that is nonnegative when the matrices are the same

\[(A, A) = \text{Tr} A^\dagger A = \sum_{i=1}^N \sum_{j=1}^L A_{ij}^* A_{ij} = \sum_{i=1}^N \sum_{j=1}^L |A_{ij}|^2 \geq 0 \quad (1.87)\]

which is zero only when \(A = 0\). So this inner product is positive definite. □

A vector space with a positive-definite inner product (1.73–1.76) is called an inner-product space, a metric space, or a pre-Hilbert space.

A sequence of vectors \(f_n\) is a Cauchy sequence if for every \(\varepsilon > 0\) there is an integer \(N(\varepsilon)\) such that \(\|f_n - f_m\| < \varepsilon\) whenever both \(n\) and \(m\) exceed \(N(\varepsilon)\). A sequence of vectors \(f_n\) converges to a vector \(f\) if for every \(\varepsilon > 0\) there is an integer \(N(\varepsilon)\) such that \(\|f - f_n\| < \varepsilon\) whenever \(n\) exceeds \(N(\varepsilon)\). An inner-product space with a norm defined as in (1.80) is complete if each of its Cauchy sequences converges to a vector in that space. A Hilbert space is a complete inner-product space. Every finite-dimensional inner-product space is complete and so is a Hilbert space. But the term Hilbert space more often is used to describe infinite-dimensional complete inner-product spaces, such as the space of all square-integrable functions (David Hilbert, 1862–1943).

Example 1.17 (The Hilbert Space of Square-Integrable Functions) For the vector space of functions (1.55), a natural inner product is

\[(f, g) = \int_a^b dx f^*(x)g(x). \quad (1.88)\]

The squared norm \(\|f\|\) of a function \(f(x)\) is

\[\|f\|^2 = \int_a^b dx |f(x)|^2. \quad (1.89)\]

A function is square integrable if its norm is finite. The space of all square-integrable functions is an inner-product space; it also is complete and so is a Hilbert space. □

Example 1.18 (Minkowski Inner Product) The Minkowski or Lorentz inner product \((p, x)\) of two 4-vectors \(p = (E/c, p_1, p_2, p_3)\) and \(x = (ct, x_1, x_2, x_3)\) is \(p \cdot x = Et\). It is indefinite, nondegenerate (1.79), and invariant under Lorentz transformations, and often is written as \(p \cdot x\) or as \(px\). If \(p\) is the 4-momentum of a freely moving physical particle of mass \(m\), then

\[p \cdot p = p^2 - E^2/c^2 = -c^2m^2 \leq 0. \quad (1.90)\]

The Minkowski inner product satisfies the rules (1.73, 1.74, and 1.79), but