Example 1.46  Suppose $A$ is the $3 \times 2$ matrix

$$A = \begin{pmatrix} r_1 & p_1 \\ r_2 & p_2 \\ r_3 & p_3 \end{pmatrix}$$

(1.385)

and the vector $|y\rangle$ is the cross-product $|y\rangle = L = r \times p$. Then no solution $|x\rangle$ exists to the equation $A|x\rangle = |y\rangle$ (unless $r$ and $p$ are parallel) because $A|x\rangle$ is a linear combination of the vectors $r$ and $p$ while $|y\rangle = L$ is perpendicular to both $r$ and $p$.

Even when the matrix $A$ is square, the equation (1.381) sometimes has no solutions. For instance, if $A$ is a square matrix that vanishes, $A = 0$, then (1.381) has no solutions whenever $|y\rangle \neq 0$. And when $N > M$, as in for instance

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(1.386)

the solution (1.384) is never unique, for we may add to it any linear combination of the vectors $|n\rangle$ that $A$ annihilates for $M < n \leq N$

$$|x\rangle = \sum_{n=1}^{\min(M,N)} \frac{\langle m_n | y \rangle}{S_n} |n\rangle + \sum_{n=M+1}^{N} x_n |n\rangle.$$  

(1.387)

These are the vectors $|n\rangle$ for $M < n \leq N$ that $A$ maps to zero since they do not occur in the sum (1.362) which stops at $n = \min(M, N) < N$.

Example 1.47 (The CKM Matrix)  In the standard model, the mass matrices of the $u$, $c$, $t$ and $d$, $s$, $b$ quarks are $3 \times 3$ complex matrices $M_u$ and $M_d$ with singular-value decompositions $M_u = U_u \Sigma_u V_u^\dagger$ and $M_d = U_d \Sigma_d V_d^\dagger$ whose singular-values are the quark masses. The unitary CKM matrix $U_u^\dagger U_d$ (Cabibbo, Kobayashi, Maskawa) describes transitions among the quarks mediated by the $W^\pm$ gauge bosons. By redefining the quark fields, one may make the CKM matrix real, apart from a phase that violates charge-conjugation-parity ($CP$) symmetry.

The adjoint of a complex symmetric matrix $M$ is its complex conjugate, $M^\dagger = M^\ast$. So by (1.351), its right singular vectors $|n\rangle$ are the eigenstates of $M^\ast M$

$$M^\ast M |n\rangle = S_n^2 |n\rangle$$

(1.388)
and by (1.366) its left singular vectors $|m_n\rangle$ are the eigenstates of $MM^*$

$$MM^*|m_n\rangle = (M^*M)^*|m_n\rangle = S_n^2|m_n\rangle.$$  

(1.389)

Thus its left singular vectors are the complex conjugates of its right singular vectors, $|m_n\rangle = |n\rangle^*$. So the unitary matrix $V$ is the complex conjugate of the unitary matrix $U$, and the SVD of $M$ is (Autonne, 1915)

$$M = U\Sigma U^\top.$$  

(1.390)

### 1.32 The Moore-Penrose Pseudoinverse

Although a matrix $A$ has an inverse $A^{-1}$ if and only if it is square and has a nonzero determinant, one may use the singular-value decomposition to make a pseudoinverse $A^+$ for an arbitrary $M \times N$ matrix $A$. If the singular-value decomposition of the matrix $A$ is

$$A = U\Sigma V^\dagger$$  

(1.391)

then the Moore-Penrose pseudoinverse (Eliakim H. Moore 1862–1932, Roger Penrose 1931–) is

$$A^+ = V\Sigma^+ U^\dagger$$  

(1.392)

in which $\Sigma^+$ is the transpose of the matrix $\Sigma$ with every nonzero entry replaced by its inverse (and the zeros left as they are). One may show that the pseudoinverse $A^+$ satisfies the four relations

$$AA^+A = A \quad \text{and} \quad A^+AA^+ = A^+$$

$$\left(AA^+\right)^\dagger = AA^+ \quad \text{and} \quad \left(A^+A\right)^\dagger = A^+A.$$  

(1.393)

and that it is the only matrix that does so.

Suppose that all the singular values of the $M \times N$ matrix $A$ are positive. In this case, if $A$ has more rows than columns, so that $M > N$, then the product $AA^+$ is the $N \times N$ identity matrix $I_N$

$$A^+A = V^\dagger\Sigma^+\Sigma V = V^\dagger I_N V = I_N$$  

(1.394)

and $AA^+$ is an $M \times M$ matrix that is not the identity matrix $I_M$. If instead $A$ has more columns than rows, so that $N > M$, then $AA^+$ is the $M \times M$ identity matrix $I_M$

$$AA^+ = U\Sigma \Sigma^+ U^\dagger = UI_M U^\dagger = I_M$$  

(1.395)

but $A^+A$ is an $N \times N$ matrix that is not the identity matrix $I_N$. If the