and the formula (1.118) for the $k$th orthonormal linear combination of the vectors $|V_k\rangle$ is

$$|U_k\rangle = \frac{|u_k\rangle}{\sqrt{\langle u_k|u_k \rangle}}. \quad (1.139)$$

The vectors $|U_k\rangle$ are not unique; they vary with the order of the $|V_k\rangle$.

Vectors and linear operators are abstract. The numbers we compute with are inner products like $\langle g|f \rangle$ and $\langle g|A|f \rangle$. In terms of $N$ orthonormal basis vectors $|n\rangle$ with $f_n = \langle n|f \rangle$ and $g^*_n = \langle g|n \rangle$, we can use the expansion (1.131) to write these inner products as

$$\langle g|f \rangle = \sum_{n=1}^{N} \langle g|n \rangle \langle n|f \rangle = \sum_{n=1}^{N} g^*_n f_n$$

$$\langle g|A|f \rangle = \sum_{n,\ell=1}^{N} \langle g|n \rangle \langle n|A|\ell \rangle \langle \ell|f \rangle = \sum_{n,\ell=1}^{N} g^*_n A_{n\ell} f_\ell$$

in which $A_{n\ell} = \langle n|A|\ell \rangle$. We often gather the inner products $f_\ell = \langle \ell|f \rangle$ into a column vector $f$ with components $f_\ell = \langle \ell|f \rangle$

$$f = \begin{pmatrix} \langle 1|f \rangle \\ \langle 2|f \rangle \\ \vdots \\ \langle N|f \rangle \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}$$

(1.141)

and the $\langle n|A|\ell \rangle$ into a matrix $A$ with matrix elements $A_{n\ell} = \langle n|A|\ell \rangle$. If we also line up the inner products $\langle g|n \rangle = (n|g)^*$ in a row vector that is the transpose of the complex conjugate of the column vector $g$

$$g^\dagger = (\langle 1|g\rangle^*, \langle 2|g\rangle^*, \ldots, \langle N|g\rangle^*) = (g_1^*, g_2^*, \ldots, g_N^*)$$

(1.142)

then we can write inner products in matrix notation as $\langle g|f \rangle = g^\dagger f$ and as $\langle g|A|f \rangle = g^\dagger Af$.

If we switch to a different basis, say from $|n\rangle$’s to $|\alpha_n\rangle$’s, then the components of the column vectors change from $f_n = \langle n|f \rangle$ to $f'_n = \langle \alpha_n|f \rangle$, and similarly those of the row vectors $g^\dagger$ and of the matrix $A$ change, but the bras, the kets, the linear operators, and the inner products $\langle g|f \rangle$ and $\langle g|A|f \rangle$