V_N and I_M for the M-dimensional space V_M the sums

$$I_N = \sum_{n=1}^{N} |n\rangle\langle n| \quad \text{and} \quad I_M = \sum_{n'=1}^{M} |m_{n'}\rangle\langle m_{n'}|. \tag{1.358}$$

The singular-value decomposition of A then is

$$A = I_M A I_N = \sum_{n'=1}^{M} |m_{n'}\rangle\langle m_{n'}|A \sum_{n=1}^{N} |n\rangle\langle n|. \tag{1.359}$$

There are $\min(M, N)$ singular values S_n all nonnegative. For the positive singular values, equations (1.352 & 1.354) show that the matrix element $\langle m_{n'}|A|n\rangle$ vanishes unless $n' = n$

$$\langle m_{n'}|A|n\rangle = \frac{1}{S_{n'}}\langle An'|An\rangle = S_{n'}\,\delta_{n'n}. \tag{1.360}$$

For the Z vanishing singular values, equation (1.353) shows that $A|n\rangle = 0$ and so

$$\langle m_{n'}|A|n\rangle = 0. \tag{1.361}$$

Thus only the $\min(M, N) - Z$ singular values that are positive contribute to the singular-value decomposition (1.359). If $N > M$, then there can be at most M nonzero eigenvalues e_n. If $N \le M$, there can be at most N nonzero e_n's. The final form of the singular-value decomposition then is a sum of dyadics weighted by the positive singular values

$$A = \sum_{n=1}^{\min(M,N)} |m_n\rangle S_n\,\langle n| = \sum_{n=1}^{\min(M,N)-Z} |m_n\rangle S_n\,\langle n|. \tag{1.362}$$

The vectors $|m_n\rangle$ and $|n\rangle$ respectively are the left and right singular vectors. The nonnegative numbers S_n are the singular values.

The linear operator A maps the $\min(M, N)$ right singular vectors $|n\rangle$ into the $\min(M, N)$ left singular vectors $S_n|m_n\rangle$ scaled by their singular values

$$A|n\rangle = S_n|m_n\rangle \tag{1.363}$$

and its adjoint $A^\dagger$ maps the $\min(M, N)$ left singular vectors $|m_n\rangle$ into the $\min(M, N)$ right singular vectors $|n\rangle$ scaled by their singular values

$$A^\dagger|m_n\rangle = S_n|n\rangle. \tag{1.364}$$

The N-dimensional vector space V_N is the **domain** of the linear operator A. If $N > M$, then A annihilates $N - M + Z$ of the basis vectors $|n\rangle$. The **null space** or **kernel** of A is the space spanned by the basis vectors $|n\rangle$ that

A annihilates. The vector space spanned by the left singular vectors $|m_n\rangle$ with nonzero singular values $S_n > 0$ is the **range** or **image** of A. It follows from the singular value decomposition (1.362) that the dimension N of the domain is equal to the dimension of the kernel $N - M + Z$ plus that of the range $M - Z$, a result called the **rank-nullity theorem**.

Incidentally, the vectors $|m_n\rangle$ are the eigenstates of the hermitian matrix $A\,A^\dagger$ as one may see from the explicit product of the expansion (1.362) with its adjoint

$$\begin{aligned} A\,A^\dagger &= \sum_{n=1}^{\min(M,N)} |m_n\rangle S_n \,\langle n| \sum_{n'=1}^{\min(M,N)} |n'\rangle S_{n'} \,\langle m_{n'}| \\ &= \sum_{n=1}^{\min(M,N)} \sum_{n'=1}^{\min(M,N)} |m_n\rangle S_n \,\delta_{nn'} S_{n'} \,\langle m_{n'}| \\ &= \sum_{n=1}^{\min(M,N)} |m_n\rangle S_n^2 \langle m_n| \end{aligned} \tag{1.365}$$

which shows that $|m_n\rangle$ is an eigenvector of $A\,A^\dagger$ with eigenvalue $e_n = S_n^2$

$$A\,A^\dagger |m_n\rangle = S_n^2 |m_n\rangle. \tag{1.366}$$

The SVD expansion (1.362) usually is written as a product of three explicit matrices, $A = U \Sigma V^\dagger$. The middle matrix Σ is an $M \times N$ matrix with the $\min(M,N)$ singular values $S_n = \sqrt{e_n}$ on its main diagonal and zeros elsewhere. By convention, one writes the S_n in decreasing order with the biggest S_n as entry Σ_{11}. The first matrix U and the third matrix $V^\dagger$ depend upon the bases one uses to represent the linear operator A. If these basis vectors are $|\alpha_k\rangle$ & $|\beta_\ell\rangle$, then

$$A_{k\ell} = \langle \alpha_k | A | \beta_\ell \rangle = \sum_{n=1}^{\min(M,N)} \langle \alpha_k | m_n \rangle S_n \,\langle n | \beta_\ell \rangle \tag{1.367}$$

so that the k, nth entry in the matrix U is $U_{kn} = \langle \alpha_k | m_n \rangle$. The columns of the matrix U are the left singular vectors of the matrix A:

$$\begin{pmatrix} U_{1n} \\ U_{2n} \\ \vdots \\ U_{Mn} \end{pmatrix} = \begin{pmatrix} \langle \alpha_1 | m_n \rangle \\ \langle \alpha_2 | m_n \rangle \\ \vdots \\ \langle \alpha_M | m_n \rangle \end{pmatrix}. \tag{1.368}$$

Similarly, the n, ℓth entry of the matrix $V^\dagger$ is $\left(V^\dagger\right)_{n,\ell} = \langle n | \beta_\ell \rangle$. Thus $V_{\ell,n} =$