \( s \) means that we’ve separated out the delta functions. The Fourier coefficients then are bounded by

\[
|f_n| \leq \frac{2\pi M}{n^{k+2}}
\]

Equation (2.58)

in which \( M \) is related to the maximum absolute values of \( f^{(k+2)}_s(x) \) and of the \( \Delta f^{(k+1)}_j \). The Fourier series of periodic \( C^k \) functions converge very rapidly if \( k \) is big.

**Example 2.6 (Fourier Series of a \( C^0 \) Function)** The function defined by

\[
f(x) = \begin{cases} 
0 & -\pi \leq x < 0 \\
x & 0 \leq x < \pi/2 \\
\pi - x & \pi/2 \leq x \leq \pi 
\end{cases}
\]

Equation (2.59)

is continuous on the interval \([ -\pi, \pi ]\) and its first derivative is piecewise