Remarkably, this translation operator is an exponential of the momentum operator $U(a) = \exp(-i\,p\,a/\hbar)$ in which $\hbar = h/2\pi = 1.054 \times 10^{-34}$ Js is Planck's constant divided by 2π.

In two-dimensions, with basis states $|x, y\rangle$ that are orthonormal in Dirac's sense, $\langle x, y|x', y'\rangle = \delta(x - x')\delta(y - y')$, the unitary operator

$$U(\theta) = \int |x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta\rangle\langle x, y|\, dxdy \tag{1.171}$$

rotates a system in space by the angle θ. This rotation operator is the exponential $U(\theta) = \exp(-i\,\theta\,L_z/\hbar)$ in which the z component of the angular momentum is $L_z = x\,p_y - y\,p_x$.

We may carry most of our intuition about matrices over to these unitary transformations that change from one infinite basis to another. But we must use common sense and keep in mind that infinite sums and integrals do not always converge.

1.18 Antiunitary, Antilinear Operators

Certain maps on states $|\psi\rangle \to |\psi'\rangle$, such as those involving time reversal, are implemented by operators K that are **antilinear**

$$K(z\psi + w\phi) = K(z|\psi\rangle + w|\phi\rangle) = z^* K|\psi\rangle + w^* K|\phi\rangle = z^* K\psi + w^* K\phi \tag{1.172}$$

and **antiunitary**

$$(K\phi, K\psi) = \langle K\phi|K\psi\rangle = (\phi, \psi)^* = \langle\phi|\psi\rangle^* = \langle\psi|\phi\rangle = (\psi, \phi)\,. \tag{1.173}$$

In Dirac notation, these rules are $K(z|\psi\rangle) = z^*\langle\psi|$ and $K(w\langle\phi|) = w^*|\phi\rangle$.

1.19 Symmetry in Quantum Mechanics

In quantum mechanics, a symmetry is a one-to-one map of states $|\psi\rangle \leftrightarrow |\psi'\rangle$ and $|\phi\rangle \leftrightarrow |\phi'\rangle$ that preserves probabilities

$$|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2. \tag{1.174}$$

Eugene Wigner (1902–1995) showed that every symmetry in quantum mechanics can be represented either by an operator U that is linear and unitary or by an operator K that is antilinear and antiunitary. The antilinear, antiunitary case seems to occur only when the symmetry involves time reversal. Most symmetries are represented by operators that are linear and unitary. Unitary operators are of great importance in quantum mechanics.