L_3 for a hydrogen atom without spin:

$$
\begin{aligned}
H|n,\ell,m\rangle &= E_n|n,\ell,m\rangle \\
\boldsymbol{L}^2|n,\ell,m\rangle &= \hbar^2\ell(\ell+1)|n,\ell,m\rangle \\
L_3|n,\ell,m\rangle &= \hbar m|n,\ell,m\rangle.
\end{aligned}
\tag{1.415}
$$

Suppose the states $|\sigma\rangle$ for $\sigma = \pm$ are the eigenstates of the third component S_3 of the operator $\boldsymbol{S}$ that represents the spin of the electron

$$
S_3|\sigma\rangle = \sigma\frac{\hbar}{2}|\sigma\rangle. \tag{1.416}
$$

Then the direct- or tensor-product states

$$
|n,\ell,m,\sigma\rangle \equiv |n,\ell,m\rangle \otimes |\sigma\rangle \equiv |n,\ell,m\rangle|\sigma\rangle \tag{1.417}
$$

represent a hydrogen atom including the spin of its electron. They are eigenvectors of all four operators H, $\boldsymbol{L}^2$, L_3, and S_3:

$$
\begin{aligned}
H|n,\ell,m,\sigma\rangle &= E_n|n,\ell,m,\sigma\rangle \qquad & \boldsymbol{L}^2|n,\ell,m,\sigma\rangle &= \hbar^2\ell(\ell+1)|n,\ell,m,\sigma\rangle \\
L_3|n,\ell,m,\sigma\rangle &= \hbar m|n,\ell,m,\sigma\rangle \qquad & S_3|n,\ell,m,\sigma\rangle &= \sigma\frac{\hbar}{2}|n,\ell,m,\sigma\rangle.
\end{aligned}
\tag{1.418}
$$

Suitable linear combinations of these states are eigenvectors of the square $\boldsymbol{J}^2$ of the composite angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ as well as of J_3, L_3, and S_3. □

Example 1.51 (Adding Two Spins) The smallest positive value of angular momentum is $\hbar/2$. The spin-one-half angular momentum operators $\boldsymbol{S}$ are represented by three 2×2 matrices

$$
S_a = \frac{\hbar}{2}\,\sigma_a \tag{1.419}
$$

in which the σ_a are the Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.420}
$$

Consider two spin operators $\boldsymbol{S}^{(1)}$ and $\boldsymbol{S}^{(2)}$ acting on two spin-one-half systems. The states $|\pm\rangle_1$ are eigenstates of $S_3^{(1)}$, and the states $|\pm\rangle_2$ are eigenstates of $S_3^{(2)}$

$$
S_3^{(1)}|\pm\rangle_1 = \pm\frac{\hbar}{2}|\pm\rangle_1 \quad \text{and} \quad S_3^{(2)}|\pm\rangle_2 = \pm\frac{\hbar}{2}|\pm\rangle_2. \tag{1.421}
$$