

Elements of Supersymmetry*

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Abstract

These notes are intended to provide an introduction to supersymmetry. They begin with supersymmetric quantum mechanics and the basic properties of spinor fields. The supersymmetry of simple theories of spin-zero and spin-one-half fields is discussed with emphasis upon the charges that generate the transformations of supersymmetry. Abelian and non-abelian supersymmetric gauge theories are characterized in their simpler and more general forms. Superfields are defined, and the concise notation they make possible is described. The minimal supersymmetric standard model is discussed with particular attention to the Higgs fields and the electro-weak superpotential.

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1 Introduction

Most articles and books on supersymmetry are written for experts. These notes on supersymmetry are explicitly intended for graduate students in experimental high-energy physics and were written as part of a seminar which Michael Gold and I taught at the University of New Mexico during the spring of 1997.

Because papers about supersymmetry are often written in superfield notation and because superfields are defined in terms of Grassmann variables, most students get the impression that supersymmetry is harder than quantum field theory. In fact supersymmetric theories are nothing more than particular quantum field theories in which the fields and parameters satisfy special relations that make the theories *simpler, more symmetric, and easier to understand*.

The relative simplicity of supersymmetry is illustrated by the case of supersymmetric quantum mechanics, which is discussed in section 2. The exact description of the ground state, which typically in ordinary quantum mechanics is impossible, becomes trivial in supersymmetric quantum mechanics.

Section 3 is a laconic list of the notation that will be employed in this work. Section 4 contains a careful discussion of spin-one-half fields. Special attention is given to left-handed and right-handed two-component spinors and to Majorana and Dirac fields.

A discussion of the simplest supersymmetric field theories is given in section 5 with emphasis upon the fermionic charges that generate the transformations of supersymmetry. The most general supersymmetric theories of scalar and spinor fields are described in section 6.

Section 7 is an explicit description of the simpler supercharges and of the magic anti-commutation relations that they satisfy. The transformations they generate are characterized in section 9. Some formal properties of the supercharges of free supersymmetric theories are discussed in section 10.

The simpler supersymmetric gauge theories and the Fayet-Iliopoulos D term are defined in section 11. More general supersymmetric field theories are described in section 12. The breaking of gauge symmetry and supersymmetry

in super *QED* is worked out. Superfield notation is introduced in section 13.

The minimal supersymmetric standard model is discussed in section 14. The three reasons why two Higgs doublets are needed rather than one as in the standard model are explained. The electro-weak superpotential is studied in some detail.

2 Supersymmetric Quantum Mechanics

Supersymmetric quantum mechanics was invented Nicolai [2] and independently by Witten [3]. A quantum-mechanical system is supersymmetric if it has N charges Q_i that commute with the hamiltonian

$$[H, Q_i] = 0 \quad \text{for } i = 1, \dots, N \quad (1)$$

and satisfy the algebra

$$\{Q_i, Q_j\} = \delta_{ij}H. \quad (2)$$

The simplest example is for $N = 2$ and describes a spin-one-half particle moving on a line. The wave function $\psi(x)$ is

$$\psi(x) = \begin{pmatrix} \langle x, + | \psi \rangle \\ \langle x, - | \psi \rangle \end{pmatrix} = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}. \quad (3)$$

The charges Q_i are

$$\begin{aligned} Q_1 &= \frac{1}{2} \left(\frac{\sigma_1 p}{\sqrt{m}} + \sigma_2 \sqrt{m} W(x) \right) \\ Q_2 &= \frac{1}{2} \left(\frac{\sigma_2 p}{\sqrt{m}} - \sigma_1 \sqrt{m} W(x) \right), \end{aligned} \quad (4)$$

where the σ 's are the Pauli matrices (36) and

$$\langle x | p | \psi \rangle = \frac{\hbar}{i} \frac{d}{dx} \psi(x). \quad (5)$$

They are hermitian

$$Q_1^\dagger = Q_1 \quad \text{and} \quad Q_2^\dagger = Q_2. \quad (6)$$

The superpotential $W(x)$ may be any function of x that grows sufficiently at large $|x|$

$$|W(x)| \rightarrow \infty \quad \text{as} \quad |x| \rightarrow \infty \quad (7)$$

to ensure that the spectrum of H is discrete.

Explicit computation shows that

$$\{Q_1, Q_1\} = 2Q_1^2 = H = \frac{p^2}{2m} + \frac{mW^2(x)}{2} + \frac{\hbar}{2}\sigma_3 \frac{dW(x)}{dx}, \quad (8)$$

that

$$\{Q_2, Q_2\} = 2Q_2^2 = H = \frac{p^2}{2m} + \frac{mW^2(x)}{2} + \frac{\hbar}{2}\sigma_3 \frac{dW(x)}{dx}, \quad (9)$$

and that

$$\{Q_1, Q_2\} = 0. \quad (10)$$

Supersymmetry is unbroken if either charge Q_1 or Q_2 annihilates a normalizable state $\psi_0(x)$

$$Q_1\psi_0(x) = 0. \quad (11)$$

In this case the state $\psi_0(x)$ has zero energy

$$H\psi_0(x) = 2Q_1^2\psi_0(x) = 0, \quad (12)$$

and the other charge Q_2 must also annihilate the state $\psi_0(x)$

$$(Q_2\psi_0, Q_2\psi_0) = (\psi_0, Q_2^\dagger Q_2\psi_0) = (\psi_0, Q_2^2\psi_0) = \left(\psi_0, \frac{1}{2}H\psi_0\right) = 0. \quad (13)$$

Since both charges annihilate the state $\psi_0(x)$, that state is left invariant under the unitary transformation

$$U(\theta) = e^{-i\theta_i Q_i}, \quad (14)$$

and so supersymmetry is unbroken.

Note that all the energies E_n must be positive or zero since

$$H = 2Q_1^2 = 2Q_1^\dagger Q_1. \quad (15)$$

The state ψ_0 is easy to find if it exists:

$$\begin{aligned}
Q_1\psi_0(x) &= 0 \\
\left(\frac{\sigma_1 p}{2\sqrt{m}} + \frac{\sigma_2\sqrt{m}W(x)}{2}\right)\psi_0(x) &= 0 \\
\sigma_1 p\psi_0(x) &= -\sigma_2 mW(x)\psi_0(x) \\
\hbar\frac{d\psi_0(x)}{dx} &= \sigma_3 mW(x)\psi_0(x) \\
\psi_0(x) &= \exp\left(\frac{m}{\hbar}\int_0^x dy W(y)\sigma_3\right)\psi_0(0). \quad (16)
\end{aligned}$$

Now if the superpotential $W(x)$ satisfies

$$W(x) \rightarrow \infty \quad \text{as} \quad x \rightarrow \infty \quad (17)$$

and

$$W(x) \rightarrow -\infty \quad \text{as} \quad x \rightarrow -\infty, \quad (18)$$

then the normalizable state ψ_0 is

$$\psi_0(x) = \exp\left(-\frac{m}{\hbar}\int_0^x dy W(y)\sigma_3\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (19)$$

If $W(x)$ is continuous, then $W(x_0) = 0$ for some x_0 .

Equivalently if the superpotential $W(x)$ satisfies

$$W(x) \rightarrow -\infty \quad \text{as} \quad x \rightarrow \infty \quad (20)$$

and

$$W(x) \rightarrow \infty \quad \text{as} \quad x \rightarrow -\infty, \quad (21)$$

then the normalizable state ψ_0 is

$$\psi_0(x) = \exp\left(\frac{m}{\hbar}\int_0^x dy W(y)\sigma_3\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (22)$$

Again if $W(x)$ is continuous, then $W(x_0) = 0$ for some x_0 .

But if the superpotential $W(x)$ satisfies

$$W(x) \rightarrow \infty \quad \text{as} \quad |x| \rightarrow \infty \quad (23)$$

or

$$W(x) \rightarrow -\infty \quad \text{as} \quad |x| \rightarrow \infty, \quad (24)$$

then there is no normalizable state ψ_0 that is annihilated by Q_1 or by Q_2 . In this case, supersymmetry is dynamically broken.

3 Notation

We shall use an Anglicized version of the notation of Bagger and Wess, which they describe in the appendices of their book [1]. Space-time indices are labeled by letters like l, m, n . Spatial indices are labeled by letters like i, j, k . The flat metric η is

$$(\eta^{mn}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (25)$$

Dotted and undotted spinor indices run from 1 to 2 and are denoted by the early letters of the English alphabet. Spinor indices are raised and lowered by the ε tensors

$$(\varepsilon^{ab}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \quad (26)$$

and

$$(\varepsilon_{ab}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2. \quad (27)$$

For example, $\psi^a = \varepsilon^{ab}\psi_b$

$$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ -\psi_1 \end{pmatrix} \quad (28)$$

and $\chi_a = \varepsilon_{ab}\chi^b$

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} -\chi^2 \\ \chi^1 \end{pmatrix}. \quad (29)$$

If the spinors ψ and χ anti-commute, then the product $\psi\chi$ is

$$\psi\chi = \psi^a\chi_a = -\psi_a\chi^a = \chi^a\psi_a = \chi\psi. \quad (30)$$

The identity

$$\psi^a\psi^b = -\frac{1}{2}\psi\psi\varepsilon^{ab} \quad (31)$$

is occasionally useful.

The hermitian conjugates of the spinors ψ and χ are $\bar{\psi}$ and $\bar{\chi}$:

$$\begin{aligned}\bar{\psi}^{\dot{a}} &= (\psi^a)^\dagger \\ \bar{\chi}_{\dot{a}} &= (\chi_a)^\dagger.\end{aligned}\tag{32}$$

Their dotted indices are raised and lowered by the tensors $\varepsilon^{\dot{a}b}$ and $\varepsilon_{\dot{a}b}$ which are equal to their undotted counterparts ε^{ab} and ε_{ab} . The product $\bar{\psi}\bar{\chi}$ is

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{a}}\bar{\chi}^{\dot{a}} = -\bar{\psi}^{\dot{a}}\bar{\chi}_{\dot{a}} = \bar{\chi}_{\dot{a}}\bar{\psi}^{\dot{a}} = \bar{\chi}\bar{\psi}.\tag{33}$$

These definitions allow us to write

$$(\chi\psi)^\dagger = (\chi^a\psi_a)^\dagger = \bar{\psi}_{\dot{a}}\bar{\chi}^{\dot{a}} = \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}\tag{34}$$

and

$$\bar{\psi}^{\dot{a}}\bar{\psi}^{\dot{b}} = \frac{1}{2}\bar{\psi}\bar{\psi}\varepsilon^{\dot{a}b}.\tag{35}$$

The Pauli matrices σ_{ab}^m are

$$\begin{aligned}\sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}\tag{36}$$

in which the rule $\sigma^0 = -I$ is occasioned by the choice of metric $\eta = (-1, 1, 1, 1)$. The barred Pauli matrices $\bar{\sigma}^{m\dot{a}b}$ are defined as

$$\bar{\sigma}^{m\dot{a}b} = \varepsilon^{\dot{a}c}\varepsilon^{bd}\sigma_{dc}^m, \quad \text{whence} \quad \sigma_{ab}^n = \varepsilon_{ac}\varepsilon_{bd}\bar{\sigma}^{ndc},\tag{37}$$

and are related to the unbarred ones by

$$\begin{aligned}\bar{\sigma}^0 &= \sigma^0 = -I \\ \vec{\bar{\sigma}} &= -\vec{\sigma}.\end{aligned}\tag{38}$$

The generators of the Lorentz group in the spinor representation are

$$\sigma^{nm}{}_a{}^c = \frac{1}{4}\left(\sigma_{ab}^n\bar{\sigma}^{mbc} - \sigma_{ab}^m\bar{\sigma}^{nbc}\right)\tag{39}$$

$$\bar{\sigma}^{nm\dot{a}}{}_{\dot{c}} = \frac{1}{4}\left(\bar{\sigma}^{n\dot{a}b}\sigma_{bc}^m - \bar{\sigma}^{m\dot{a}b}\sigma_{bc}^n\right).\tag{40}$$

The Pauli matrices satisfy the relations

$$(\sigma^m \bar{\sigma}^n + \sigma^n \bar{\sigma}^m)_a^b = -2\eta^{mn} \delta_a^b, \quad (41)$$

$$(\bar{\sigma}^m \sigma^n + \bar{\sigma}^n \sigma^m)^{\dot{a}}_{\dot{b}} = -2\eta^{mn} \delta_{\dot{b}}^{\dot{a}}, \quad (42)$$

the trace identities

$$\text{Tr } \sigma^m \bar{\sigma}^n = -2\eta^{mn} \quad \text{and} \quad \text{Tr } \sigma^m \sigma^n = 2\delta^{mn}, \quad (43)$$

and the completeness relation

$$\sigma_{a\dot{c}}^m \bar{\sigma}_m^{\dot{b}c} = -2\delta_a^b \delta_{\dot{c}}^{\dot{d}}, \quad (44)$$

which implies that

$$(\psi\phi) \bar{\chi}_{\dot{b}} = -\frac{1}{2} \phi \sigma^n \bar{\chi} (\psi \sigma_n)_{\dot{b}} = -\frac{1}{2} \phi \sigma^n \bar{\chi} \psi^a \sigma_{nab}. \quad (45)$$

The adjoint of this relation is

$$\chi_b (\bar{\phi} \bar{\psi}) = -\frac{1}{2} (\sigma_n \bar{\psi})_b \chi \sigma^n \phi. \quad (46)$$

Other useful identities are

$$(\bar{\xi} \bar{\sigma}^m)_a = \varepsilon_{ab} (\bar{\xi} \bar{\sigma}^m)^b = -\sigma_{a\dot{c}}^m \bar{\xi}^{\dot{c}}, \quad (47)$$

$$(\bar{\sigma}^m \xi)^{\dot{a}} = -(\xi \sigma^m)^{\dot{a}} = -\varepsilon^{\dot{a}c} \xi^d \sigma_{dc}^m, \quad (48)$$

$$\chi \sigma^n \bar{\psi} = -\bar{\psi} \bar{\sigma}^n \chi, \quad (49)$$

$$\chi \sigma^m \bar{\sigma}^n \psi = \psi \sigma^n \bar{\sigma}^m \chi, \quad (50)$$

$$(\chi \sigma^m \bar{\psi})^\dagger = \psi \sigma^m \bar{\chi}, \quad (51)$$

and

$$(\chi \sigma^m \bar{\sigma}^n \psi)^\dagger = \bar{\psi} \bar{\sigma}^n \sigma^m \bar{\chi}. \quad (52)$$

4 Spinor Fields

All known matter fields are of spin one-half. Yet spinor fields are confusing, and the authors of most books choose to assume that the topic has been worked out somewhere else.

4.1 The Right-Handed Massless Field

The Lagrange density

$$\mathcal{L}_K = \frac{i}{2} \partial_n \bar{\psi} \bar{\sigma}^n \psi - \frac{i}{2} \bar{\psi} \bar{\sigma}^n \partial_n \psi \quad (53)$$

describes a massless two-component spinor field ψ of spin one-half which satisfies the equation

$$\bar{\sigma}^n \partial_n \psi = 0. \quad (54)$$

This field transforms according to the $(\frac{1}{2}, 0)$ representation of the Lorentz group and so carries undotted indices. We may expand ψ in terms of the operators $b(p, s)$ that annihilate the particles of the field ψ and the operators $b_c^\dagger(p, s)$ that create the particles that are the anti-particles of the particles of the field ψ :

$$\psi_a(x) = \sum_s \int \frac{d^3 p}{\sqrt{(2\pi)^3}} \left[e^{ip \cdot x} u_a(p, s) b(p, s) + e^{-ip \cdot x} v_a(p, s) b_c^\dagger(p, s) \right], \quad (55)$$

in which $p^0 = |\vec{p}|$. We shall see that the states $b^\dagger(p, s)|0\rangle$ and $b_c^\dagger(p, s)|0\rangle$ are different, because they have opposite helicity whether or not the field ψ carries a charge. But circumstances and conventions will determine whether we call these states particles and anti-particles or just different helicity states of the same particle.

Since the free field ψ satisfies the free-field equation (54), the spinors $u(p, s)$ and $v(p, s)$ must be eigenvectors of $\vec{\sigma} \cdot \vec{p}$

$$\vec{\sigma} \cdot \vec{p} u(p, s) = p^0 u(p, s) \quad \text{and} \quad \vec{\sigma} \cdot \vec{p} v(p, s) = p^0 v(p, s) \quad (56)$$

with positive eigenvalue p^0 . Now $\vec{\sigma} \cdot \vec{p}$ has two eigenvalues $\pm |\vec{p}|$, and there is only one eigenvector of $\vec{\sigma} \cdot \vec{p}$ with positive eigenvalue $p^0 = |\vec{p}|$. Normalized to unity, this eigenvector $u(p)$ is

$$u(p) = \frac{1}{\sqrt{2p^0(p^0 + p^3)}} \begin{pmatrix} p^0 + p^3 \\ p^1 + ip^2 \end{pmatrix} \quad (57)$$

apart from a phase factor.

Thus in the expansion (55) only one helicity state survives, and the expansion of the massless field ψ takes the form

$$\psi_a(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} \left[e^{ip \cdot x} u_a(p, \frac{1}{2}) b(p, \frac{1}{2}) + e^{-ip \cdot x} v_a(p, -\frac{1}{2}) b_c^\dagger(p, -\frac{1}{2}) \right] \quad (58)$$

in which $u_a(p, \frac{1}{2}) = v_a(p, -\frac{1}{2}) = u_a(p)$. Equivalently, we may use the briefer notation

$$\psi_a(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} u_a(p) \left[e^{ip \cdot x} b(p) + e^{-ip \cdot x} b_c^\dagger(p) \right]. \quad (59)$$

We impose the anti-commutation relations

$$\{b(p, s), b^\dagger(k, s')\} = \delta_{ss'} \delta(\vec{p} - \vec{k}) \quad \text{and} \quad \{b_c(p, s), b_c^\dagger(k, s')\} = \delta_{ss'} \delta(\vec{p} - \vec{k}) \quad (60)$$

or equivalently

$$\{b(p), b^\dagger(k)\} = \delta(\vec{p} - \vec{k}) \quad \text{and} \quad \{b_c(p), b_c^\dagger(k)\} = \delta(\vec{p} - \vec{k}) \quad (61)$$

and require all other anti-commutators of b and b_c operators to vanish. It follows then from these relations and from the explicit form (57) of the spinor $u(p)$ that the field ψ satisfies the equal-time anti-commutation relations

$$\{\psi_a(t, \vec{x}), \psi_b^\dagger(t, \vec{y})\} = \delta_{ab} \delta(\vec{x} - \vec{y}), \quad (62)$$

and

$$\{\psi_a(t, \vec{x}), \psi_b(t, \vec{y})\} = 0. \quad (63)$$

The spin part of the angular-momentum operator is

$$\vec{\Sigma} = \int d^3x : \psi^\dagger(x) \frac{1}{2} \vec{\sigma} \psi(x) : \quad (64)$$

in which the colons indicate (fermionic) normal ordering. By using the anti-commutation relations (62) and (63), one may show that the spin operator obeys the angular-momentum commutation relations

$$[\Sigma_i, \Sigma_j] = i \epsilon_{ijk} \Sigma_k \quad (65)$$

in which the Levi-Civita tensor ϵ_{ijk} is totally anti-symmetric with $\epsilon_{123} = 1$. If we substitute the briefer expansion (59) of the field ψ into the spin operator Σ , then we may write it as

$$\vec{\Sigma} = \int d^3k \left[b^\dagger(k) b(k) - b_c^\dagger(k) b_c(k) \right] u^\dagger(k) \frac{1}{2} \vec{\sigma} u(k). \quad (66)$$

By using the anti-commutation relations (61) and the relation $\vec{p} \cdot \vec{\sigma} u(p) = |\vec{p}|u(p)$, we may show that the state $b^\dagger(p)|0\rangle$ is an eigen-state of the helicity operator $\vec{P} \cdot \vec{\Sigma}$ with eigen-value $|\vec{p}|/2$:

$$\begin{aligned}
& \vec{P} \cdot \vec{\Sigma} b^\dagger(p) |0\rangle \\
&= \int d^3k \left[b^\dagger(k)b(k) - b_c^\dagger(k)b_c(k) \right] u^\dagger(k) \frac{1}{2} \vec{P} \cdot \vec{\sigma} u(k) b^\dagger(p) |0\rangle \\
&= \int d^3k b^\dagger(k)b(k) u^\dagger(k) \frac{1}{2} \vec{p} \cdot \vec{\sigma} u(k) b^\dagger(p) |0\rangle \\
&= \int d^3k b^\dagger(k) \delta(\vec{k} - \vec{p}) u^\dagger(k) \frac{1}{2} \vec{p} \cdot \vec{\sigma} u(k) |0\rangle \\
&= \left(u^\dagger(p) \frac{1}{2} \vec{p} \cdot \vec{\sigma} u(p) \right) b^\dagger(p) |0\rangle = \left(\frac{|\vec{p}|}{2} \right) b^\dagger(p) |0\rangle. \tag{67}
\end{aligned}$$

We say that the ψ -particle and the state $|b, \vec{p}\rangle$ are right handed, or have positive helicity, because their momentum and spin are parallel. A similar computation for the state $b_c^\dagger(p)|0\rangle$ gives

$$\vec{P} \cdot \vec{\Sigma} b_c^\dagger(p) |0\rangle = - \left(\frac{|\vec{p}|}{2} \right) b_c^\dagger(p) |0\rangle. \tag{68}$$

The anti- ψ -particle and the state $|b_c, \vec{p}\rangle$ are said to be left handed, or to have negative helicity, because their momentum and spin are anti-parallel.

The field ψ is called right handed because it annihilates particles that are right handed and creates particles that are left handed. Since the state $|b, \vec{p}\rangle$ and the state $|b_c, \vec{p}\rangle$ have opposite helicities, they can not be the same. It is conventional to distinguish the two states by the words “particle” and “anti-particle” if the field ψ carries a conserved charge. For instance, we speak of the electron and the anti-electron, or positron, but not of the photon and the anti-photon.

4.2 The Left-Handed Massless Field

If the spinor ψ_a is right handed, then the conjugate spinor $\bar{\psi}^{\dot{a}}$ is left handed. We might emphasize this change in handedness by using the letter χ instead of ψ and writing explicitly

$$\bar{\chi}^{\dot{a}} = \varepsilon^{\dot{a}b} \bar{\psi}_b = \varepsilon^{\dot{a}b} (\psi_b)^\dagger \tag{69}$$

$$\chi^a = \varepsilon^{ab} \psi_b. \quad (70)$$

By making these substitutions in the action density (53), using the identity (49), and ignoring any c-number terms arising from the use of that identity, we may arrive at the Lagrange density for a massless, left-handed, two-component spinor field $\bar{\chi}$

$$\mathcal{L}_K = \frac{i}{2} \partial_n \chi \sigma^n \bar{\chi} - \frac{i}{2} \chi \sigma^n \partial_n \bar{\chi}. \quad (71)$$

This field of spin one-half satisfies the equation

$$\sigma^n \partial_n \bar{\chi} = 0. \quad (72)$$

The bar indicates that the field $\bar{\chi}$ transforms according to the $(0, \frac{1}{2})$ representation of the Lorentz group and so carries dotted indices. (Whether this dotty notation is worth the effort is unclear.) Now in the briefer expansion

$$\bar{\chi}_a(x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3}} v_a(p) \left[e^{ip \cdot x} c(p) + e^{-ip \cdot x} c_c^\dagger(p) \right], \quad (73)$$

the spinor $v(p)$ must be an eigenvector of $\vec{\sigma} \cdot \vec{p}$ with eigenvalue $-p^0 = -|\vec{p}|$

$$\vec{\sigma} \cdot \vec{p} v(p) = -|\vec{p}| v(p). \quad (74)$$

Normalized to unity, this eigenvector $v(p)$ is

$$v(p) = \frac{1}{\sqrt{2p^0(p^0 + p^3)}} \begin{pmatrix} p^1 - ip^2 \\ -p^0 - p^3 \end{pmatrix} \quad (75)$$

up to an arbitrary phase factor here chosen so that

$$v(p)^{\dot{a}} = \varepsilon^{\dot{a}b} (u_b)^* \quad (76)$$

in conformity with (69).

We impose the anti-commutation relations

$$\{c(p, s), c^\dagger(k, s')\} = \delta_{ss'} \delta(\vec{p} - \vec{k}) \quad \text{and} \quad \{c_c(p, s), c_c^\dagger(k, s')\} = \delta_{ss'} \delta(\vec{p} - \vec{k}) \quad (77)$$

or equivalently

$$\{c(p), c^\dagger(k)\} = \delta(\vec{p} - \vec{k}) \quad \text{and} \quad \{c_c(p), c_c^\dagger(k)\} = \delta(\vec{p} - \vec{k}) \quad (78)$$

which together with the explicit form (75) of the spinor $v(p)$ imply that the field χ satisfies the equal-time anti-commutation relations

$$\{\chi_a(t, \vec{x}), \chi_b^\dagger(t, \vec{y})\} = \delta_{ab} \delta(\vec{x} - \vec{y}), \quad (79)$$

and

$$\{\chi_a(t, \vec{x}), \chi_b(t, \vec{y})\} = 0. \quad (80)$$

The spin part of the angular momentum operator is

$$\vec{\Sigma} = \int d^3x : \chi(x) \frac{1}{2} \vec{\sigma} \bar{\chi}(x) : \quad (81)$$

in which the colons indicate (fermionic) normal ordering. As in the case of the right-handed field, one may derive the eigen-value equations

$$\vec{P} \cdot \vec{\Sigma} c^\dagger(p) |0\rangle = - \left(\frac{|\vec{p}|}{2} \right) c^\dagger(p) |0\rangle \quad (82)$$

$$\vec{P} \cdot \vec{\Sigma} c_c^\dagger(p) |0\rangle = \left(\frac{|\vec{p}|}{2} \right) c_c^\dagger(p) |0\rangle \quad (83)$$

which show that the particles annihilated by the field χ are left handed and that their anti-particles are right handed. That is, the state $c^\dagger(p)|0\rangle$ is left handed, or of negative helicity, in the sense that its momentum and its spin are in opposite directions, while the state $c_c^\dagger(p)|0\rangle$ is right handed, or of positive helicity, because its momentum is parallel to its spin.

4.3 Majorana Mass

Let us now return to the massless right-handed action density (53) and add to it a Majorana-mass term and its hermitian conjugate

$$\mathcal{L}_K = \frac{i}{2} \partial_n \bar{\psi} \bar{\sigma}^n \psi - \frac{i}{2} \bar{\psi} \bar{\sigma}^n \partial_n \psi - \frac{1}{2} m \psi \psi - \frac{1}{2} m \bar{\psi} \bar{\psi}, \quad (84)$$

in which we have absorbed a possible phase in the Majorana mass m into the definition of the field ψ . If the mass m is really a constant, and not a field, then the field ψ can not carry a charge, at least not one whose conservation is formally implied by a symmetry of the action. It is possible that the mass m is actually a field \hat{m} that assumes a mean value m in the vacuum. And in this case, the field ψ may carry a charge q if the mass field \hat{m} carries charge $-2q$. But if the mass field \hat{m} acquires the non-zero mean value m in the vacuum, then the symmetry of the action that formally implies the conservation of the charge q would be spontaneously broken. So the field ψ must either be neutral or the carrier of a charge of a broken symmetry.

We shall consider here the case in which m is really a constant. The field ψ then satisfies the equation

$$-i\vec{\sigma}^{n\dot{a}b}\partial_n\psi_b = m\bar{\psi}^{\dot{a}} = m\varepsilon^{\dot{a}b}\bar{\psi}_b. \quad (85)$$

In a simpler undotted notation, with $\varepsilon^{\dot{a}b} = i\sigma_{ab}^2 = i\sigma_{2ab}$, this equation is

$$i(\partial_0 + \vec{\sigma} \cdot \nabla)\psi = im\sigma_2\psi^\dagger. \quad (86)$$

If we try to solve this equation by expanding the field ψ in the form

$$\psi_a(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3}} \left[e^{ip \cdot x} u_a(p, s) b(p, s) - i\sigma_{2ab} e^{-ip \cdot x} v_b^*(p, s) b_c^\dagger(p, s) \right], \quad (87)$$

then we find

$$(p^0 - \vec{\sigma} \cdot \vec{p}) u(p, s) b(p, s) = mv(p, s) b_c(p, s) \quad (88)$$

$$(-p^0 + \vec{\sigma} \cdot \vec{p}) (-i\sigma^2 v^*(p, s)) b_c^\dagger(p, s) = im\sigma^2 u^*(p, s) b^\dagger(p, s). \quad (89)$$

Evidently the particles described by this field, ψ , must be the same as their anti-particles

$$b(p, s) = b_c(p, s). \quad (90)$$

By using the familiar relation $\sigma^i \sigma^j = \delta^{ij} + i\varepsilon_{ijk} \sigma^k$, we may rewrite the spinor equations as

$$(p^0 - \vec{\sigma} \cdot \vec{p}) u(p, s) = mv(p, s) \quad (91)$$

$$(p^0 + \vec{\sigma} \cdot \vec{p}) v(p, s) = mu(p, s). \quad (92)$$

By combining these equations, we find

$$\begin{aligned} \left((p^0)^2 - (\vec{p})^2 \right) u(p, s) &= \left(p^0 + \vec{\sigma} \cdot \vec{p} \right) \left(p^0 - \vec{\sigma} \cdot \vec{p} \right) u(p, s) \\ &= m \left(p^0 + \vec{\sigma} \cdot \vec{p} \right) v(p, s) = m^2 u(p, s), \end{aligned} \quad (93)$$

which shows that $(p^0)^2 = (\vec{p})^2 + m^2$. The spinor equations (91) and (92) together form the “positive-mass” Dirac equation

$$(ip^n \gamma_n + m) U(p, s) = 0, \quad (94)$$

and the spinors $u(p, s)$ and $v(p, s)$ may be identified with the lower and upper components of its solution $U(p, s)$

$$U(p, s) = \begin{pmatrix} v(p, s) \\ u(p, s) \end{pmatrix}. \quad (95)$$

The addition of Majorana mass terms has converted the action density of a massless right-handed field to that of a massive field whose particles and anti-particles are the same or differ by a charge of a broken symmetry.

We may also add Majorana mass terms to the action density (71) of the left-handed field χ

$$\mathcal{L}_K = \frac{i}{2} \partial_n \chi \sigma^n \bar{\chi} - \frac{i}{2} \chi \sigma^n \partial_n \bar{\chi} - \frac{1}{2} m \chi \chi - \frac{1}{2} m \bar{\chi} \bar{\chi}, \quad (96)$$

in which we have absorbed a possible phase in the Majorana mass m into the definition of the field χ . The equation of motion is now

$$-i \sigma_{ab}^n \partial_n \bar{\chi}^b = m \chi_a = m \varepsilon_{ab} \chi^b. \quad (97)$$

4.4 The Dirac Field

The original spin-one-half field is the Dirac field which is a four-component combination of the right-handed field ψ and the left-handed field $\bar{\chi}$. In the massless case, the action density for the Dirac field is just the sum of (71) for the left-handed field $\bar{\chi}$ and of (53) for the right-handed field ψ

$$\mathcal{L}_K = \frac{i}{2} \partial_n \chi \sigma^n \bar{\chi} - \frac{i}{2} \chi \sigma^n \partial_n \bar{\chi} + \frac{i}{2} \partial_n \bar{\psi} \bar{\sigma}^n \psi - \frac{i}{2} \bar{\psi} \bar{\sigma}^n \partial_n \psi, \quad (98)$$

and little is gained by the combination

$$\Psi_l = \begin{pmatrix} \bar{\chi}^{\dot{a}} \\ \psi_b \end{pmatrix}, \quad (99)$$

except that the two field equations (54) and (72) can be written as

$$\gamma^n \partial_n \Psi(x) = 0. \quad (100)$$

The γ matrices are required to satisfy the anti-commutation relations

$$\{\gamma^n, \gamma^m\} = 2g^{nm}. \quad (101)$$

A useful set of γ matrices is

$$\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \quad (102)$$

The real point of the Dirac field is that it provides a simple way of giving mass to a field that carries a conserved charge and whose particles and anti-particles are different. In terms of the two-component fields ψ and χ , the action density for the massive case is after an integration by parts

$$\mathcal{L}_D = -i\chi\sigma^n\partial_n\bar{\chi} - i\bar{\psi}\bar{\sigma}^n\partial_n\psi - m\chi\psi - m\bar{\psi}\bar{\chi}, \quad (103)$$

which in a simpler notation is

$$\mathcal{L}_D = i\bar{\chi}^\dagger (\partial_0 - \vec{\sigma} \cdot \nabla) \bar{\chi} + i\psi^\dagger (\partial_0 + \vec{\sigma} \cdot \nabla) \psi - m\bar{\chi}^\dagger\psi - m\psi^\dagger\bar{\chi}. \quad (104)$$

In Dirac's notation this action density is

$$\mathcal{L}_D = -\bar{\Psi} (\gamma^n \partial_n + m) \Psi \quad (105)$$

in which $\bar{\Psi} \equiv i\Psi^\dagger\gamma^0$. The Dirac field of mass m satisfies

$$(\gamma^n \partial_n + m) \Psi = 0. \quad (106)$$

The corresponding two-component equations are

$$-i\sigma_{ab}^n \partial_n \bar{\chi}^{\dot{b}} - m\psi_a = 0 \quad (107)$$

$$-i\bar{\sigma}^{n\dot{a}b} \partial_n \psi_b - m\bar{\chi}^{\dot{a}} = 0. \quad (108)$$

The expansion of the Dirac field in terms of operators $b(p, s)$ that annihilate the particles of the field Ψ and operators $b_c^\dagger(p, s)$ that create their anti-particles is

$$\Psi_l(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[U_l(p, s) e^{ipx} b(p, s) + V_l(p, s) e^{-ipx} b_c^\dagger(p, s) \right] \quad (109)$$

in which the sum over spin states s is from $-1/2$ to $1/2$ and the index l runs from 1 to 4. The spinors $U(p, s)$ and $V(p, s)$ are normalized to $\delta_{ss'}$ and satisfy the equations

$$(ip^n \gamma_n + m) U(p, s) = 0 \quad \text{and} \quad (ip^n \gamma_n - m) V(p, s) = 0 \quad (110)$$

as well as

$$\sum_{s=-\frac{1}{2}}^{\frac{1}{2}} U(p, s) U^\dagger(p, s) = \frac{i}{2p^0} (-ip^n \gamma_n + m) \gamma^0 \quad (111)$$

and

$$\sum_{s=-\frac{1}{2}}^{\frac{1}{2}} V(p, s) V^\dagger(p, s) = \frac{i}{2p^0} (-ip^n \gamma_n - m) \gamma^0. \quad (112)$$

4.5 The Seesaw Mechanism

If the fields χ and ψ do not carry any conserved charge, as is the case for the neutrino, for instance, then we may add Majorana mass terms to the action density (103). Let us allow for the possibility of two different Majorana masses, m_χ and m_ψ , and let us use m_d for the Dirac mass. Because we have three mass parameters and only two fields, we should consider these masses to be complex. The notation will be slightly clearer if we use right-handed fields:

$$\begin{aligned} \mathcal{L}_D = & -i\bar{\chi}\bar{\sigma}^n \partial_n \chi - i\bar{\psi}\bar{\sigma}^n \partial_n \psi - m_d \chi \psi - m_d^* \bar{\psi} \bar{\chi} \\ & - \frac{1}{2} m_\chi \chi \chi - \frac{1}{2} m_\chi^* \bar{\chi} \bar{\chi} - \frac{1}{2} m_\psi \psi \psi - \frac{1}{2} m_\psi^* \bar{\psi} \bar{\psi}. \end{aligned} \quad (113)$$

If the complex parameter $m_\chi m_d^* + m_d m_\psi^*$ is real, then the matrix

$$M = \begin{pmatrix} m_\chi & m_d \\ m_d & m_\psi \end{pmatrix} \quad (114)$$

is normal, $[M, M^\dagger] = 0$, and may be diagonalized by a unitary matrix. In this case the masses are

$$m_{\pm} = \frac{1}{2} \left| m_{\chi} + m_{\psi} \pm \sqrt{(m_{\chi} - m_{\psi})^2 + 4m_d^2} \right|. \quad (115)$$

If one of the three masses vanishes, then by redefining the two fields we may make both of the non-zero masses real. If the fields χ and ψ respectively represent the charge conjugate of the left-handed neutrino and the right-handed neutrino, then in the standard model the Majorana mass term m_{χ} must vanish by gauge invariance. Thus in this case we may take both m_{ψ} and m_d to be real. The physical masses then are

$$\pm m_{\pm} = \frac{1}{2} \left(m_{\psi} \pm \sqrt{m_{\psi}^2 + 4m_d^2} \right), \quad (116)$$

and their product is

$$m_+ m_- = m_d^2. \quad (117)$$

Thus if the Majorana mass term is very much bigger than the Dirac mass term, $m_{\psi} \gg m_d$, then

$$m_+ \approx m_{\psi} \quad \text{and} \quad m_- \approx \frac{m_d^2}{m_{\psi}}, \quad (118)$$

which would nicely explain why the masses of the neutrinos are so tiny.

4.6 The General Neutral Spinor Field

The normal case forms a set of measure zero in the space of theories. To analyze the general case of three complex mass parameters, we use the undotted notation (104) in which the action density is

$$\begin{aligned} \mathcal{L}_N = & i\chi^\dagger (\partial_0 + \vec{\sigma} \cdot \nabla) \chi + i\psi^\dagger (\partial_0 + \vec{\sigma} \cdot \nabla) \psi + im_d \chi \sigma_2 \psi - im_d^* \psi^\dagger \sigma_2 \chi^\dagger \\ & + \frac{i}{2} m_{\chi} \chi \sigma_2 \chi - \frac{i}{2} m_{\chi}^* \chi^\dagger \sigma_2 \chi^\dagger + \frac{i}{2} m_{\psi} \psi \sigma_2 \psi - \frac{i}{2} m_{\psi}^* \psi^\dagger \sigma_2 \psi^\dagger. \end{aligned} \quad (119)$$

The equations of motion are:

$$i(\partial_0 + \vec{\sigma} \cdot \nabla) \chi = im_d^* \sigma_2 \psi^\dagger + im_{\chi}^* \sigma_2 \chi^\dagger \quad (120)$$

$$i(\partial_0 + \vec{\sigma} \cdot \nabla) \psi = im_d^* \sigma_2 \chi^\dagger + im_{\psi}^* \sigma_2 \psi^\dagger. \quad (121)$$

Let us expand the fields as in Eq.(87):

$$\psi_a(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3}} \left[e^{ip \cdot x} u_a(p, s) b(p, s) - i\sigma_{2ab} e^{-ip \cdot x} v_b^*(p, s) b_c^\dagger(p, s) \right] \quad (122)$$

and

$$\chi_a(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3}} \left[e^{ip \cdot x} w_a(p, s) c(p, s) - i\sigma_{2ab} e^{-ip \cdot x} z_b^*(p, s) c_c^\dagger(p, s) \right]. \quad (123)$$

By substituting these expansions into the free-field equations of motion (120–121), taking the natural four-dimensional Fourier transforms, and using the rule $\sigma_2 \vec{\sigma}^* \sigma_2 = -\vec{\sigma}$, we find four equations:

$$\begin{aligned} (p^0 - \vec{p} \cdot \vec{\sigma}) w(p, s) c(p, s) &= m_d^* v(p, s) b_c(p, s) + m_\chi^* z(p, s) c_c(p, s) \\ (p^0 - \vec{p} \cdot \vec{\sigma}) u(p, s) b(p, s) &= m_d^* z(p, s) c_c(p, s) + m_\psi^* v(p, s) b_c(p, s) \\ (p^0 + \vec{p} \cdot \vec{\sigma}) z(p, s) c_c(p, s) &= m_d u(p, s) b(p, s) + m_\chi w(p, s) c(p, s) \\ (p^0 + \vec{p} \cdot \vec{\sigma}) v(p, s) b_c(p, s) &= m_d w(p, s) c(p, s) + m_\psi u(p, s) b(p, s) \end{aligned} \quad (124)$$

which when combined as in Eq.(93) imply that

$$\begin{aligned} (m^2 - |m_\chi|^2 - |m_d|^2) w(p, s) c(p, s) &= (m_\psi m_d^* + m_d m_\chi^*) u(p, s) b(p, s) \\ (m^2 - |m_\psi|^2 - |m_d|^2) u(p, s) b(p, s) &= (m_\chi m_d^* + m_d m_\psi^*) w(p, s) c(p, s) \\ (m^2 - |m_\chi|^2 - |m_d|^2) z(p, s) c_c(p, s) &= (m_\chi m_d^* + m_d m_\psi^*) v(p, s) b_c(p, s) \\ (m^2 - |m_\psi|^2 - |m_d|^2) v(p, s) b_c(p, s) &= (m_\psi m_d^* + m_d m_\chi^*) z(p, s) c_c(p, s). \end{aligned} \quad (125)$$

These eight relations tell us that the operators $b(p, s)$, $c(p, s)$, $b(p, s)_c$, and $c(p, s)_c$ all describe the same self-conjugate particles and that the physical mass m must satisfy the quartic equation

$$(m^2 - |m_\chi|^2 - |m_d|^2) (m^2 - |m_\psi|^2 - |m_d|^2) = |m_\chi m_d^* + m_d m_\psi^*|^2 \quad (126)$$

whose solutions

$$m_\pm^2 = |m_d|^2 + \frac{1}{2} \left(|m_\chi|^2 + |m_\psi|^2 \pm \sqrt{(|m_\chi|^2 - |m_\psi|^2)^2 + 4|m_\chi m_d^* + m_d m_\psi^*|^2} \right) \quad (127)$$

depend upon the relative phases among the three complex mass parameters m_d, m_χ , and m_ψ . If the parameter $m_\chi m_d^* + m_d m_\psi^*$ is real, then this expression agrees with the simpler formula (115). When combined into two four-component spinors, the fields $\bar{\chi}_\pm$ and ψ_\pm represent two self-conjugate particles, one of mass m_+ and the other of mass m_- .

5 Simple Chiral Multiplets

5.1 A Right-Handed Spinor and a Spinless Boson

The susy action density for a right-handed spinor field ψ not interacting with a complex spin-zero field A is

$$\mathcal{L}_K = \frac{i}{2} \partial_n \bar{\psi} \bar{\sigma}^n \psi - \frac{i}{2} \bar{\psi} \bar{\sigma}^n \partial_n \psi - \partial_n \bar{A} \partial^n A + \bar{F} F. \quad (128)$$

The field equations are

$$i \bar{\sigma}^n \partial_n \psi = 0 \quad (129)$$

$$\partial_n \partial^n A = 0 \quad (130)$$

$$F = 0. \quad (131)$$

By ignoring total derivatives and using the constraint (131), we may write the kinetic Lagrange density (128) as

$$\mathcal{L}_K = -i \bar{\psi} \bar{\sigma}^n \partial_n \psi - \partial_n \bar{A} \partial^n A. \quad (132)$$

5.2 Susy Invariance of the Free Field Equations

The susy transformations on the scalar field A , the spinor field ψ_a , and the auxiliary field F are

$$\delta A = \sqrt{2} \xi \psi \quad (133)$$

$$\delta \psi = i \sqrt{2} \sigma^m \bar{\xi} \partial_m A + \sqrt{2} \xi F \quad (134)$$

$$\delta F = i \sqrt{2} \bar{\xi} \bar{\sigma}^m \partial_m \psi. \quad (135)$$

The conjugate equations are

$$\delta\bar{A} = \sqrt{2}\bar{\xi}\bar{\psi} \quad (136)$$

$$\delta\bar{\psi} = -i\sqrt{2}\xi\sigma^m\partial_m\bar{A} + \sqrt{2}\bar{\xi}\bar{F} \quad (137)$$

$$\delta\bar{F} = -i\sqrt{2}\partial_m\bar{\psi}\bar{\sigma}^m\xi. \quad (138)$$

The factors of $\sqrt{2}$ in these equations are due to a convention that probably ought to be changed.

Under the susy transformation (133–138), the change in the Lagrange density

$$\mathcal{L}_K = \frac{i}{2}\partial_n\bar{\psi}\bar{\sigma}^n\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^n\partial_n\psi - \partial_n\bar{A}\partial^n A + \bar{F}F \quad (139)$$

is

$$\begin{aligned} \delta\mathcal{L}_K &= \frac{i}{2}\partial_n\delta\bar{\psi}\bar{\sigma}^n\psi + \frac{i}{2}\partial_n\bar{\psi}\bar{\sigma}^n\delta\psi - \frac{i}{2}\delta\bar{\psi}\bar{\sigma}^n\partial_n\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^n\partial_n\delta\psi \\ &\quad - \partial_n\delta\bar{A}\partial^n A - \partial_n\bar{A}\partial^n\delta A + \delta\bar{F}F + \bar{F}\delta F \end{aligned} \quad (140)$$

or

$$\begin{aligned} \delta\mathcal{L}_K &= \frac{i}{2}\partial_n \left(-i\sqrt{2}\xi\sigma^m\partial_m\bar{A} + \sqrt{2}\bar{\xi}\bar{F} \right) \bar{\sigma}^n\psi \\ &\quad + \frac{i}{2}\partial_n\bar{\psi}\bar{\sigma}^n \left(i\sqrt{2}\sigma^m\bar{\xi}\partial_m A + \sqrt{2}\bar{\xi}F \right) \\ &\quad - \frac{i}{2} \left(-i\sqrt{2}\xi\sigma^m\partial_m\bar{A} + \sqrt{2}\bar{\xi}\bar{F} \right) \bar{\sigma}^n\partial_n\psi \\ &\quad - \frac{i}{2}\bar{\psi}\bar{\sigma}^n\partial_n \left(i\sqrt{2}\sigma^m\bar{\xi}\partial_m A + \sqrt{2}\bar{\xi}F \right) \\ &\quad - \partial_n \left(\sqrt{2}\bar{\xi}\bar{\psi} \right) \partial^n A - \partial_n\bar{A}\partial^n \left(\sqrt{2}\bar{\xi}\psi \right) \\ &\quad + \left(-i\sqrt{2}\partial_m\bar{\psi}\bar{\sigma}^m\xi \right) F + \bar{F} \left(i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m\psi \right). \end{aligned} \quad (141)$$

After a partial cancellation of terms involving ξF , the part of $\delta\mathcal{L}_K$ that depends upon ξ is

$$\begin{aligned} \delta\mathcal{L}_{K\xi} &= \frac{1}{\sqrt{2}}\xi\sigma^m\bar{\sigma}^n\psi\partial_n\partial_m\bar{A} - \frac{1}{\sqrt{2}}\xi\sigma^m\bar{\sigma}^n\partial_n\psi\partial_m\bar{A} - \sqrt{2}\xi\partial^n\psi\partial_n\bar{A} \\ &\quad - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\sigma}^n\xi\partial_n F - \frac{i}{\sqrt{2}}\partial_n\bar{\psi}\bar{\sigma}^n\xi F. \end{aligned} \quad (142)$$

We now use the identity (41) and the commutativity of partial derivatives to rewrite the first term of the preceding expression

$$\begin{aligned}\frac{1}{\sqrt{2}}\xi\sigma^m\bar{\sigma}^n\psi\partial_n\partial_m\bar{A} &= \frac{1}{\sqrt{2}}\xi(-\sigma^n\bar{\sigma}^m - 2\eta^{mn}I)\psi\partial_n\partial_m\bar{A} \\ &= -\frac{1}{\sqrt{2}}\xi\sigma^m\bar{\sigma}^n\psi\partial_n\partial_m\bar{A} - \sqrt{2}\xi\psi\partial_n\partial^n\bar{A}.\end{aligned}\quad (143)$$

Thus the change in the action density $\delta\mathcal{L}_{\mathcal{K}\xi}$ is

$$\begin{aligned}\delta\mathcal{L}_{\mathcal{K}\xi} &= -\frac{1}{\sqrt{2}}\xi\sigma^m\bar{\sigma}^n\psi\partial_n\partial_m\bar{A} - \sqrt{2}\xi\psi\partial_n\partial^n\bar{A} \\ &\quad - \frac{1}{\sqrt{2}}\xi\sigma^m\bar{\sigma}^n\partial_n\psi\partial_m\bar{A} - \sqrt{2}\xi\partial^n\psi\partial_n\bar{A} \\ &\quad - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\sigma}^n\xi\partial_nF - \frac{i}{\sqrt{2}}\partial_n\bar{\psi}\bar{\sigma}^n\xi F.\end{aligned}\quad (144)$$

So the change $\delta\mathcal{L}_{\mathcal{K}\xi}$ is the total divergence

$$\delta\mathcal{L}_{\mathcal{K}\xi} = \partial_n K_{\mathcal{K}\xi}^n \quad (145)$$

of the current

$$K_{\mathcal{K}\xi}^n = -\frac{1}{\sqrt{2}}\xi\sigma^m\bar{\sigma}^n\psi\partial_m\bar{A} - \sqrt{2}\xi\psi\partial^n\bar{A} - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\sigma}^n\xi F. \quad (146)$$

The change in the free action density $\delta\mathcal{L}_K$ is the divergence

$$\delta\mathcal{L}_K = \partial_n K_K^n \quad (147)$$

of the current

$$K_K^n = K_{K\xi}^n + K_{K\bar{\xi}}^n \quad (148)$$

in which $K_{K\bar{\xi}}^n = (K_{K\xi}^n)^\dagger$. Thus the equations of motion of the free field theory are invariant under the susy transformation (133–138).

The Noether current associated with the susy transformation (133–138) of the free action density is

$$\begin{aligned}J_K^n &= \frac{i}{2}\delta\bar{\psi}\bar{\sigma}^n\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^n\delta\psi - \delta\bar{A}\partial^n A - \partial^n\bar{A}\delta A \\ &= \frac{i}{2}\left(-i\sqrt{2}\xi\sigma^m\partial_m\bar{A} + \sqrt{2}\bar{\xi}\bar{F}\right)\bar{\sigma}^n\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^n\left(i\sqrt{2}\sigma^m\bar{\xi}\partial_m A + \sqrt{2}\xi F\right) \\ &\quad - \sqrt{2}\bar{\xi}\bar{\psi}\partial^n A - \partial^n\bar{A}\sqrt{2}\xi\psi.\end{aligned}\quad (149)$$

The part depending on ξ is

$$J_{K\xi}^n = \frac{1}{\sqrt{2}}\xi\sigma^m\bar{\sigma}^n\psi\partial_m\bar{A} - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\sigma}^n\xi F - \sqrt{2}\xi\psi\partial^n\bar{A}. \quad (150)$$

The Noether current J_K^n satisfies

$$\delta\mathcal{L}_K = \partial_n J_K^n, \quad (151)$$

and so the difference $J_K^n - K_K^n$ of the two currents

$$S_K^n = J_K^n - K_K^n \quad (152)$$

is conserved

$$\partial_n S_K^n = 0. \quad (153)$$

The current S_K^n is the supercurrent of the free Lagrange density \mathcal{L}_K . The part $S_{K\xi}^n$ that depends upon ξ is simply

$$S_{K\xi}^n = \sqrt{2}\xi\sigma^m\bar{\sigma}^n\psi\partial_m\bar{A}. \quad (154)$$

Thus the quantity ξQ_K is

$$\xi Q_K = \int d^3x S_{K\xi}^0 = \int d^3x \sqrt{2}\xi\sigma^m\bar{\sigma}^0\psi\partial_m\bar{A}. \quad (155)$$

And so the supercharges Q_{Ka} of the free theory are

$$\begin{aligned} Q_{Ka} &= \sqrt{2} \int d^3x \sigma_{ab}^m \bar{\sigma}^{0bc} \psi_c \partial_m \bar{A} \\ &= -\sqrt{2} \int d^3x \sigma_{ab}^m \psi_b \partial_m \bar{A}. \end{aligned} \quad (156)$$

5.3 A Left-Handed Spinor and a Spinless Boson

The susy action density for a left-handed spinor $\bar{\chi}$ not interacting with a complex spin-zero field A is

$$\mathcal{L}_K = \frac{i}{2}\partial_n\chi\sigma^n\bar{\chi} - \frac{i}{2}\chi\sigma^n\partial_n\bar{\chi} - \partial_n\bar{A}\partial^n A + \bar{F}F. \quad (157)$$

The field equations are

$$i\sigma^n\partial_n\bar{\chi} = 0 \quad (158)$$

$$\partial_n\partial^n A = 0 \quad (159)$$

$$F = 0. \quad (160)$$

By ignoring total derivatives and using the constraint (160), we may write the kinetic Lagrange density (157) as

$$\mathcal{L}_K = -i\chi\sigma^n\partial_n\bar{\chi} - \partial_n\bar{A}\partial^n A. \quad (161)$$

The susy transformations for the fields A , χ^a , and F are

$$\delta A = \sqrt{2}\chi\xi \quad (162)$$

$$\delta\chi = -i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m A + \sqrt{2}\xi F \quad (163)$$

$$\delta F = -i\sqrt{2}\partial_m\chi\sigma^m\bar{\xi} \quad (164)$$

and for their conjugates are

$$\delta\bar{A} = \sqrt{2}\bar{\xi}\bar{\chi} \quad (165)$$

$$\delta\bar{\chi} = i\sqrt{2}\bar{\sigma}^m\xi\partial_m\bar{A} + \sqrt{2}\bar{\xi}\bar{F} \quad (166)$$

$$\delta\bar{F} = i\sqrt{2}\xi\sigma^m\partial_m\bar{\chi}. \quad (167)$$

These transformations may seem different from those of equations (133–d bar F). But if we use the identities (47–49) to write them for the fields A , χ_a , and F , then we find that they are in effect the same:

$$\delta A = \sqrt{2}\xi\chi \quad (168)$$

$$\delta\chi = i\sqrt{2}\sigma^m\bar{\xi}\partial_m A + \sqrt{2}\xi F \quad (169)$$

$$\delta F = i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m\chi. \quad (170)$$

$$\delta\bar{A} = \sqrt{2}\bar{\xi}\bar{\chi} \quad (171)$$

$$\delta\bar{\chi} = -i\sqrt{2}\xi\sigma^m\partial_m\bar{A} + \sqrt{2}\bar{\xi}\bar{F} \quad (172)$$

$$\delta\bar{F} = -i\sqrt{2}\partial_m\bar{\chi}\bar{\sigma}^m\xi. \quad (173)$$

The action density (157) changes only by a total divergence under the susy transformations (162–167). Indeed the change in the action density

(157) is

$$\begin{aligned}\delta\mathcal{L}_K &= \frac{i}{2}\partial_n\delta\chi\sigma^n\bar{\chi} + \frac{i}{2}\partial_n\chi\sigma^n\delta\bar{\chi} - \frac{i}{2}\delta\chi\sigma^n\partial_n\bar{\chi} - \frac{i}{2}\chi\sigma^n\partial_n\delta\bar{\chi} \\ &\quad - \partial_n\delta\bar{A}\partial^n A - \partial_n\bar{A}\partial^n\delta A + \delta\bar{F}F + \bar{F}\delta F.\end{aligned}\quad (174)$$

Explicitly this change is

$$\begin{aligned}\delta\mathcal{L}_K &= \frac{i}{2}\partial_n\left(-i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m A + \sqrt{2}\xi F\right)\sigma^n\bar{\chi} \\ &\quad + \frac{i}{2}\partial_n\chi\sigma^n\left(i\sqrt{2}\bar{\sigma}^m\xi\partial_m\bar{A} + \sqrt{2}\bar{\xi}\bar{F}\right) \\ &\quad - \frac{i}{2}\left(-i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m A + \sqrt{2}\xi F\right)\sigma^n\partial_n\bar{\chi} \\ &\quad - \frac{i}{2}\chi\sigma^n\partial_n\left(i\sqrt{2}\bar{\sigma}^m\xi\partial_m\bar{A} + \sqrt{2}\bar{\xi}\bar{F}\right) \\ &\quad - \partial_n\left(\sqrt{2}\bar{\xi}\bar{\chi}\right)\partial^n A - \partial_n\bar{A}\partial^n\left(\sqrt{2}\chi\xi\right) \\ &\quad + \left(i\sqrt{2}\xi\sigma^m\partial_m\bar{\chi}\right)F + \bar{F}\left(-i\sqrt{2}\partial_m\chi\sigma^m\bar{\xi}\right).\end{aligned}\quad (175)$$

The part that depends upon ξ is

$$\begin{aligned}\delta\mathcal{L}_{K\xi} &= \frac{i}{\sqrt{2}}\partial_n\xi F\sigma^n\bar{\chi} - \frac{1}{\sqrt{2}}\partial_n\chi\sigma^n\bar{\sigma}^m\xi\partial_m\bar{A} \\ &\quad - \frac{i}{\sqrt{2}}\xi F\sigma^n\partial_n\bar{\chi} + \frac{1}{\sqrt{2}}\chi\sigma^n\partial_n\bar{\sigma}^m\xi\partial_m\bar{A} \\ &\quad - \sqrt{2}\partial_n\bar{A}\partial^n\chi\xi + i\sqrt{2}\xi\sigma^m\partial_m\bar{\chi}F\end{aligned}\quad (176)$$

or

$$\begin{aligned}\delta\mathcal{L}_{K\xi} &= -\frac{1}{\sqrt{2}}\partial_n\chi\sigma^n\bar{\sigma}^m\xi\partial_m\bar{A} + \frac{1}{\sqrt{2}}\chi\sigma^n\bar{\sigma}^m\xi\partial_n\partial_m\bar{A} \\ &\quad - \sqrt{2}\partial_n\bar{A}\partial^n\chi\xi + \frac{i}{\sqrt{2}}\partial_n\left(\xi F\sigma^n\bar{\chi}\right).\end{aligned}\quad (177)$$

By using formula (41), we may write this change as the divergence

$$\delta\mathcal{L}_{K\xi} = \partial_n K_{K\xi}^n \quad (178)$$

of the current

$$K_{K\xi}^n = -\frac{1}{\sqrt{2}}\chi\sigma^n\bar{\sigma}^m\xi\partial_m\bar{A} - \sqrt{2}\bar{A}\partial^n\chi\xi + \frac{i}{\sqrt{2}}\xi\sigma^n\bar{\chi}F, \quad (179)$$

which shows that the space-time integral of the action density (157) is invariant under the susy transformations (162–167).

The Noether current of the action density (157) under the susy transformations (162–167) is

$$\begin{aligned}
J_K^n &= \frac{i}{2} \delta\chi \sigma^n \bar{\chi} - \frac{i}{2} \chi \sigma^n \delta\bar{\chi} - \delta\bar{A} \partial^n A - \partial^n \bar{A} \delta A \\
&= \frac{i}{2} \left(-i\sqrt{2}\bar{\xi} \bar{\sigma}^m \partial_m A + \sqrt{2}\xi F \right) \sigma^n \bar{\chi} - \frac{i}{2} \chi \sigma^n \left(i\sqrt{2}\bar{\sigma}^m \xi \partial_m \bar{A} + \sqrt{2}\bar{\xi} \bar{F} \right) \\
&\quad - \left(\sqrt{2}\bar{\xi} \bar{\chi} \right) \partial^n A - \partial^n \bar{A} \left(\sqrt{2}\xi \chi \right). \tag{180}
\end{aligned}$$

The part of this current that depends upon ξ is

$$J_{K\xi}^n = \frac{i}{\sqrt{2}} F \xi \sigma^n \bar{\chi} + \frac{1}{\sqrt{2}} \chi \sigma^n \bar{\sigma}^m \xi \partial_m \bar{A} - \sqrt{2} \partial^n \bar{A} \xi \chi. \tag{181}$$

The susy current S_ξ^n is then

$$S_\xi^n = J_{K\xi}^n - K_{K\xi}^n = \sqrt{2} \chi \sigma^n \bar{\sigma}^m \xi \partial_m \bar{A}. \tag{182}$$

The super-charges of the free theory are therefore

$$Q_K^a = \sqrt{2} \int d^3x \left(\chi \sigma^0 \bar{\sigma}^m \right)^a \partial_m \bar{A} = \sqrt{2} \int d^3x \chi^c \sigma_{cb}^0 \bar{\sigma}^{mba} \partial_m \bar{A}. \tag{183}$$

In a simpler notation, the spinor of susy charges of a left-handed spinor $\bar{\chi}$ not interacting with a complex scalar field A is

$$Q_K = \sqrt{2} \int d^3x \chi (\partial_0 + \vec{\sigma} \cdot \nabla) \bar{A}. \tag{184}$$

6 General Chiral Multiplets

6.1 A Pair of Right-Handed Fields

For a single right-handed spin-one-half field ψ and a single complex scalar field A , the most general chiral multiplet is described by the Lagrange density

$$\begin{aligned}
\mathcal{L} &= \frac{i}{2} \partial_n \bar{\psi} \bar{\sigma}^n \psi - \frac{i}{2} \bar{\psi} \bar{\sigma}^n \partial_n \psi - \partial_n \bar{A} \partial^n A + \bar{F} F \\
&\quad + F W' + \bar{F} \bar{W}' - \frac{1}{2} W'' \psi \psi - \frac{1}{2} \bar{W}'' \bar{\psi} \bar{\psi} \tag{185}
\end{aligned}$$

in which $W(A)$ is an analytic function of the complex scalar field A and the primes indicate differentiation with respect to the field A or \bar{A}

$$W' = \frac{\partial W(A)}{\partial A}, \quad \bar{W}' = \frac{\partial \bar{W}(\bar{A})}{\partial \bar{A}}, \quad \text{etc.} \quad (186)$$

The constraints on the auxiliary field F are

$$F = -\bar{W}' \quad \text{and} \quad \bar{F} = -W'. \quad (187)$$

By implementing these constraints and ignoring total derivatives, we may write the Lagrange density (185) as

$$\mathcal{L} = -i\bar{\psi}\bar{\sigma}^n\partial_n\psi - \partial_n\bar{A}\partial^n A - |W'|^2 - \frac{1}{2}W''\psi\psi - \frac{1}{2}\bar{W}''\bar{\psi}\bar{\psi}, \quad (188)$$

in which the term $|W'|^2$ is the self-interaction of the field A and the last two terms are Yukawa interactions. The field equations are

$$-i\bar{\sigma}^n\partial_n\psi = \bar{W}''\bar{\psi} \quad (189)$$

$$\partial_n\partial^n A = \bar{W}'''W' + \frac{1}{2}\bar{W}'''\bar{\psi}\bar{\psi}. \quad (190)$$

If the theory is to be renormalizable, then the function $W(A)$ should be a polynomial in the field A that is at most cubic

$$W(A) = cA + \frac{1}{2}mA^2 + \frac{1}{3}gA^3. \quad (191)$$

in which we have omitted a possible constant term because only derivatives of W appear in the lagrangian. By a field redefinition $A' = A + d$, where d is a constant, one may also remove the linear term cA , as is usually done. For this case,

$$W(A) = \frac{1}{2}mA^2 + \frac{1}{3}gA^3, \quad (192)$$

the Lagrange density (188) is

$$\mathcal{L} = -i\bar{\psi}\bar{\sigma}^n\partial_n\psi - \partial_n\bar{A}\partial^n A - |mA + gA^2|^2 - \frac{1}{2}(m + 2gA)\psi\psi - \frac{1}{2}(\bar{m} + 2\bar{g}\bar{A})\bar{\psi}\bar{\psi}, \quad (193)$$

and the equations of motion are

$$-i\bar{\sigma}^n\partial_n\psi = (\bar{m} + 2\bar{g}\bar{A})\bar{\psi} \quad (194)$$

$$\partial_n\partial^n A = (\bar{m} + 2\bar{g}\bar{A})(mA + gA^2) + \bar{g}\bar{\psi}\bar{\psi}. \quad (195)$$

6.2 Many Pairs of Right-Handed Fields

The most general right-handed chiral multiplet consists of n copies of the spinor, complex scalar, and complex auxiliary fields ψ_i , A_i , and F_i of the simplest chiral multiplet. The most general chiral Lagrange density is

$$\begin{aligned} \mathcal{L} = & \frac{i}{2} \partial_n \bar{\psi}_i \bar{\sigma}^n \psi_i - \frac{i}{2} \bar{\psi}_i \bar{\sigma}^n \partial_n \psi_i - \partial_n \bar{A}_i \partial^n A_i + \bar{F}_i F_i \\ & + F_i W_i + \bar{F}_i \bar{W}_i - \frac{1}{2} W_{ij} \psi_i \psi_j - \frac{1}{2} \bar{W}_{ij} \bar{\psi}_i \bar{\psi}_j \end{aligned} \quad (196)$$

in which sums are understood over repeated indices and in which $W(A)$ is an analytic function of the n complex scalar fields A_i with

$$W_i = \frac{\partial W}{\partial A_i} \quad \text{and} \quad W_{ij} = \frac{\partial^2 W}{\partial A_i \partial A_j}. \quad (197)$$

As we shall see, the action density (196) changes only by a total derivative when its scalar fields A_i , spinor fields ψ_i , and auxiliary fields F_i are subjected to the supersymmetry transformations

$$\delta A_i = \sqrt{2} \xi \psi_i \quad (198)$$

$$\delta \psi_i = i \sqrt{2} \sigma^m \bar{\xi} \partial_m A_i + \sqrt{2} \xi F_i \quad (199)$$

$$\delta F_i = i \sqrt{2} \bar{\xi} \bar{\sigma}^m \partial_m \psi_i \quad (200)$$

and to the conjugate susy transformations

$$\delta \bar{A}_i = \sqrt{2} \bar{\xi} \bar{\psi}_i \quad (201)$$

$$\delta \bar{\psi}_i = -i \sqrt{2} \xi \sigma^m \partial_m \bar{A}_i + \sqrt{2} \bar{\xi} \bar{F}_i \quad (202)$$

$$\delta \bar{F}_i = -i \sqrt{2} \partial_m \bar{\psi}_i \bar{\sigma}^m \xi. \quad (203)$$

We may express the auxiliary fields F_i in terms of the scalar fields A_i by resolving the constraints

$$0 = \frac{\partial \mathcal{L}}{\partial \bar{F}_i} = F_i + \bar{W}_i \quad \text{and} \quad 0 = \frac{\partial \mathcal{L}}{\partial F_i} = \bar{F}_i + W_i. \quad (204)$$

Eliminating the auxiliary fields and dropping total derivatives, we may write the general chiral action density as

$$\mathcal{L} = -\bar{\psi}_i \bar{\sigma}^n \partial_n \psi_i - \partial_n \bar{A}_i \partial^n A_i - |W_i|^2 - \frac{1}{2} W_{ij} \psi_i \psi_j - \frac{1}{2} \bar{W}_{ij} \bar{\psi}_i \bar{\psi}_j. \quad (205)$$

If the theory is to be renormalizable, then the function $W(A)$ should be a polynomial in the n scalar fields A_i that is no higher than cubic

$$W(A) = c_i A_i + \frac{1}{2} m_{ij} A_i A_j + \frac{1}{3} g_{ijk} A_i A_j A_k. \quad (206)$$

Since only derivatives of W appear in the lagrangian, we have omitted a constant term. The tensors m_{ij} and g_{ijk} must be totally symmetric in their indices; any anti-symmetric piece would not contribute to W . By redefining the scalar fields, one may remove the linear terms

$$W(A) = \frac{1}{2} m_{ij} A_i A_j + \frac{1}{3} g_{ijk} A_i A_j A_k. \quad (207)$$

6.3 Invariance of the Right-Handed Chiral Action

It will be useful to separate the Lagrange density (196) into the kinetic part \mathcal{L}_K and the interaction part \mathcal{L}_I

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_I \quad (208)$$

where

$$\mathcal{L}_K = \frac{i}{2} \partial_n \bar{\psi}_i \bar{\sigma}^n \psi_i - \frac{i}{2} \bar{\psi}_i \bar{\sigma}^n \partial_n \psi_i - \partial_n \bar{A}_i \partial^n A_i + \bar{F}_i F_i \quad (209)$$

and

$$\mathcal{L}_I = F_i W_i + \bar{F}_i \bar{W}_i - \frac{1}{2} W_{ij} \psi_i \psi_j - \frac{1}{2} \bar{W}_{ij} \bar{\psi}_i \bar{\psi}_j. \quad (210)$$

The kinetic Lagrange density (209) is the sum of n copies of the Lagrange density (128). And the susy transformation (198–203) is n copies of the susy transformation (133–138). Thus just as (128) is invariant under (133–138), so too the kinetic Lagrange density (209) changes only by a total derivative under the susy transformation (198–203).

We shall now examine the change in the rest of the lagrangian. Under the susy transformation (198–203), the part of the change in the interaction Lagrange density \mathcal{L}_I that involves ξ is

$$\begin{aligned} \delta \mathcal{L}_{I\xi} &= F_i W_{ij} \delta A_j + \delta \bar{F}_i \bar{W}_i - \frac{1}{2} W_{ijk} \delta A_i \psi_j \psi_k - W_{ij} \psi_i \delta \psi_j - \bar{W}_{ij} \delta \bar{\psi}_i \bar{\psi}_j \\ &= F_i W_{ij} \sqrt{2} \xi \psi_j - i \sqrt{2} \partial_m \bar{\psi}_i \bar{\sigma}^m \xi \bar{W}_i - \frac{1}{\sqrt{2}} W_{ijk} \xi \psi_i \psi_j \psi_k \\ &\quad - W_{ij} \psi_j \sqrt{2} \xi F_i + i \bar{W}_{ij} \sqrt{2} \xi \sigma^m (\partial_m \bar{A}_i) \bar{\psi}_j. \end{aligned}$$

in which we drop terms proportional to $\bar{\xi}$. The first term cancels the fourth term.

The term $W_{ijk}\xi\psi_i\psi_j\psi_k$ vanishes because W_{ijk} is symmetric. Let us write this term more explicitly as $\mathcal{W}_{ijk}\xi^a\psi_{ai}\psi_j^b\psi_{bk}$. Since the spinor indices a and b can assume only the two values 1 and 2, it follows that in every term at least two of the spinor indices must be the same. The symmetry in i, j, k of \mathcal{W}_{ijk} then implies that such terms, *e.g.*, $\mathcal{W}_{ijk}\psi_{1i}\psi_{1j}$, must vanish. In particular

$$\mathcal{W}_{ijk}\psi_{1i}\psi_{1j} = \frac{1}{2}(\mathcal{W}_{ijk} + \mathcal{W}_{jik})\psi_{1i}\psi_{1j} = \frac{1}{2}(\mathcal{W}_{ijk}\psi_{1i}\psi_{1j} + \mathcal{W}_{jik}\psi_{1i}\psi_{1j}) \quad (211)$$

so that

$$\mathcal{W}_{ijk}\psi_{1i}\psi_{1j} = \frac{1}{2}(\mathcal{W}_{ijk}\psi_{1i}\psi_{1j} + \mathcal{W}_{ijk}\psi_{1j}\psi_{1i}) = \frac{1}{2}\mathcal{W}_{ijk}(\psi_{1i}\psi_{1j} + \psi_{1j}\psi_{1i}) = 0. \quad (212)$$

Thus using (49), we see that the part of the change $\delta\mathcal{L}_{\mathcal{I}\xi}$ in the interaction Lagrange density that depends upon ξ is

$$\delta\mathcal{L}_{\mathcal{I}\xi} = -i\sqrt{2}\bar{\psi}_i\bar{\sigma}^m\xi\bar{W}_{ij}\partial_m\bar{A}_j - i\sqrt{2}\partial_m\bar{\psi}_i\bar{\sigma}^m\xi\bar{W}_i \quad (213)$$

which is the total divergence

$$\delta\mathcal{L}_{\mathcal{I}\xi} = \partial_m K_{\mathcal{I}\xi}^m \quad (214)$$

of the current

$$K_{\mathcal{I}\xi}^m = -i\sqrt{2}\bar{\psi}_i\bar{\sigma}^m\xi\bar{W}_i. \quad (215)$$

The change $\delta\mathcal{L}_{\mathcal{I}}$ is the divergence

$$\delta\mathcal{L}_{\mathcal{I}} = \partial_n K_{\mathcal{I}}^n = \partial_n \left(K_{\mathcal{I}\xi}^n + (K_{\mathcal{I}\xi}^n)^\dagger \right). \quad (216)$$

Since the change in the full action density (208) is a total divergence

$$\delta\mathcal{L} = \partial_n (K_K^n + K_{\mathcal{I}}^n), \quad (217)$$

the action and the equations of motion are invariant under the susy transformation (133–138).

The species index i essentially serves to label the copies; we shall suppress it in what follows, except when it is worth noticing.

6.4 Supercharges of the Right-Handed Chiral Theory

Because there are no derivatives of the fields in the interaction Lagrange density \mathcal{L}_I , the Noether current of the interacting theory is the same as that of the free theory. Thus the part of the supercurrent S^n that depends upon ξ is

$$\begin{aligned} S_\xi^n &= J_{K\xi}^n - K_{K\xi}^n - K_I^n \\ &= S_{K\xi}^n - K_I^n \\ &= \sqrt{2}\xi\sigma^m\bar{\sigma}^n\psi\partial_m\bar{A} + i\sqrt{2}\bar{\psi}\bar{\sigma}^n\xi\bar{W}'. \end{aligned} \quad (218)$$

The divergence of this susy current is the difference between two equal expressions for the change in the action density — the Noether form and the more straightforward form obtained by substituting the changes in the fields. The susy current S_ξ^n therefore is conserved:

$$\partial_n S_\xi^n = 0. \quad (219)$$

The supercharge ξQ of this theory is the spatial integral of the time component of this conserved current

$$\xi Q = \int d^3x S_\xi^0 = \sqrt{2} \int d^3x \left(\xi\sigma^m\bar{\sigma}^0\psi\partial_m\bar{A} + i\bar{\psi}\bar{\sigma}^0\xi\bar{W}' \right). \quad (220)$$

By the identity (49) we have $\bar{\psi}\bar{\sigma}^0\xi = -\xi\sigma^0\bar{\psi}$, and so the supercharges Q_a are

$$Q_a = \sqrt{2} \int d^3x \left(\sigma_{ab}^m \bar{\sigma}^{0bc} \psi_c \partial_m \bar{A} - i \sigma_{ab}^0 \bar{\psi}^b \bar{W}' \right). \quad (221)$$

In a mixed notation the supercharges are

$$Q_a = -\sqrt{2} \int d^3x \left(\sigma_{ab}^m \psi_b \partial_m \bar{A} - i \bar{\psi}^{\dot{a}} \bar{W}' \right) \quad (222)$$

$$= -\sqrt{2} \int d^3x \left(\sigma_{ab}^m \psi_b \partial_m \bar{A} - i \varepsilon^{\dot{a}b} \bar{\psi}_b \bar{W}' \right) \quad (223)$$

or more simply

$$Q = \sqrt{2} \int d^3x \left[\left(\partial_0 \bar{A} - \vec{\sigma} \cdot \nabla \bar{A} \right) \psi - \bar{W}' \sigma^2 \bar{\psi} \right]. \quad (224)$$

The adjoint supercharges $\bar{Q}_{\dot{a}}$ are

$$\bar{Q}_{\dot{a}} = (Q_a)^\dagger = \sqrt{2} \int d^3x \left(\bar{\psi}_{\dot{c}} \bar{\sigma}^{0\dot{c}b} \sigma_{b\dot{a}}^m \partial_m A + i \psi^b \sigma_{b\dot{a}}^0 W' \right). \quad (225)$$

In a mixed notation they are

$$\bar{Q}_{\dot{a}} = -\sqrt{2} \int d^3x \left(\bar{\psi}_{\dot{b}} \sigma_{b\dot{a}}^m \partial_m A + i \varepsilon^{\dot{a}b} \psi_b W' \right) \quad (226)$$

or more simply

$$\bar{Q}_{\dot{a}} = \sqrt{2} \int d^3x \left[\bar{\psi} (\partial_0 - \vec{\sigma} \cdot \nabla) A + W' \sigma^2 \psi \right]. \quad (227)$$

6.5 A Pair of Left-Handed Fields

The action density for a left-handed spin-one-half field $\bar{\chi}$ interacting with a complex scalar field A is

$$\mathcal{L}_L = \frac{i}{2} \partial_n \chi \sigma^n \bar{\chi} - \frac{i}{2} \chi \sigma^n \partial_n \bar{\chi} - \partial_n \bar{A} \partial^n A + \bar{F} F \quad (228)$$

$$+ F W' + \bar{F} \bar{W}' - \frac{1}{2} W'' \chi \chi - \frac{1}{2} \bar{W}'' \bar{\chi} \bar{\chi}, \quad (229)$$

in which $W(A)$ is a cubic polynomial in the field A . The equations of motion are

$$-i \sigma^n \partial_n \bar{\chi} = W'' \chi \quad (230)$$

$$- \partial_n \partial^n A = F W'' - \frac{1}{2} W''' \chi \chi \quad (231)$$

$$F = -\bar{W}'. \quad (232)$$

If we implement the constraint (232) and ignore total derivatives, then we may write the action density as

$$\mathcal{L}_L = -i \chi \sigma^n \partial_n \bar{\chi} - \partial_n \bar{A} \partial^n A - |W'|^2 - \frac{1}{2} W'' \chi \chi - \frac{1}{2} \bar{W}'' \bar{\chi} \bar{\chi} \quad (233)$$

and the equations of motion as

$$-i \sigma^n \partial_n \bar{\chi} = W'' \chi \quad (234)$$

$$\partial_n \partial^n A = \bar{W}' W'' + \frac{1}{2} W''' \chi \chi. \quad (235)$$

6.6 Invariance of the Left-Handed Chiral Action

Under the susy transformations (162–167), the change of the interaction part (229) of the action density is

$$\delta\mathcal{L}_{LI} = \delta FW' + FW''\delta A + \delta\bar{F}\bar{W}' + \bar{F}\bar{W}''\delta\bar{A} - W''\delta\chi\chi - \bar{W}''\bar{\chi}\delta\bar{\chi} \quad (236)$$

in which we have ignored the terms proportional to the product of three χ 's, which vanish as shown earlier (211–212). This change is

$$\begin{aligned} \delta\mathcal{L}_{LI} = & \left(-i\sqrt{2}\partial_m\chi\sigma^m\bar{\xi}\right)W' + FW''\left(\sqrt{2}\chi\xi\right) + \left(i\sqrt{2}\xi\sigma^m\partial_m\bar{\chi}\right)\bar{W}' \\ & + \bar{F}\bar{W}''\left(\sqrt{2}\bar{\xi}\bar{\chi}\right) - W''\left(-i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m A + \sqrt{2}\xi F\right)\chi \\ & - \bar{W}''\bar{\chi}\left(i\sqrt{2}\bar{\sigma}^m\xi\partial_m\bar{A} + \sqrt{2}\bar{\xi}\bar{F}\right). \end{aligned} \quad (237)$$

Of this change, the part that depends upon ξ is

$$\begin{aligned} \delta\mathcal{L}_{LI\xi} = & FW''\left(\sqrt{2}\chi\xi\right) + \left(i\sqrt{2}\xi\sigma^m\partial_m\bar{\chi}\right)\bar{W}' - W''\sqrt{2}\xi F\chi \\ & - \bar{W}''\bar{\chi}\left(i\sqrt{2}\bar{\sigma}^m\xi\partial_m\bar{A}\right). \end{aligned} \quad (238)$$

By using the identity (49), we may write this change as the divergence

$$\delta\mathcal{L}_{LI\xi} = \partial_n K_{LI\xi}^n \quad (239)$$

of the current

$$K_{LI\xi}^n = i\sqrt{2}\xi\sigma^n\bar{\chi}\bar{W}'. \quad (240)$$

This result and the earlier one (179) establish the invariance of the action of this theory under susy transformations.

6.7 Supercharges of the Left-Handed Chiral Theory

The interaction (229) contains no derivatives of χ or of A and so generates no extra Noether current. The difference between the Noether current (181) and sum of the currents $K_{K\xi}^n$ (179) and $K_{LI\xi}^n$ (240) is therefore the susy current of the interacting theory (228–229):

$$S_\xi^n = J_{K\xi}^n - K_{K\xi}^n - K_{LI\xi}^n = \sqrt{2}\chi\sigma^n\bar{\sigma}^m\xi\partial_m\bar{A} - i\sqrt{2}\xi\sigma^n\bar{\chi}\bar{W}', \quad (241)$$

and it is conserved

$$\partial_n S_\xi^m = 0. \quad (242)$$

The supercharges Q_a of the left-handed chiral theory are given by

$$\xi Q = \int d^3x S_\xi^0 = \sqrt{2} \int d^3x \left(\chi \sigma^0 \bar{\sigma}^m \xi \partial_m \bar{A} - i \xi \sigma^0 \bar{\chi} \bar{W}' \right). \quad (243)$$

The identity (50) implies that $\chi \sigma^0 \bar{\sigma}^m \xi = \xi \sigma^m \bar{\sigma}^0 \chi$, and so the supercharges Q_a are

$$Q_a = \sqrt{2} \int d^3x \left(\sigma_{ab}^m \bar{\sigma}^{0bc} \chi_c \partial_m \bar{A} - i \sigma_{ab}^0 \bar{\chi}^b \bar{W}' \right). \quad (244)$$

Since $\sigma^0 = \bar{\sigma}^0 = -I$, we may write the Q_a in a mixed notation as

$$Q_a = -\sqrt{2} \int d^3x \left(\sigma_{ab}^m \chi_b \partial_m \bar{A} - i \bar{\chi}^a \bar{W}' \right) \quad (245)$$

$$= -\sqrt{2} \int d^3x \left(\sigma_{ab}^m \chi_b \partial_m \bar{A} - i \varepsilon^{ab} \bar{\chi}_b \bar{W}' \right) \quad (246)$$

or in a simpler notation as

$$Q = \sqrt{2} \int d^3x \left[\left(\partial_0 \bar{A} - \vec{\sigma} \cdot \nabla \bar{A} \right) \chi - \bar{W}' \sigma^2 \bar{\chi} \right]. \quad (247)$$

The adjoint supercharges are

$$\bar{Q}_{\dot{a}} = \sqrt{2} \int d^3x \left(\bar{\chi}_{\dot{c}} \bar{\sigma}^{0\dot{c}b} \sigma_{b\dot{a}}^m \partial_m A + i \chi^b \sigma_{b\dot{a}}^0 W' \right) \quad (248)$$

or more simply

$$\bar{Q} = \sqrt{2} \int d^3x \left[\bar{\chi} (\partial_0 A - \vec{\sigma} \cdot \nabla A) + W' \sigma^2 \chi \right]. \quad (249)$$

7 The Supercharges

Supercharges satisfy the remarkable {anti-}commutation relations

$$\{Q_a, \bar{Q}_{\dot{b}}\} = 2\sigma_{ab}^m P_m \quad (250)$$

$$\{Q_a, Q_b\} = \{\bar{Q}_{\dot{a}}, \bar{Q}_{\dot{b}}\} = 0 \quad (251)$$

$$[Q_a, P_n] = [\bar{Q}_{\dot{a}}, P_n] = [P_m, P_n] = 0. \quad (252)$$

In particular

$$\begin{aligned} \{Q_a, \bar{Q}_{\dot{a}}\} &= Q_1 (Q_1)^\dagger + (Q_1)^\dagger Q_1 + Q_2 (Q_2)^\dagger + (Q_2)^\dagger Q_2 \\ &= 2\text{Tr}(\sigma^m) P_m = -4P_0 = 4P^0 = 4H. \end{aligned} \quad (253)$$

The factors of 2 and 4 that litter these equations are due to the extra factors of $\sqrt{2}$ in the conventional definition of the susy transformation (133–138) of the chiral multiplet.

By using the canonical commutation relations, one may show that the right-handed supercharges

$$Q_a = \sqrt{2} \int d^3x \left(\sigma_{ab}^m \bar{\sigma}^{0bc} \psi_c \partial_m \bar{A} - i \sigma_{ab}^0 \bar{\psi}^b \bar{W}' \right) \quad (254)$$

$$\bar{Q}_{\dot{a}} = \sqrt{2} \int d^3x \left(\bar{\psi}_{\dot{c}} \bar{\sigma}^{0cb} \sigma_{ba}^m \partial_m A + i \psi^b \sigma_{ba}^0 W' \right), \quad (255)$$

whose various forms were listed in eqs.(221–227), actually do satisfy the anti-commutation relations (250, 252). The the canonical equal-time commutation and anti-commutation relations obeyed by the fields of the right-handed chiral multiplet are:

$$[A(t, \vec{x}), \partial_0 \bar{A}(t, \vec{y})] = i\delta(\vec{x} - \vec{y}) \quad (256)$$

$$[\bar{A}(t, \vec{x}), \partial_0 A(t, \vec{y})] = i\delta(\vec{x} - \vec{y}) \quad (257)$$

$$\{\psi_a(t, \vec{x}), \bar{\psi}_b(t, \vec{y})\} = \delta_{ab} \delta(\vec{x} - \vec{y}), \quad (258)$$

$$[A(t, \vec{x}), A(t, \vec{y})] = [A(t, \vec{x}), \bar{A}(t, \vec{y})] = [\bar{A}(t, \vec{x}), \bar{A}(t, \vec{y})] = 0, \quad (259)$$

$$[\partial_0 A(t, \vec{x}), \partial_0 A(t, \vec{y})] = [\partial_0 A(t, \vec{x}), \partial_0 \bar{A}(t, \vec{y})] = [\partial_0 \bar{A}(t, \vec{x}), \partial_0 \bar{A}(t, \vec{y})] = 0, \quad (260)$$

and

$$\{\psi_a(t, \vec{x}), \psi_b(t, \vec{y})\} = \{\bar{\psi}_{\dot{a}}(t, \vec{x}), \bar{\psi}_{\dot{b}}(t, \vec{y})\} = 0. \quad (261)$$

The anti-commutator (250) is

$$\{Q_a, \bar{Q}_{\dot{b}}\} = 2 \int d^3x d^3y \left\{ \left(\sigma_{ac}^m \psi_c \partial_m \bar{A} - i \varepsilon^{ac} \bar{\psi}_{\dot{c}} \bar{W}' \right), \left(\bar{\psi}_{\dot{d}} \sigma_{db}^n \partial_n A + i \varepsilon^{bd} \psi_d W' \right) \right\} \quad (262)$$

the sum of four terms. The simplest term arises from the anti-commutator of the \bar{W}' term with the W' term

$$\bar{W}W_{ab} \equiv 2 \int d^3x d^3y \left\{ -i\varepsilon^{ac} \bar{\psi}_c(x) \bar{W}'(x), i\varepsilon^{bd} \psi_d(y) W'(y) \right\}. \quad (263)$$

By using the anti-commutation relation (258), we may write this term as

$$\begin{aligned} \bar{W}W_{ab} &= 2 \int d^3x d^3y \varepsilon^{ac} \varepsilon^{bd} \bar{W}'(x) W'(y) \delta_{cd} \delta(\vec{x} - \vec{y}) \\ &= 2 \int d^3x \varepsilon^{ac} \varepsilon^{bc} |W'(x)|^2 \\ &= 2\delta_{ab} \int d^3x |W'(x)|^2. \end{aligned} \quad (264)$$

The next simplest term arises from the anti-commutator of the first \bar{W}' term with the ∂A term

$$\begin{aligned} \bar{W}A_{ab} &\equiv 2 \int d^3x d^3y \left\{ -i\varepsilon^{\dot{a}c} \bar{\psi}_c(x) \bar{W}', \bar{\psi}_d(y) \sigma_{\dot{a}b}^n \partial_n A(y) \right\} \\ &= -2i \int d^3x d^3y \left(\varepsilon^{\dot{a}c} \bar{\psi}_c(x) \bar{W}' \bar{\psi}_d(y) \sigma_{\dot{a}b}^n \partial_n A(y) \right. \\ &\quad \left. + \bar{\psi}_d(y) \sigma_{\dot{a}b}^n \partial_n A(y) \varepsilon^{\dot{a}c} \bar{\psi}_c(x) \bar{W}' \right). \end{aligned} \quad (265)$$

Due to the vanishing of the anti-commutator (261), we may write these terms as

$$\bar{W}A_{ab} = -2i \int d^3x d^3y \varepsilon^{\dot{a}c} \bar{\psi}_c(x) \bar{\psi}_d(y) \sigma_{\dot{a}b}^n [\bar{W}', \partial_n A(y)]. \quad (266)$$

By referring to the commutators (256) and (259), we see that only the $n = 0$ part survives

$$\begin{aligned} \bar{W}A_{ab} &= -2i \int d^3x d^3y \varepsilon^{\dot{a}c} \bar{\psi}_c(x) \bar{\psi}_d(y) \sigma_{\dot{a}b}^0 [\bar{W}', \partial_0 A(y)] \\ &= 2 \int d^3x d^3y \varepsilon^{\dot{a}c} \bar{\psi}_c(x) \bar{\psi}_d(y) \sigma_{\dot{a}b}^0 \bar{W}'' \delta(\vec{x} - \vec{y}) \\ &= 2 \int d^3x \varepsilon^{\dot{a}c} \bar{\psi}_c(x) \bar{\psi}_d(x) \sigma_{\dot{a}b}^0 \bar{W}'' \\ &= 2 \int d^3x \bar{\psi}_b(x) \bar{\psi}^{\dot{a}}(x) \bar{W}''. \end{aligned} \quad (267)$$

By completing an analogous argument, the reader may show that the adjoint term $\bar{A}W$ is

$$\begin{aligned} \bar{A}W_{ab} &\equiv 2i \int d^3x d^3y \left\{ \sigma_{\dot{a}c}^m \psi_c \partial_m \bar{A}, i\varepsilon^{bd} \psi_d W' \right\} \\ &= 2 \int d^3x \psi^a(x) \psi_b(x) W''. \end{aligned} \quad (268)$$

The last term is the full anti-commutator of the free theory

$$\begin{aligned}
\bar{A}A_{ab} &\equiv 2 \int d^3x d^3y \left\{ \sigma_{ac}^m \psi_c(x) \partial_m \bar{A}(x), \bar{\psi}_d(y) \sigma_{db}^n \partial_n A(y) \right\} \\
&= 2 \int d^3x d^3y \left(\bar{\psi}_d(y) \sigma_{db}^n \partial_n A(y) \sigma_{ac}^m \psi_c(x) \partial_m \bar{A}(x) \right. \\
&\quad \left. + \sigma_{ac}^m \psi_c(x) \partial_m \bar{A}(x) \bar{\psi}_d(y) \sigma_{db}^n \partial_n A(y) \right) \\
&= 2 \int d^3x d^3y \left(\bar{\psi}_d(y) \sigma_{db}^n \sigma_{ac}^m \psi_c(x) \left[\partial_n A(y), \partial_m \bar{A}(x) \right] \right. \\
&\quad \left. + \sigma_{ac}^m \sigma_{db}^n \delta_{cd} \delta(\vec{x} - \vec{y}) \partial_m \bar{A}(x) \partial_n A(y) \right). \quad (269)
\end{aligned}$$

By using the commutation relations (256) and (259), we see that the first term is non-zero if both $n = i \neq 0$ and $m = 0$ or if both $n = 0$ and $m = j \neq 0$. In the former case, we integrate by parts over y ; in the latter case, we integrate by parts over x

$$\begin{aligned}
\bar{A}A_{ab} &= 2 \int d^3x d^3y \left(-\partial_i \bar{\psi}_d(y) \sigma_{db}^i \sigma_{ac}^0 \psi_c(x) \left[A(y), \partial_0 \bar{A}(x) \right] \right. \\
&\quad \left. - \bar{\psi}_d(y) \sigma_{db}^0 \sigma_{ac}^j \partial_j \psi_c(x) \left[\partial_0 A(y), \bar{A}(x) \right] \right. \\
&\quad \left. + \sigma_{ac}^m \sigma_{cb}^n \delta(\vec{x} - \vec{y}) \partial_m \bar{A}(x) \partial_n A(y) \right) \quad (270)
\end{aligned}$$

$$\begin{aligned}
&= 2 \int d^3x d^3y \left(-\partial_i \bar{\psi}_d(y) \sigma_{db}^i \sigma_{ac}^0 \psi_c(x) i \delta(\vec{x} - \vec{y}) \right. \\
&\quad \left. - \bar{\psi}_d(y) \sigma_{db}^0 \sigma_{ac}^j \partial_j \psi_c(x) (-i) \delta(\vec{x} - \vec{y}) \right. \\
&\quad \left. + \sigma_{ac}^m \sigma_{cb}^n \delta(\vec{x} - \vec{y}) \partial_m \bar{A}(x) \partial_n A(y) \right) \\
&= 2 \int d^3x \left(-i \partial_i \bar{\psi}_d(x) \sigma_{db}^i \sigma_{ac}^0 \psi_c(x) + i \bar{\psi}_d(x) \sigma_{db}^0 \sigma_{ac}^j \partial_j \psi_c(x) \right. \\
&\quad \left. + \sigma_{ac}^m \sigma_{cb}^n \partial_m \bar{A}(x) \partial_n A(x) \right) \\
&= 2 \int d^3x \left(i \partial_i \bar{\psi}_d(x) \sigma_{db}^i \psi_a(x) - i \bar{\psi}_b(x) \sigma_{ac}^i \partial_i \psi_c(x) \right. \\
&\quad \left. + \sigma_{ac}^m \sigma_{cb}^n \partial_m \bar{A}(x) \partial_n A(x) \right). \quad (271)
\end{aligned}$$

Collecting our four terms (264), (267), (268), and (271), we find for the anti-commutator (262) of Q_a with \bar{Q}_b the expression

$$\begin{aligned}
\{Q_a, \bar{Q}_b\} &= 2 \int d^3x \left(i \partial_i \bar{\psi}_d(x) \sigma_{db}^i \psi_a(x) - i \bar{\psi}_b(x) \sigma_{ac}^i \partial_i \psi_c(x) \right. \\
&\quad \left. + \sigma_{ac}^m \sigma_{cb}^n \partial_m \bar{A}(x) \partial_n A(x) + \delta_{ab} |W'(x)|^2 \right. \\
&\quad \left. + \psi^a(x) \psi_b(x) W'' + \bar{\psi}_b(x) \bar{\psi}^a(x) \bar{W}'' \right). \quad (272)
\end{aligned}$$

By setting $a = b$, summing from 1 to 2, and using the trace identity (43), we obtain

$$\begin{aligned} \{Q_a, \bar{Q}_a\} &= 4 \int d^3x \left(\frac{i}{2} \partial_i \bar{\psi}(x) \sigma^i \psi(x) - \frac{i}{2} \bar{\psi}(x) \sigma^i \partial_i \psi(x) + \partial_n \bar{A}(x) \partial_n A(x) \right. \\ &\quad \left. + \frac{1}{2} \psi \psi W'' + \frac{1}{2} \bar{\psi} \bar{\psi} \bar{W}'' + |W'(x)|^2 \right). \end{aligned} \quad (273)$$

Thus the famous relation

$$\{Q_a, \bar{Q}_a\} = 4H. \quad (274)$$

We may select the momentum operator P_3 by taking the trace with σ^3 :

$$\begin{aligned} \{Q_a, \bar{Q}_b\} \sigma_{ba}^3 &= 2 \int d^3x \left(i \partial_i \bar{\psi}_d(x) \sigma_{db}^i \sigma_{ba}^3 \psi_a(x) - i \bar{\psi}_b(x) \sigma_{ba}^3 \sigma_{ac}^i \partial_i \psi_c(x) \right. \\ &\quad \left. + \sigma_{ac}^m \sigma_{cb}^n \sigma_{ba}^3 \partial_m \bar{A}(x) \partial_n A(x) + \delta_{ab} \sigma_{ba}^3 |W'(x)|^2 \right. \\ &\quad \left. + \psi^a(x) \sigma_{ba}^3 \psi_b(x) W'' + \bar{\psi}_i(x) \sigma_{ba}^3 \bar{\psi}^i(x) \bar{W}'' \right). \end{aligned} \quad (275)$$

The terms involving the potential W all vanish. The kinetic term for the scalar field A contains the trace $\text{Tr}(\sigma^m \sigma^n \sigma^3) = -2\delta_{m0} \delta_{n3} - 2\delta_{m3} \delta_{n0}$. Thus integrating by parts, we have

$$\begin{aligned} \{Q_a, \bar{Q}_b\} \sigma_{ba}^3 &= 2 \int d^3x \left(i \partial_i \bar{\psi} \{ \sigma^i, \sigma^3 \} \psi - 2 \partial_0 \bar{A} \partial_3 A - 2 \partial_3 \bar{A} \partial_0 A \right) \\ &= 4 \int d^3x \left(i \partial_3 \bar{\psi} \psi - \partial_0 \bar{A} \partial_3 A - \partial_3 \bar{A} \partial_0 A \right), \end{aligned} \quad (276)$$

which is the relation

$$\{Q_a, \bar{Q}_b\} \sigma_{ba}^3 = 4P_3. \quad (277)$$

To verify that the supercharges anti-commute, we use the commutation relations (259) and (261) to reduce $\{Q_a, Q_b\}$ to

$$\begin{aligned} \{Q_a, Q_b\} &= 2 \int d^3x d^3y \left(\{ \sigma_{ac}^m \psi_c(x) \partial_m \bar{A}(x), -i \varepsilon^{bd} \bar{\psi}_d(y) \bar{W}' \} \right. \\ &\quad \left. + \{ -i \varepsilon^{ac} \bar{\psi}_c(x) \bar{W}', \sigma_{bd}^n \psi_d(y) \partial_n \bar{A}(y) \} \right). \end{aligned} \quad (278)$$

Now using the anti-commutation relation (258), we get

$$\begin{aligned} \{Q_a, Q_b\} &= 2 \int d^3x d^3y \left(\partial_m \bar{A}(x) \bar{W}' \sigma_{ac}^m (-i) \varepsilon^{bd} \delta_{cd} \delta(\vec{x} - \vec{y}) \right. \\ &\quad \left. + \bar{W}' \partial_n \bar{A}(y) (-i) \varepsilon^{ac} \sigma_{bd}^n \delta_{cd} \delta(\vec{x} - \vec{y}) \right) \\ &= -2i \int d^3x \partial_n \bar{A}(x) \bar{W}' \left(\sigma_{ac}^n \varepsilon^{bc} + \varepsilon^{ac} \sigma_{bc}^n \right). \end{aligned} \quad (279)$$

For $n = 0$, the σ 's and ε 's come to $\varepsilon^{b\dot{a}} + \varepsilon^{\dot{a}b} = 0$, and so only the spatial terms survive

$$\{Q_a, Q_b\} = -2i \int d^3x \partial_i \bar{A}(x) \bar{W}' \left(\sigma_{ac}^i \varepsilon^{b\dot{c}} + \varepsilon^{\dot{a}c} \sigma_{bc}^i \right). \quad (280)$$

But this quantity is a space integral of a total spatial divergence, and so it vanishes

$$\{Q_a, Q_b\} = -2i \int d^3x \partial_i \left(\bar{W} \left(\sigma_{ac}^i \varepsilon^{b\dot{c}} + \varepsilon^{\dot{a}c} \sigma_{bc}^i \right) \right) = 0. \quad (281)$$

It is worth checking, however, whether the surface terms that we here neglect as well as those that we neglected in deriving eq.(270) from eq.(269) actually do vanish. In particular, when susy is spontaneously broken, the minimum of $|W'|$ is positive, and there is at least one massless fermion.

8 The Supercharges As Generators

8.1 The Supercharges Lack Auxiliary Fields

The field equations

$$0 = \frac{\partial \mathcal{L}}{\partial \bar{F}} = F + \bar{W}' \quad \text{and} \quad 0 = \frac{\partial \mathcal{L}}{\partial F} = \bar{F} + W' \quad (282)$$

express the auxiliary fields F and \bar{F} in terms of the fields A and \bar{A} as

$$F = -\bar{W}' \quad \text{and} \quad \bar{F} = -W'. \quad (283)$$

The supercharges Q_a and \bar{Q}_i involve the fields A and ψ explicitly

$$Q_a = -\sqrt{2} \int d^3x \left(\sigma_{ab}^m \psi_b \partial_m \bar{A} - i \varepsilon^{a\dot{b}} \bar{\psi}_b \bar{W}' \right) \quad (284)$$

$$\bar{Q}_i = -\sqrt{2} \int d^3x \left(\bar{\psi}_b \sigma_{ba}^m \partial_m A + i \varepsilon^{a\dot{b}} \psi_b W' \right). \quad (285)$$

But in these formulas, the auxiliary fields F and \bar{F} appear only as $-\bar{W}'$ and $-W'$.

8.2 The Supercharges Generate Susy Transformations

If as the generator $G(\xi)$ of a susy transformation, we use

$$G(\xi) = \xi Q + \bar{Q}\bar{\xi}, \quad (286)$$

then we find that the change in the field $A(x)$ is

$$\begin{aligned} \delta_\xi A(x) \equiv [iG(\xi), A(x)] &= i\sqrt{2} \int d^3y \xi^a \sigma_{ab}^m \psi_b(y) [A(x), \partial_m \bar{A}(y)] \\ &= -\sqrt{2} \int d^3y \xi^a \sigma_{ab}^0 \psi_b(y) \delta(\vec{x} - \vec{y}) \\ &= \sqrt{2} \xi^a \psi_a(x) = \sqrt{2} \xi \psi, \end{aligned} \quad (287)$$

which is (133). But if we apply $G(\xi)$ to the field $\psi(x)$, then we get

$$\begin{aligned} \delta_\xi \psi_a(x) &\equiv [iG(\xi), \psi_a(x)] \\ &= \sqrt{2} \int d^3y [\psi_a(x), \xi^b \varepsilon^{bc} \bar{\psi}_c(y) \bar{W}' + i\bar{\psi}_c(y) \sigma_{cb}^m \bar{\xi}^b \partial_m A(y)] \\ &= \sqrt{2} (-\xi^b \varepsilon^{ba} \bar{W}' + i\sigma_{ab}^m \bar{\xi}^b \partial_m A(x)) \\ &= \sqrt{2} (\xi^b \varepsilon_{ba} \bar{W}' + i\sigma_{ab}^m \bar{\xi}^b \partial_m A(x)) \\ &= \sqrt{2} i\sigma_{ab}^m \bar{\xi}^b \partial_m A(x) - \sqrt{2} \xi_a \bar{W}' \end{aligned} \quad (288)$$

which agrees with (134) if we use $-\bar{W}' = F$.

8.3 Iterated Susy Transformations

We may now ask what happens when we perform a second susy transformation

$$\begin{aligned} \delta_\eta \delta_\xi A(x) &\equiv [iG(\eta), [iG(\xi), A(x)]] = [iG(\eta), \sqrt{2} \xi^a \psi_a(x)] \\ &= \sqrt{2} \xi^a [iG(\eta), \psi_a(x)] = 2i\xi^a \sigma_{ab}^m \bar{\eta}^b \partial_m A(x) - 2\xi^a \eta_a \bar{W}'. \end{aligned} \quad (289)$$

So the difference $\delta_\eta \delta_\xi A - \delta_\xi \delta_\eta A$ is

$$\begin{aligned} [\delta_\eta, \delta_\xi] A(x) &\equiv [iG(\eta), [iG(\xi), A(x)]] - [iG(\xi), [iG(\eta), A(x)]] \\ &= 2i\xi \sigma^m \bar{\eta} \partial_m A(x) - 2\xi \eta \bar{W}' - 2i\eta \sigma^m \bar{\xi} \partial_m A(x) + 2\eta \xi \bar{W}' \\ &= 2i (\xi \sigma^m \bar{\eta} - \eta \sigma^m \bar{\xi}) \partial_m A(x). \end{aligned} \quad (290)$$

The algebra closes on the field A , as it must. For by the Jacobi identity, the difference

$$[\delta_\eta, \delta_\xi] A(x) = [iG(\eta), [iG(\xi), A(x)]] - [iG(\xi), [iG(\eta), A(x)]] \quad (291)$$

is

$$[\delta_\eta, \delta_\xi] A(x) = [A(x), [iG(\xi), iG(\eta)]] . \quad (292)$$

Now by using the definition (286) of the generators and successively eqs.(251) and (250) of the susy algebra, we may write the commutator of the generators as

$$\begin{aligned} [iG(\xi), iG(\eta)] &= [i\xi Q + i\bar{Q}\bar{\xi}, i\eta Q + i\bar{Q}\bar{\eta}] \\ &= [\eta Q, \bar{Q}\bar{\xi}] - [\xi Q, \bar{Q}\bar{\eta}] \\ &= \eta^a \{Q_a, \bar{Q}_b\} \bar{\xi}^b - \xi^a \{Q_a, \bar{Q}_b\} \bar{\eta}^b \\ &= \eta^a 2\sigma_{ab}^n P_n \bar{\xi}^b - \xi^a 2\sigma_{ab}^n P_n \bar{\eta}^b \\ &= 2\eta\sigma^n P_n \bar{\xi} - 2\xi\sigma^n P_n \bar{\eta}. \end{aligned} \quad (293)$$

By virtue of the commutation relations

$$\partial_n A(x) = i[A(x), P_n], \quad (294)$$

the difference

$$[\delta_\eta, \delta_\xi] A(x) = [A(x), 2\eta\sigma^n P_n \bar{\xi} - 2\xi\sigma^n P_n \bar{\eta}] \quad (295)$$

is

$$[\delta_\eta, \delta_\xi] A(x) = 2i (\xi\sigma^n \bar{\eta} - \eta\sigma^n \bar{\xi}) \partial_n A(x), \quad (296)$$

which is eq.(290).

To see how the algebra closes on the field ψ , we examine the double susy transformation

$$\begin{aligned} \delta_\eta \delta_\xi \psi_a(x) &\equiv [iG(\eta), [iG(\xi), \psi_a(x)]] \\ &= [iG(\eta), \sqrt{2}i\sigma_{ab}^m \bar{\xi}^b \partial_m A(x) - \sqrt{2}\xi_a \bar{W}'] \\ &= \sqrt{2}i\sigma_{ab}^m \bar{\xi}^b \partial_m [iG(\eta), A(x)] - \sqrt{2}\xi_a [iG(\eta), \bar{W}'] \\ &= \sqrt{2}i\sigma_{ab}^m \bar{\xi}^b \partial_m \sqrt{2}\eta^c \psi_c(x) - \sqrt{2}\xi_a \bar{W}'' [iG(\eta), \bar{A}] \\ &= 2i\sigma_{ab}^m \bar{\xi}^b \eta^c \partial_m \psi_c(x) - \sqrt{2}\xi_a \bar{W}'' \sqrt{2}\bar{\psi}_c \bar{\eta}^c \\ &= 2i\sigma_{ab}^m \bar{\xi}^b \eta^c \partial_m \psi_c(x) - 2\xi_a \bar{\eta}_c \bar{\psi}^c \bar{W}'' . \end{aligned} \quad (297)$$

The Heisenberg equation of motion for the field ψ is

$$-i\bar{\sigma}^n \partial_n \psi - \bar{W}'' \bar{\psi} = 0. \quad (298)$$

So we may write the second variation $\delta_\eta \delta_\xi \psi_a(x)$ as

$$\delta_\eta \delta_\xi \psi_a(x) = 2i\sigma_{ab}^m \bar{\xi}^b \eta^c \partial_m \psi_c(x) + 2i\xi_a \bar{\eta}_c \bar{\sigma}^{cd} \partial_n \psi_d(x). \quad (299)$$

We may manipulate the first term into the successive forms

$$\begin{aligned} 2i\sigma_{ab}^m \bar{\xi}^b \eta^c \partial_m \psi_c &= -2i\eta\sigma^m \bar{\xi} \partial_m \psi_a + \sum_{c \neq a} \left(2i\sigma_{ab}^m \bar{\xi}^b \eta^c \partial_m \psi_c + 2i\eta^c \sigma_{cb}^m \bar{\xi}^b \partial_m \psi_a \right) \\ &= -2i\eta\sigma^m \bar{\xi} \partial_m \psi_a + 2i \sum_{c \neq a} \left(\sigma_{cb}^m \partial_m \psi_a - \sigma_{ab}^m \partial_m \psi_c \right) \eta^c \bar{\xi}^b. \end{aligned} \quad (300)$$

The terms in the restricted sum with $c = a$ actually cancel

$$\begin{aligned} 2i \sum_{c \neq a} \left(\sigma_{cb}^m \partial_m \psi_a - \sigma_{ab}^m \partial_m \psi_c \right) \eta^c \bar{\xi}^b \\ &= 2i \left(\sigma_{cb}^m \partial_m \psi_a - \sigma_{ab}^m \partial_m \psi_c \right) \eta^c \bar{\xi}^b \\ &= 2i\varepsilon_{bd} \left(\varepsilon_{ce} \bar{\sigma}^{mde} \partial_m \psi_a - \varepsilon_{ae} \bar{\sigma}^{mde} \partial_m \psi_c \right) \eta^c \bar{\xi}^b. \end{aligned} \quad (301)$$

Both for $a = 1$ and for $a = 2$, this sum is

$$\begin{aligned} 2i \sum_{c \neq a} \left(\sigma_{cb}^m \partial_m \psi_a - \sigma_{ab}^m \partial_m \psi_c \right) \eta^c \bar{\xi}^b &= 2i\varepsilon_{bd} \bar{\sigma}^{mdc} \partial_m \psi_c \varepsilon_{fa} \eta^f \bar{\xi}^b \\ &= 2i\bar{\sigma}^{mdc} \partial_m \psi_c \eta_a \bar{\xi}_b. \end{aligned} \quad (302)$$

By inserting Kronecker deltas, and next using first (44) and then (49) and last (46), we may write this last term as

$$\begin{aligned} 2i\bar{\sigma}^{mdc} \partial_m \psi_c \eta_a \bar{\xi}_b &= 2i\bar{\sigma}^{mdc} \partial_m \psi_c \eta_g \bar{\xi}_h \delta_a^g \delta_b^h \\ &= i\sigma_{ad}^n \bar{\sigma}^{mdc} \partial_m \psi_c \bar{\xi}_h \bar{\sigma}_n^{hg} \eta_g \\ &= -i \left(\sigma^n \bar{\sigma}^m \partial_m \psi \right)_a \eta \sigma_n \bar{\xi} \\ &= 2i\eta_a \bar{\xi} \bar{\sigma}^m \partial_m \psi(x). \end{aligned} \quad (303)$$

Gathering the results (299–303), we may write the second variation of the field $\psi_a(x)$ in the symmetrical form

$$\delta_\eta \delta_\xi \psi_a(x) = -2i\eta\sigma^m \bar{\xi} \partial_m \psi_a(x) + 2i\eta_a \bar{\xi} \bar{\sigma}^m \partial_m \psi(x) + 2i\xi_a \bar{\eta} \bar{\sigma}^m \partial_m \psi(x). \quad (304)$$

Thus the double difference $\delta_\eta \delta_\xi \psi - \delta_\xi \delta_\eta \psi$ is

$$[\delta_\eta, \delta_\xi] \psi_a(x) = 2i \left(\xi \sigma^m \bar{\eta} - \eta \sigma^m \bar{\xi} \right) \partial_m \psi_a(x), \quad (305)$$

which shows that the algebra also closes on the field ψ , as it must.

9 Susy without Grassmann Variables

Exponentials of the generator $G(\xi) = \xi Q + \bar{Q}\bar{\xi}$ are not unitary operators because they involve Grassmann variables. Can one avoid these anti-commuting variables?

Let us consider using generators $G(z)$ that are complex linear forms in the supercharges Q and \bar{Q}

$$G(z) = zQ + \bar{Q}\bar{z} \quad (306)$$

where z^a is a complex spinor.

Now the change in the field $A(x)$ is

$$\begin{aligned} dA(x) \equiv [iG(z), A(x)] &= -i\sqrt{2} \int d^3y z^a \sigma_{ab}^m \psi_b(y) [\partial_m \bar{A}(y), A(x)] \\ &= -\sqrt{2} \int d^3y z^a \sigma_{ab}^0 \psi_b(y) \delta(\vec{x} - \vec{y}) \\ &= \sqrt{2} z^a \psi_a(x) = \sqrt{2} z^a \psi_a, \end{aligned} \quad (307)$$

which is the same as (287) except that the Grassmann spinor ξ has been replaced by the complex spinor z . The conjugate change is

$$d\bar{A} = \sqrt{2} \bar{\psi}_a \bar{z}^a. \quad (308)$$

This procedure will not work, however, for the Fermi field ψ . Instead we must write $d_z \psi$ as an anti-commutator. There are several ways of doing this. If we want a single rule for the change in the product of two spinor fields irrespective of whether they transform like ψ or like $\bar{\chi}$, then we can not have $d\psi$ be the adjoint of $d\bar{\psi}$. We choose to have $d\psi$ be the adjoint of $d\bar{\psi}$. We shall have four different rules for the change in the product of two spinor fields. We define

$$\begin{aligned} d\psi_a(x) &\equiv -\{iG(z), \psi_a(x)\} \\ &= \sqrt{2} \int d^3y \left\{ z^b \varepsilon^{bc} \bar{\psi}_c(y) \bar{W}' + i \bar{\psi}_c(y) \sigma_{cb}^m \bar{z}^b \partial_m A(y), \psi_a(x) \right\} \\ &= \sqrt{2} \left(z^b \varepsilon^{ba} \bar{W}' + i \sigma_{ab}^m \bar{z}^b \partial_m A(x) \right) \\ &= \sqrt{2} \left(-z^b \varepsilon_{ba} \bar{W}' + i \sigma_{ab}^m \bar{z}^b \partial_m A(x) \right) \\ &= \sqrt{2} i \sigma_{ab}^m \bar{z}^b \partial_m A(x) + \sqrt{2} z_a \bar{W}'. \end{aligned} \quad (309)$$

The change in the conjugate $\bar{\psi}$ is the conjugate of the change in ψ :

$$\begin{aligned} d\bar{\psi}_a &\equiv \{iG(z), \bar{\psi}_a\} = (-\{iG(z), \psi_a(x)\})^\dagger = (d_z \psi_a)^\dagger \\ &= -\sqrt{2}iz^b \sigma_{b\dot{a}}^m \partial_m \bar{A} + \sqrt{2}\bar{z}_{\dot{a}} W'. \end{aligned} \quad (310)$$

Although these formulas differ from expression (288) for $\delta\psi$ and its conjugate for $\delta\bar{\psi}$ by the signs of their second terms, and of course by the replacement of a Grassmann spinor ξ by a complex one z , we shall see that these sign differences are appropriate and that supersymmetry can be implemented by unitary transformations acting on the states and physical operators of the theory.

The key point is that the physical operators of the theory contain even powers of the Fermi fields. Thus the change in the generic product $\psi\phi$ is

$$\begin{aligned} d(\psi\phi) &= [iG(z), \psi\phi] \\ &= iG(z)\psi\phi - \psi\phi iG(z) \\ &= iG(z)\psi\phi + \psi iG(z)\phi - \psi iG(z)\phi - \psi\phi iG(z) \\ &= \{iG(z), \psi\}\phi - \psi\{iG(z), \phi\} \\ &= -d_z\psi\phi + \psi d_z\phi, \end{aligned} \quad (311)$$

in which the spinor indices, which may be different, are suppressed. It is easy to see that the other three rules are:

$$d(\bar{\psi}\bar{\phi}) = [iG(z), \bar{\psi}\bar{\phi}] = d\bar{\psi}\bar{\phi} - \bar{\psi}d\bar{\phi} \quad (312)$$

$$d(\bar{\psi}\phi) = [iG(z), \bar{\psi}\phi] = d\bar{\psi}\phi + \bar{\psi}d\phi \quad (313)$$

$$d(\psi\bar{\phi}) = [iG(z), \psi\bar{\phi}] = -d\psi\bar{\phi} - \psi d\bar{\phi}. \quad (314)$$

Let us consider now the effect of these transformations on the action density (188) or equivalently on the general chiral action density (196) with the index i suppressed and with the auxiliary field F replaced by $-\bar{W}'$,

$$\mathcal{L} = \frac{i}{2}\partial_n \bar{\psi}\bar{\sigma}^n \psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^n \partial_n \psi - \partial_n \bar{A}\partial^n A - |W'|^2 - \frac{1}{2}W''\psi\psi - \frac{1}{2}\bar{W}''\bar{\psi}\bar{\psi}. \quad (315)$$

The change in \mathcal{L} due to the changes $d_z A$ and $d_z \psi$ and their conjugates is

$$d\mathcal{L} = [iG(z), \mathcal{L}] = \frac{i}{2}\partial_n d_z \bar{\psi}\bar{\sigma}^n \psi + \frac{i}{2}\partial_n \bar{\psi}\bar{\sigma}^n d_z \psi - \frac{i}{2}d_z \bar{\psi}\bar{\sigma}^n \partial_n \psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^n \partial_n d_z \psi$$

$$\begin{aligned}
& -\partial_n d_z \bar{A} \partial^n A - \partial_n \bar{A} \partial^n d_z A - \bar{W}' W'' d_z A - W' \bar{W}'' d_z \bar{A} \\
& - \frac{1}{2} W''' d_z A \psi \psi - \frac{1}{2} \bar{W}''' d_z \bar{A} \bar{\psi} \bar{\psi} + \frac{1}{2} W'' d_z \psi \psi - \frac{1}{2} W'' \psi d_z \psi \\
& - \frac{1}{2} \bar{W}'' d_z \bar{\psi} \bar{\psi} + \frac{1}{2} \bar{W}'' \bar{\psi} d_z \bar{\psi}. \tag{316}
\end{aligned}$$

The part of $d\mathcal{L}$ that depends upon z is

$$\begin{aligned}
d_z \mathcal{L} &= \frac{i}{2} \partial_n (-\sqrt{2} i z \sigma^m \partial_m \bar{A}) \bar{\sigma}^n \psi + \frac{i}{2} \partial_n \bar{\psi} \bar{\sigma}^n (\sqrt{2} z \bar{W}') \\
& - \frac{i}{2} (-\sqrt{2} i z \sigma^m \partial_m \bar{A}) \bar{\sigma}^n \partial_n \psi - \frac{i}{2} \bar{\psi} \bar{\sigma}^n \partial_n (\sqrt{2} z \bar{W}') \\
& - \partial_n \bar{A} \partial^n (\sqrt{2} z \psi) - \bar{W}' W'' (\sqrt{2} z \psi) \\
& - \frac{1}{2} W''' (\sqrt{2} z \psi) \psi \psi + \frac{1}{2} W'' (\sqrt{2} z \bar{W}') \psi - \frac{1}{2} W'' \psi (\sqrt{2} z \bar{W}') \\
& - \frac{1}{2} \bar{W}'' (-\sqrt{2} i z \sigma^m \partial_m \bar{A}) \bar{\psi} + \frac{1}{2} \bar{W}'' \bar{\psi} (-\sqrt{2} i z \sigma^m \partial_m \bar{A}). \tag{317}
\end{aligned}$$

As in eqs.(211) and (212), the term proportional to W''' vanishes. Also the two terms proportional to $\bar{W}' W''$ cancel. Finally the last two terms may be written as

$$\frac{i}{2} \bar{\psi}^{\dot{a}} \sigma_{\dot{b}\dot{a}}^m z^b \partial_m \bar{W}' = \frac{i}{2} \bar{\psi}_{\dot{c}} \varepsilon^{\dot{a}\dot{c}} \sigma_{\dot{b}\dot{a}}^m \varepsilon^{bd} z_d \partial_m \bar{W}' = \frac{i}{2} \bar{\psi} \bar{\sigma}^m z \partial_m \bar{W}' \tag{318}$$

and as

$$-\frac{i}{2} \bar{\psi}_{\dot{a}} z^b \sigma_{\dot{b}\dot{c}}^m \varepsilon^{\dot{a}\dot{c}} \partial_m \bar{W}' = -\frac{i}{2} \bar{\psi}_{\dot{a}} \varepsilon^{\dot{a}\dot{c}} \varepsilon^{bd} \sigma_{\dot{b}\dot{c}}^m z_d \partial_m \bar{W}' = \frac{i}{2} \bar{\psi} \bar{\sigma}^m z \partial_m \bar{W}'. \tag{319}$$

So the change $d_z \mathcal{L}$ in the action density is

$$\begin{aligned}
d_z \mathcal{L} &= \frac{1}{\sqrt{2}} z \sigma^m \bar{\sigma}^n \psi \partial_n \partial_m \bar{A} - \frac{1}{\sqrt{2}} z \sigma^m \bar{\sigma}^n \partial_n \psi \partial_m \bar{A} \\
& - \sqrt{2} z \partial^n \psi \partial_n \bar{A} + \frac{i}{2} \partial_n (\bar{\psi} \bar{\sigma}^n z \bar{W}'). \tag{320}
\end{aligned}$$

Just as in eqs.(142) and (146), we may write this change $d_z \mathcal{L}$ as the total divergence

$$d_z \mathcal{L} = \partial_n K_z^n \tag{321}$$

of the current

$$K_z^n = -\frac{1}{\sqrt{2}} z \sigma^m \bar{\sigma}^n \partial_m \bar{A} - \sqrt{2} z \psi \partial^n \bar{A} + \frac{i}{2} \bar{\psi} \bar{\sigma}^n z \bar{W}', \tag{322}$$

which shows that the action is invariant under the unitary transformation

$$U(z) = e^{-iG(z)} \quad (323)$$

at least for infinitesimal values of the complex spinor z .

Thus unitary operators without Grassmann variables can implement supersymmetry transformations upon the action and other operators that involve only even powers of Fermi fields. But the effect (309) of a susy transformation upon a single Fermi field does not seem to admit such a representation.

Also if we define physical states $|\psi\rangle$ as those that under a rotation of angle 2π about any axis $\hat{\theta}$ suffer at most a phase change

$$e^{-i2\pi\hat{\theta}\cdot\vec{J}}|\psi\rangle = e^{i\phi}|\psi\rangle, \quad (324)$$

then the unitary operator (323), being the exponential of an operator $G(z)$ that is odd or fermionic, transforms all physical states that are not annihilated by $G(z)$ into states that are not physical.

10 The Free Supercharges Q_K

10.1 Expansions

We may expand the supercharges Q_{Ka} of the free theory in terms of creation and annihilation operators by using the expansions of the field operators ψ_a and A . Since the free field ψ_a satisfies the free-field equation

$$-i\vec{\sigma}^n\partial_n\psi = 0, \quad (325)$$

we may expand it as

$$\psi_a(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} \left[e^{ip\cdot x} u(p)_a b(p) + e^{-ip\cdot x} v(p)_a c^\dagger(p) \right] \quad (326)$$

in which the spinors $u(p)$ and $v(p)$ are eigenvectors of $\vec{\sigma}\cdot\vec{p}$

$$\vec{\sigma}\cdot\vec{p}u(p) = |\vec{p}|u(p) \quad \text{and} \quad \vec{\sigma}\cdot\vec{p}v(p) = |\vec{p}|v(p), \quad (327)$$

which are normalized $u(p)^\dagger u(p) = v(p)^\dagger v(p) = 1$. The expansion of the scalar field A is

$$A(x) = \int \frac{d^3 k}{\sqrt{(2\pi)^3 2|\vec{k}|}} \left[e^{ik \cdot x} a(k) + e^{-ik \cdot x} a_c^\dagger(k) \right], \quad (328)$$

in which the operator a_c^\dagger creates the particle that is the anti-particle of the particle created by a^\dagger . By substituting the expansion (326) and the adjoint of (328) into the formula (156) for the supercharges of the free theory, we find after some elementary manipulations

$$Q_{\mathcal{K}a} = 2i \int d^3 p \sqrt{|\vec{p}|} \left[u(p)_a b(p) a^\dagger(p) - v(p)_a c^\dagger(p) a_c(p) \right]. \quad (329)$$

Since every term in the supercharge $Q_{\mathcal{K}a}$ and in its adjoint $\bar{Q}_{\mathcal{K}\dot{a}}$ contains an annihilation operator, it is clear that both $Q_{\mathcal{K}a}$ and $\bar{Q}_{\mathcal{K}\dot{a}} = Q_{\mathcal{K}a}^\dagger$ annihilate the no-particle state $|0\rangle$, which is the vacuum of the free-field theory

$$Q_{\mathcal{K}a}|0\rangle = 0 \quad \text{and} \quad Q_{\mathcal{K}a}^\dagger|0\rangle = 0, \quad (330)$$

which incidentally shows that susy is unbroken in the free theory.

10.2 Functional Methods

It may be useful to see how the equations (330) look in terms of the wave function of the vacuum of the free theory. Let the state $|\mathcal{A}\rangle$ be a simultaneous eigenstate of the field operators A and \bar{A} at time $t = 0$

$$A(0, \vec{x})|\mathcal{A}\rangle = \mathcal{A}(\vec{x})|\mathcal{A}\rangle \quad \text{and} \quad \bar{A}(0, \vec{x})|\mathcal{A}\rangle = \bar{\mathcal{A}}(\vec{x})|\mathcal{A}\rangle. \quad (331)$$

To define the analogous state for the Fermi field ψ , we follow Weinberg [4], *mutatis mutandis*, and write

$$|\varphi\rangle = \exp \left(\int d^3 x \psi_a^\dagger(0, \vec{x}) \varphi_a(\vec{x}) \right) \left(\prod_{\vec{x}b} \psi_b(0, \vec{x}) \right) |0\rangle \quad (332)$$

in which the sum over a is from 1 to 2. The state $|\varphi\rangle$ is an eigenstate of the operator $\psi_a(0, \vec{x})$

$$\psi_a(0, \vec{x})|\varphi\rangle = \varphi_a(\vec{x})|\varphi\rangle. \quad (333)$$

We may also construct the state

$$|\bar{\varphi}\rangle = \exp\left(\int d^3x \bar{\varphi}_a(\vec{x})\psi_a(0, \vec{x})\right) \left(\prod_{\vec{x}b} \psi_b^\dagger(0, \vec{x})\right) |0\rangle \quad (334)$$

which is an eigenstate of $\psi_a^\dagger(0, \vec{x})$

$$\psi_a^\dagger(0, \vec{x})|\bar{\varphi}\rangle = \bar{\varphi}_a(\vec{x})|\bar{\varphi}\rangle. \quad (335)$$

We shall write the adjoint states as

$$\langle\bar{\varphi}| = \langle 0| \left(\prod_{\vec{x}b} \psi_b^\dagger(0, \vec{x})\right) \exp\left(\int d^3x \bar{\varphi}_a(\vec{x})\psi_a(0, \vec{x})\right) \quad (336)$$

which satisfies

$$\langle\bar{\varphi}|\bar{\psi}_a(0, \vec{x}) = (\psi_a(0, \vec{x})|\varphi\rangle)^\dagger = \langle\bar{\varphi}|\bar{\varphi}_a(\vec{x}) \quad (337)$$

and as

$$\langle\varphi| = \langle 0| \left(\prod_{\vec{x}b} \psi_b(0, \vec{x})\right) \exp\left(\int d^3x \psi_a^\dagger(0, \vec{x})\varphi_a(\vec{x})\right) \quad (338)$$

which satisfies

$$\langle\varphi|\psi_a(0, \vec{x}) = (\psi_a^\dagger(0, \vec{x})|\bar{\varphi}\rangle)^\dagger = \langle\varphi|\varphi_a(\vec{x}). \quad (339)$$

But to obtain the effect from the right of ψ on the state $\langle\bar{\varphi}|$ or of ψ^\dagger on the state $\langle\varphi|$, one differentiates

$$\langle\bar{\varphi}|\psi_a(0, \vec{x}) = \frac{\delta}{\delta\bar{\varphi}_a} \langle\bar{\varphi}| \quad \text{and} \quad \langle\varphi|\psi_a^\dagger(0, \vec{x}) = -\frac{\delta}{\delta\varphi_a} \langle\varphi| \quad (340)$$

Both the states $|\varphi\rangle$ and the states $|\bar{\varphi}\rangle$ form complete sets of states.

Let us now compute the inner product of the no-particle state $|0\rangle$ with the eigenstates of the Fermi fields ψ and ψ^\dagger . To this end it will be useful first to calculate the integrals that occur in the exponentials (332) and (334). We may expand the field ψ at time $t = 0$ in terms of the annihilation operators $b(\vec{p})$ and the creation operators $c^\dagger(-\vec{p})$

$$\psi_a(0, \vec{x}) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} e^{i\vec{p}\cdot\vec{x}} \left[u(\vec{p})_a b(\vec{p}) + v(-\vec{p})_a c^\dagger(-\vec{p}) \right]. \quad (341)$$

We may also expand the Grassmann field $\varphi(\vec{x})$ as

$$\varphi_a(\vec{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3}} e^{i\vec{k}\cdot\vec{x}} \sum_{s=1}^2 w(\vec{k}, s)_a \varphi(k, s) \quad (342)$$

in which the variables $\varphi(k, s)$ are grassmannian and the vectors $w(\vec{k}, s)$ are chosen to obey the orthonormality relations

$$\begin{aligned} \sum_{a=1}^2 w^*(\vec{p}, 1)_a u(\vec{p})_a &= 1 & \text{and} & \quad \sum_{a=1}^2 w^*(\vec{p}, 1)_a v(-\vec{p})_a = 0 \\ \sum_{a=1}^2 w^*(\vec{p}, 2)_a u(\vec{p})_a &= 0 & \text{and} & \quad \sum_{a=1}^2 w^*(\vec{p}, 2)_a v(-\vec{p})_a = 1. \end{aligned} \quad (343)$$

Then the integrals are

$$\int d^3x \psi_a^\dagger(0, \vec{x}) \varphi_a(\vec{x}) = \int d^3p \left[b^\dagger(\vec{p}) \varphi(\vec{p}, 1) + c(-\vec{p}) \varphi(\vec{p}, 2) \right] \quad (344)$$

and its adjoint

$$\int d^3x \bar{\varphi}_a(\vec{x}) \psi_a(0, \vec{x}) = \int d^3p \left[\bar{\varphi}(\vec{p}, 1) b(\vec{p}) + \bar{\varphi}(\vec{p}, 2) c^\dagger(-\vec{p}) \right], \quad (345)$$

in which sums over a from 1 to 2 are understood.

We may now evaluate the inner product

$$\langle 0|\varphi\rangle = \langle 0|\exp\left(\int d^3x \psi_a^\dagger(0, \vec{x}) \varphi_a(\vec{x})\right) \left(\prod_{\vec{x}b} \psi_b(0, \vec{x})\right) |0\rangle. \quad (346)$$

In the infinite product, only the negative-frequency parts

$$\psi_b^{(-)}(0, \vec{x}) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} e^{i\vec{p}\cdot\vec{x}} v(-\vec{p})_a c^\dagger(-\vec{p}) \quad (347)$$

survive

$$\langle 0|\varphi\rangle = \langle 0|\exp\left(\int d^3x \psi_a^\dagger(0, \vec{x}) \varphi_a(\vec{x})\right) \left(\prod_{\vec{x}b} \psi_b^{(-)}(0, \vec{x})\right) |0\rangle. \quad (348)$$

Substituting our formula (344) for the integral, we get

$$\langle 0|\varphi\rangle = \langle 0|\exp\left(\int d^3p \left[b^\dagger(\vec{p})\varphi(\vec{p}, 1) + c(-\vec{p})\varphi(\vec{p}, 2)\right]\right) \left(\prod_{\vec{x}b} \psi_b^{(-)}(0, \vec{x})\right) |0\rangle. \quad (349)$$

If at each momentum \vec{p} we now expand the exponential, remembering that φ is Grassmann, then we find

$$\langle 0|\varphi\rangle = \langle 0|\left(\prod_{\vec{p}} \left[1 + d^3p \left(b^\dagger(\vec{p})\varphi(\vec{p}, 1) + c(-\vec{p})\varphi(\vec{p}, 2)\right)\right]\right) \left(\prod_{\vec{x}a} \psi_a^{(-)}(0, \vec{x})\right) |0\rangle. \quad (350)$$

None of the $b^\dagger(\vec{p})$ terms survive, and so apart from an overall factor independent of φ , we have

$$\langle 0|\varphi\rangle = \prod_{\vec{p}} \varphi(\vec{p}, 2). \quad (351)$$

By a similar argument we find

$$\begin{aligned} \langle 0|\bar{\varphi}\rangle &= \langle 0|\exp\left(\int d^3x \bar{\varphi}_a(\vec{x})\psi_a(0, \vec{x})\right) \left(\prod_{\vec{x}b} \psi_b^\dagger(0, \vec{x})\right) |0\rangle \\ &= \langle 0|\exp\left(\int d^3p \left[\bar{\varphi}(\vec{p}, 1)b(\vec{p}) + \bar{\varphi}(\vec{p}, 2)c^\dagger(-\vec{p})\right]\right) \left(\prod_{\vec{x}b} \psi_b^{(+)\dagger}(0, \vec{x})\right) |0\rangle \\ &= \prod_{\vec{p}} \bar{\varphi}(\vec{p}, 1). \end{aligned} \quad (352)$$

The bosonic wave-function of the vacuum is well known to be

$$\langle A, \bar{A}|0\rangle = \exp\left(-\int d^3x \bar{A}(\vec{x})\sqrt{-\nabla^2}A(\vec{x})\right) \quad (353)$$

or equivalently

$$\langle A, \bar{A}|0\rangle = \exp\left(-\int d^3x A(\vec{x})\sqrt{-\nabla^2}\bar{A}(\vec{x})\right). \quad (354)$$

Thus the full wave-functions of the vacuum are

$$\langle \varphi, A, \bar{A}|0\rangle = \left(\prod_{\vec{p}} \varphi(\vec{p}, 1)\right) \exp\left(-\int d^3x A(\vec{x})\sqrt{-\nabla^2}\bar{A}(\vec{x})\right) \quad (355)$$

and

$$\langle \bar{\varphi}, A, \bar{A} | 0 \rangle = \left(\prod_{\vec{p}} \bar{\varphi}(\vec{p}, 2) \right) \exp \left(- \int d^3 x \bar{A}(\vec{x}) \sqrt{-\nabla^2} A(\vec{x}) \right). \quad (356)$$

We may now verify in this functional formalism that the free supercharges $Q_{\mathcal{K}a}$ and $\bar{Q}_{\mathcal{K}\dot{a}}$ annihilate the no-particle state $|0\rangle$. By referring to eq.(156), we have

$$\langle \varphi, A, \bar{A} | Q_{\mathcal{K}a} | 0 \rangle = \langle \varphi, A, \bar{A} | (-\sqrt{2}) \int d^3 x \sigma_{ab}^m \psi_b(0, \vec{x}) \partial_m \bar{A}(0, \vec{x}) | 0 \rangle. \quad (357)$$

The time derivative $\partial_0 \bar{A}(\vec{x})$ is the canonical momentum operator $\pi(\vec{x})$ which may be represented as a functional derivative

$$\langle \varphi, A, \bar{A} | \partial_0 \bar{A}(0, \vec{x}) = \langle \varphi, A, \bar{A} | \pi(0, \vec{x}) = \frac{\delta}{i\delta A(\vec{x})} \langle \varphi, A, \bar{A} |. \quad (358)$$

Thus we have

$$\langle \varphi, A, \bar{A} | Q_{\mathcal{K}a} | 0 \rangle = -\sqrt{2} \langle \varphi, A, \bar{A} | \int d^3 x \sigma_{ab}^i \psi_b(0, \vec{x}) \partial_i \bar{A}(0, \vec{x}) - \psi_a(0, \vec{x}) \pi(0, \vec{x}) | 0 \rangle. \quad (359)$$

Now using the eigenvalue relation (338) and the functional relation (358), we find

$$\langle \varphi, A, \bar{A} | Q_{\mathcal{K}a} | 0 \rangle = -\sqrt{2} \int d^3 x \left[\sigma_{ab}^i \varphi_b(\vec{x}) \partial_i \bar{A}(\vec{x}) - \varphi_a(\vec{x}) \frac{\delta}{i\delta A(\vec{x})} \right] \langle \varphi, A, \bar{A} | Q_{\mathcal{K}a} | 0 \rangle. \quad (360)$$

By using our formula (355) for the wave-function of the vacuum, we have

$$\begin{aligned} \langle \varphi, A, \bar{A} | Q_{\mathcal{K}a} | 0 \rangle &= -\sqrt{2} \int d^3 x \left[\sigma_{ab}^i \varphi_b(\vec{x}) \partial_i \bar{A}(\vec{x}) - \varphi_a(\vec{x}) \frac{\delta}{i\delta A(\vec{x})} \right] \\ &\quad \times \left(\prod_{\vec{p}} \varphi(\vec{p}, 1) \right) \exp \left(- \int d^3 y A(\vec{y}) \sqrt{-\nabla^2} \bar{A}(\vec{y}) \right) \\ &= -\sqrt{2} \int d^3 x \left[\sigma_{ab}^i \varphi_b(\vec{x}) \partial_i \bar{A}(\vec{x}) - i\varphi_a(\vec{x}) \sqrt{-\nabla^2} \bar{A}(\vec{x}) \right] \\ &\quad \times \left(\prod_{\vec{p}} \varphi(\vec{p}, 1) \right) \exp \left(- \int d^3 y A(\vec{y}) \sqrt{-\nabla^2} \bar{A}(\vec{y}) \right) \end{aligned}$$

or after integrating by parts

$$\langle \varphi, A, \bar{A} | Q_{\mathcal{K}a} | 0 \rangle = i\sqrt{2} \int d^3x \bar{A}(\vec{x}) \left[(-i\vec{\sigma} \cdot \nabla + \sqrt{-\nabla^2}) \varphi(\vec{x}) \right]_a \langle \varphi, A, \bar{A} | Q_{\mathcal{K}a} | 0 \rangle. \quad (361)$$

The effect of the differential operator $-i\vec{\sigma} \cdot \nabla + \sqrt{-\nabla^2}$ on the Grassmann field φ is

$$(-i\vec{\sigma} \cdot \nabla + \sqrt{-\nabla^2}) \varphi(\vec{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3}} (\vec{\sigma} \cdot \vec{k} + |\vec{k}|) e^{i\vec{k} \cdot \vec{x}} \sum_{s=1}^2 w(\vec{k}, s) \varphi(k, s). \quad (362)$$

By their definitions (343), the spinors $w(\vec{k}, 1)$ and $w(\vec{k}, 2)$ respond to $\vec{\sigma} \cdot \vec{k} + |\vec{k}|$ the same ways as do the spinors $u(\vec{k})$ and $v(-\vec{k})$ respectively. Thus by the spinor eigenvalue equations (327), the operator $-i\vec{\sigma} \cdot \nabla + \sqrt{-\nabla^2}$ projects out the part of φ that is proportional to $w(\vec{k}, 1)$

$$(-i\vec{\sigma} \cdot \nabla + \sqrt{-\nabla^2}) \varphi(\vec{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3}} 2|\vec{k}| e^{i\vec{k} \cdot \vec{x}} w(\vec{k}, 1) \varphi(\vec{k}, 1). \quad (363)$$

Hence on the vacuum the free supercharge Q_K produces a doubling of the Grassmann variable $\varphi(\vec{k}, 1)$

$$\begin{aligned} \langle \varphi, A, \bar{A} | Q_{\mathcal{K}a} | 0 \rangle &= i\sqrt{2} \int d^3x \bar{A}(\vec{x}) \int \frac{d^3k}{\sqrt{(2\pi)^3}} 2|\vec{k}| e^{i\vec{k} \cdot \vec{x}} w(\vec{k}, 1)_a \varphi(\vec{k}, 1) \\ &\times \left(\prod_{\vec{p}} \varphi(\vec{p}, 1) \right) e^{-\int d^3y A(\vec{y}) \sqrt{-\nabla^2} \bar{A}(\vec{y})} = 0 \end{aligned} \quad (364)$$

which vanishes.

The free adjoint supercharges $Q_{\mathcal{K}}^\dagger$ may be shown to annihilate the free vacuum $|0\rangle$ by similar functional manipulations. Indeed because $\langle \bar{\varphi}, A, \bar{A} | 0 \rangle$ is proportional by (356) to the product of $\bar{\varphi}(\vec{k}, 2)$ for all \vec{k} , the matrix element

$$\begin{aligned} \langle \bar{\varphi}, A, \bar{A} | Q_{\mathcal{K}a}^\dagger | 0 \rangle &= -\sqrt{2} \int d^3x \langle \bar{\varphi}, A, \bar{A} | \bar{\psi}_b \sigma_{ba}^m \partial_m A | 0 \rangle \\ &= -\sqrt{2} \int d^3x \left[\bar{\varphi} \left(\vec{\sigma} \cdot \nabla A - \frac{\delta}{i\delta \bar{A}} \right) \right]_a \langle \bar{\varphi}, A, \bar{A} | 0 \rangle \end{aligned}$$

$$\begin{aligned}
&= i\sqrt{2} \int d^3x \left[\bar{\varphi} \left(i\vec{\sigma} \cdot \nabla A + \sqrt{-\nabla^2} A \right) \right]_{\dot{a}} \langle \bar{\varphi}, A, \bar{A} | 0 \rangle \\
&= i\sqrt{2} \int d^3x A \left(-i\partial_i \bar{\varphi} \sigma^i + \sqrt{-\nabla^2} \bar{\varphi} \right)_{\dot{a}} \langle \bar{\varphi}, A, \bar{A} | 0 \rangle \\
&= i\sqrt{2} \int \frac{d^3x d^3k}{\sqrt{(2\pi)^3}} A e^{-i\vec{k} \cdot \vec{x}} 2|\vec{k}| w_{\dot{a}}^\dagger(\vec{k}, 2) \bar{\varphi}(\vec{k}, 2) \langle \bar{\varphi}, A, \bar{A} | 0 \rangle \\
&= 0
\end{aligned} \tag{365}$$

is proportional to a sum of squares of $\bar{\varphi}(\vec{k}, 2)$ and therefore vanishes.

11 Abelian Supersymmetric Gauge Theories

11.1 The Vector Multiplet

The action density of the supersymmetric $U(1)$ gauge theory is

$$\mathcal{L} = -\frac{1}{4} v_{mn} v^{mn} - i\bar{\lambda} \bar{\sigma}^m \partial_m \lambda + \frac{1}{2} D^2 \tag{366}$$

in which v_n is the gauge field and

$$v_{mn} = \partial_m v_n - \partial_n v_m \tag{367}$$

is the Maxwell field-strength tensor, which is often written as F_{mn} . Because the gaugino or photino field λ is, like v_n , in the adjoint representation, its covariant derivative is the same as its ordinary derivative. The theory thus describes two non-interacting free fields after the constraint $D = 0$ on the auxiliary field D is implemented.

The fields v_m , λ , and D form part of a vector supermultiplet. One may make the other fields of the supermultiplet vanish by performing a supergauge transformation to the Wess-Zumino gauge. Under a general supersymmetry transformation, these extra fields do not remain zero. But if one augments an arbitrary supersymmetry transformation by a related supergauge transformation, one may keep the extra fields zero and restore the Wess-Zumino gauge. The action density (366) changes by at most a total derivative under

such an augmented supersymmetry transformation

$$\delta v_m = -i\bar{\lambda}\bar{\sigma}^m\xi + i\bar{\xi}\bar{\sigma}^m\lambda \quad (368)$$

$$\delta\lambda = \sigma^{mn}\xi v_{mn} + i\xi D \quad (369)$$

$$\delta D = -\xi\sigma^m\partial_m\bar{\lambda} - \partial_m\lambda\sigma^m\bar{\xi} \quad (370)$$

and is invariant under the ordinary gauge transformation

$$\delta v_m = \partial_m\omega \quad (371)$$

$$\delta\lambda = 0 \quad (372)$$

$$\delta D = 0 \quad (373)$$

in which the function $\omega(x)$ is a scalar.

11.2 The Fayet-Iliopoulos D Term

Because the auxiliary field D changes only by a space-time derivative under the supersymmetry transformation (370) and not at all under the gauge transformation (373), the space-time integral of D is invariant and may be added to the action. This extra piece ξD in the action, conventionally multiplied by the constant ξ , is known as the Fayet-Iliopoulos D term.

With the Fayet-Iliopoulos D term, the action density

$$\mathcal{L} = -\frac{1}{4}v_{mn}v^{mn} - i\bar{\lambda}\bar{\sigma}^m\partial_m\lambda + \frac{1}{2}D^2 + \xi D \quad (374)$$

entails for the auxiliary field D the constraint

$$D = -\xi, \quad (375)$$

and may be written as

$$\mathcal{L} = -\frac{1}{4}v_{mn}v^{mn} - i\bar{\lambda}\bar{\sigma}^m\partial_m\lambda - \frac{1}{2}\xi^2. \quad (376)$$

The energy density thus acquires the positive term

$$V = \frac{1}{2}\xi^2, \quad (377)$$

and supersymmetry is spontaneously broken.

11.3 The General Abelian Gauge Theory

The action density for a general supersymmetric abelian gauge theory is

$$\mathcal{L} = -\frac{1}{4}v_{mn}v^{mn} \quad (378)$$

$$-i\bar{\lambda}\bar{\sigma}^m\partial_m\lambda \quad (379)$$

$$+\frac{1}{2}D^2 \quad (380)$$

$$+\xi D \quad (381)$$

$$-\overline{\mathcal{D}_m A^i} \mathcal{D}^m A^i \quad (382)$$

$$-i\bar{\psi}^i\bar{\sigma}^m\mathcal{D}_m\psi^i \quad (383)$$

$$+\bar{F}_i F_i \quad (384)$$

$$-i\sqrt{2}g_i(\bar{A}_i\psi_i\lambda - \bar{\lambda}\bar{\psi}_i A_i) \quad (385)$$

$$-g_i D\bar{A}_i A_i \quad (386)$$

$$+m_{ij}\left(A_i F_j - \frac{1}{2}\psi_i\psi_j\right) + \bar{m}_{ij}\left(\bar{A}_i\bar{F}_j - \frac{1}{2}\bar{\psi}_i\bar{\psi}_j\right) \quad (387)$$

$$+g_{ijk}(F_i A_j A_k - \psi_i\psi_j A_k) + \bar{g}_{ijk}(\bar{F}_i\bar{A}_j\bar{A}_k - \bar{\psi}_i\bar{\psi}_j\bar{A}_k), \quad (388)$$

in which v_m is the gauge field, λ is the gaugino field, and the chiral fields ψ_i and A_i carry charge g_i . The indices i, j , and k which label chiral multiplets with abelian charges g_i, g_j , and g_k are summed over. The symmetric tensors m_{ij} and g_{ijk} must be invariant under the action of the gauge group. The covariant derivatives[‡] are

$$\mathcal{D}_m A_i = \partial_m A_i - ig_i v_m A_i \quad (389)$$

$$\mathcal{D}_m \psi_i = \partial_m \psi_i - ig_i v_m \psi_i \quad (390)$$

$$\mathcal{D}_m \bar{\psi}_i = \partial_m \bar{\psi}_i + ig_i v_m \bar{\psi}_i \quad (391)$$

$$v_{mn} = \partial_m v_n^a - \partial_n v_m^a. \quad (392)$$

[‡]The notation used here for the covariant derivatives follows that of Weinberg [5, 6] in which the covariant derivative of a field that annihilates a particle of charge q is $(\partial_m - iqA_m)\psi$.

11.4 Super QED

An important example of an abelian gauge theory is super QED with a Fayet-Iliopoulos D term. The action density is a special case of the general abelian action density (378–388) in which the index i runs from 1 to 2

$$\begin{aligned}
\mathcal{L}_{sqed} = & -\frac{1}{4}v_{mn}v^{mn} - i\bar{\lambda}\bar{\sigma}^m\partial_m\lambda + \frac{1}{2}D^2 + \xi D \\
& - \overline{\mathcal{D}_m A^i} \mathcal{D}^m A^i - i\bar{\psi}^i\bar{\sigma}^m\mathcal{D}_m\psi^i + \bar{F}_i F_i \\
& - i\sqrt{2}g_i \left(\bar{A}_i\psi_i\lambda - \bar{\lambda}\bar{\psi}_i A_i \right) - g_i D \bar{A}_i A_i \\
& + m_{ij} \left(A_i F_j - \frac{1}{2}\psi_i\psi_j \right) + \bar{m}_{ij} \left(\bar{A}_i \bar{F}_j - \frac{1}{2}\bar{\psi}_i\bar{\psi}_j \right) \quad (393)
\end{aligned}$$

with ψ^1 being the right-handed electron and ψ^2 being the right-handed positron. Thus $g_1 = -q < 0$ and $\mathcal{D}_m\psi^1 = (\partial_m + iqv_m)\psi^1$; $g_2 = q > 0$ and $\mathcal{D}_m\psi^2 = (\partial_m - iqv_m)\psi^2$; and the only non-zero values of the symmetric tensor m_{ij} are $m_{12} = m_{21} = m$.

We may also write this action density in terms of the right- and left-handed electron fields $e_R = \psi^1$ and $\bar{e}_L = \bar{\psi}^2$. By using the identity (49), integrating by parts, and dropping surface terms, we see that the action density

$$-i\bar{\psi}^2\bar{\sigma}^m\mathcal{D}_m\psi^2 = -i\bar{\psi}^2\bar{\sigma}^m(\partial_m - iqv_m)\psi^2 \quad (394)$$

is equivalent to

$$-i\psi^2\sigma^m\mathcal{D}_m\bar{\psi}^2 = -i\psi^2\sigma^m(\partial_m + iqv_m)\bar{\psi}^2. \quad (395)$$

So if we denote the right-handed selectron field as $\tilde{e}_R = A^1$ and the left-handed selectron field as $\tilde{e}_L = \bar{A}^2$, then we may write the action density (393) in the form

$$\begin{aligned}
\mathcal{L}_{sqed} = & -\frac{1}{4}v_{mn}v^{mn} - i\bar{\lambda}\bar{\sigma}^m\partial_m\lambda + \frac{1}{2}D^2 + \xi D \\
& - \overline{\mathcal{D}_m \tilde{e}_R} \mathcal{D}^m \tilde{e}_R - \overline{\mathcal{D}_m \tilde{e}_L} \mathcal{D}^m \tilde{e}_L - i\bar{e}_R\bar{\sigma}^m\mathcal{D}_m e_R - ie_L\sigma^m\mathcal{D}_m \bar{e}_L \\
& + \bar{e}_R\bar{e}_R + \bar{e}_L\bar{e}_L + i\sqrt{2}q \left(\bar{e}_R e_R \lambda - \bar{\lambda}\bar{e}_R \tilde{e}_R \right) \\
& - i\sqrt{2}q \left(\tilde{e}_L e_L \lambda - \bar{\lambda}\bar{e}_L \bar{\tilde{e}}_L \right) + eD(\bar{e}_R\bar{e}_R - \bar{e}_L\bar{\tilde{e}}_L) \\
& + m \left(\tilde{e}_R\bar{\tilde{e}}_L + \bar{\tilde{e}}_L e_R - e_R e_L \right) + \bar{m} \left(\bar{\tilde{e}}_R\bar{e}_L + \tilde{e}_L\bar{\tilde{e}}_R - \bar{e}_R\bar{e}_L \right) \quad (396)
\end{aligned}$$

in which we have used for the auxiliary fields of the right- and left-handed electron the notation $\dot{e}_R = F_1$ and $\dot{e}_L = \bar{F}_2$. Because all the matter fields in this expression have negative charge $-q$, all the covariant derivatives in it are $\mathcal{D}_m = \partial_m + iqv_m$. It is clear that we may choose the phase of the fields e_R , \dot{e}_R , and \tilde{e}_R so as to render the mass parameter m real and non-negative whatever the arbitrary phases of the corresponding left-handed-electron fields might be. So from now on we shall take $m \geq 0$.

The constraints on the auxiliary fields are:

$$\begin{aligned} D &= -\xi - q \left(|\tilde{e}_R|^2 - |\tilde{e}_L|^2 \right) \\ \dot{e}_R &= -m\tilde{e}_L \\ \dot{e}_L &= -m\tilde{e}_R, \end{aligned} \quad (397)$$

and when these constraints are enforced the action density takes the form

$$\begin{aligned} \mathcal{L}_{sqed} &= -\frac{1}{4}v_{mn}v^{mn} - i\bar{\lambda}\bar{\sigma}^m\partial_m\lambda \\ &\quad - \overline{\mathcal{D}_m\tilde{e}_R} \mathcal{D}^m\tilde{e}_R - \overline{\mathcal{D}_m\tilde{e}_L} \mathcal{D}^m\tilde{e}_L - i\bar{e}_R\bar{\sigma}^m\mathcal{D}_m e_R - ie_L\sigma^m\mathcal{D}_m\bar{e}_L \\ &\quad + i\sqrt{2}q \left(\overline{\tilde{e}_R}e_R\lambda - \bar{\lambda}\bar{e}_R\tilde{e}_R \right) - i\sqrt{2}q \left(\tilde{e}_L e_L\lambda - \bar{\lambda}\bar{e}_L\tilde{e}_L \right) \\ &\quad - m(e_R e_L - \bar{e}_R\bar{e}_L) \\ &\quad - \frac{1}{2} \left[\xi + q \left(|\tilde{e}_R|^2 - |\tilde{e}_L|^2 \right) \right]^2 - m^2 \left(|\tilde{e}_R|^2 + |\tilde{e}_L|^2 \right). \end{aligned} \quad (398)$$

Thus the scalar potential is

$$V = \frac{1}{2} \left[\xi + q \left(|\tilde{e}_R|^2 - |\tilde{e}_L|^2 \right) \right]^2 + m^2 \left(|\tilde{e}_R|^2 + |\tilde{e}_L|^2 \right). \quad (399)$$

If $\xi = 0$, but $m \neq 0$, then the mean values $\langle \tilde{e}_R \rangle$ and $\langle \tilde{e}_L \rangle$ of the selectron fields in the vacuum vanish, and neither supersymmetry nor gauge symmetry is broken. If $\xi \neq 0$, but $m = 0$, then gauge symmetry, but not supersymmetry, is broken in a three-parameter valley of vacua which satisfy

$$\langle \Omega | \left(|\tilde{e}_L|^2 - |\tilde{e}_R|^2 \right) | \Omega \rangle = \frac{\xi}{q}. \quad (400)$$

When both ξ and m are non-zero with $|m|^2 > |\xi q|$, then the mean values $\langle \tilde{e}_R \rangle$ and $\langle \tilde{e}_L \rangle$ both vanish in the vacuum, and gauge symmetry is exact, but the mean value of the potential in the vacuum is

$$\langle V \rangle = \frac{1}{2}\xi^2, \quad (401)$$

and supersymmetry is broken. On the other hand, when both ξ and m are non-zero with $m^2 < |\xi q|$, then for $\xi > 0$ there is a one-parameter ring of vacua Ω in which the mean values of the scalar fields satisfy

$$\langle \Omega | \tilde{e}_R | \Omega \rangle = 0 \quad (402)$$

$$|\langle \Omega | \tilde{e}_L | \Omega \rangle| = \sqrt{\frac{\xi}{q} - \frac{m^2}{q^2}}. \quad (403)$$

The $U(1)$ gauge symmetry is broken because this mean value gives mass to the gauge boson v_m , which absorbs the derivative of the phase of the left-handed selectron field as its longitudinal component. But the overall phase of the left-handed selectron field is still arbitrary; we now choose it so that its mean value in the vacuum is non-negative:

$$h = \langle \Omega | \tilde{e}_L | \Omega \rangle \geq 0. \quad (404)$$

For $\xi < 0$ one may use these formulae provided one interchanges 1 with 2 and replaces ξ by $|\xi|$. In these vacua the mean value of the potential is

$$\langle V \rangle = \frac{m^2 |\xi|}{q} - \frac{m^4}{2q^2} > 0, \quad (405)$$

and so both gauge symmetry and supersymmetry are broken.

11.5 The Goldstino

When supersymmetry is spontaneously broken, some of the fermions acquire masses along with some of the gauge bosons, but at least one of the fermions remains massless. This massless fermion is called the goldstino. We may illustrate this effect by computing the tree-level masses of the various particles of super QED for $\xi > 0$ and $m > 0$.

In the case $m^2 > \xi q > 0$, the mean values of the selectron fields in the vacuum vanish. Thus the gauge symmetry is unbroken, and the gauge boson v_m remains massless. Also there is no mixing between the electrons e_L and e_R and the photino λ , and so the photino remains massless: it is the goldstino. The right- and left-handed electron fields form a Dirac electron of mass

$$m_e = m. \quad (406)$$

The scalar potential V contains the mass terms

$$(m^2 + \xi q) |\tilde{e}_R|^2 + (m^2 - \xi q) |\tilde{e}_L|^2. \quad (407)$$

Because in (398) the kinetic action of the complex selectron fields contains no prefactor of $1/2$, we may identify the masses of the selectrons as

$$m_{\tilde{e}_R}^2 = m^2 + \xi q \quad (408)$$

and

$$m_{\tilde{e}_L}^2 = m^2 - \xi q. \quad (409)$$

In the case $\xi q > m^2 > 0$, both gauge symmetry and supersymmetry are broken. The simplest mass to identify is that of the gauge boson v_m . In the action density (398), the kinetic term of the left-handed selectron, $-\overline{\mathcal{D}_m \tilde{e}_L} \mathcal{D}^m \tilde{e}_L$, contains the mass term of the gauge boson

$$-q^2 h^2 v_m v^m = -\frac{1}{2} m_v^2 v_m v^m \quad (410)$$

which arises from the mean value

$$h = \langle \Omega | \tilde{e}_L | \Omega \rangle = \sqrt{\frac{\xi}{q} - \frac{m^2}{q^2}} \geq 0 \quad (411)$$

of the left-handed selectron which is given by (403) and which we have chosen to be real and non-negative (404). Thus the mass squared of the gauge boson v_m is

$$m_v^2 = 2q^2 h^2 = 2q\xi - 2m^2, \quad (412)$$

the $U(1)$ gauge symmetry is broken, and charge is not conserved.

To find the masses of the selectrons, we write the left-handed selectron as

$$\tilde{e}_L = h + r \quad (413)$$

in which the field r is hermitian. Then by expressing formula (399) for the potential V in terms of r , h , and \tilde{e}_R , we see that the terms linear in r cancel by (403), and that V contains the mass terms

$$2m^2 |\tilde{e}_R|^2 + 2q^2 h^2 r^2 = m_{\tilde{e}_R}^2 |\tilde{e}_R|^2 + m_r^2 r^2. \quad (414)$$

Because in (398) the kinetic action of the complex selectron fields contains no prefactor of 1/2, there is none in this equation. Thus the mass squared of the right-handed selectron is

$$m_{\bar{e}_R}^2 = 2m^2, \quad (415)$$

and that of the radial part r of the left-handed selectron is

$$m_r^2 = 2q^2 h^2 = 2q\xi - 2m^2 = m_v^2. \quad (416)$$

To find the masses of the fermions of this model, we write down their linearized equations of motion by differentiating the action density:

$$\begin{aligned} i\bar{\sigma}^m \partial_m \lambda &= i\sqrt{2}qh \bar{e}_L \\ i\bar{\sigma}^m \partial_m e_R &= -m \bar{e}_L \\ i\bar{\sigma}^m \partial_m e_L &= i\sqrt{2}qh \bar{\lambda} - m \bar{e}_R \\ i\sigma^m \partial_m \bar{\lambda} &= -i\sqrt{2}qh e_L \\ i\sigma^m \partial_m \bar{e}_R &= -m e_L \\ i\sigma^m \partial_m \bar{e}_L &= -i\sqrt{2}qh \lambda - m e_R. \end{aligned} \quad (417)$$

The Pauli identity (41) implies that the product of the two differential operators in the above sextet of equations is the operator of Le Rond d'Alembert

$$i\sigma^m \partial_m i\bar{\sigma}^n \partial_n = \eta^{mn} \partial_m \partial_n = \square. \quad (418)$$

Thus the second-order linearized field equations are:

$$\begin{aligned} \square \lambda &= 2q^2 h^2 \lambda - i\sqrt{2}qhm e_R = m_v^2 \lambda - im_v m e_R \\ \square e_R &= i\sqrt{2}qhm \lambda + m^2 e_R = im_v m \lambda + m^2 e_R \\ \square e_L &= (m^2 + 2q^2 h^2) e_L = (m^2 + m_v^2) e_L. \end{aligned} \quad (419)$$

It is clear that the squared mass of e_L is

$$m_{e_L}^2 = m^2 + m_v^2. \quad (420)$$

The eigenvalues of the matrix of squared masses

$$\begin{pmatrix} m_v^2 & -im_v m \\ im_v m & m^2 \end{pmatrix} \quad (421)$$

are 0 and

$$m_{e'}^2 = m^2 + m_v^2 = m_{e_L}^2. \quad (422)$$

The massive eigenvector is the field e'_R

$$e'_R = \frac{m_v \lambda + i m e_R}{m_{e'}} \quad (423)$$

which with the field $\bar{e}'_L = \bar{e}_L$ forms a Dirac spinor e' of mass $m_{e'}$. It is interesting to note that although the coupling of the left-handed electron to the massive photon is $-q$, that of the right-handed electron is less

$$-q_R = -q \frac{m}{m_{e'}} = \frac{-q}{\sqrt{1 + \frac{m_v^2}{m^2}}} = -q \frac{m}{\sqrt{2\xi q - m^2}} \quad (424)$$

and small if $\xi q \gg m^2$.

The massless eigenvector is the Goldstone spinor or goldstino

$$\psi = \frac{i m \lambda + m_v e_R}{m_{e'}}. \quad (425)$$

Let us add up the squared masses of the bosons and subtract the squared masses of the fermions; we find

$$\begin{aligned} \sum_i (-1)^{2j_i} (2j_i + 1) m_i^2 &= 2m_{\bar{e}_R}^2 + m_r^2 - 4m_{e'}^2 + 3m_v^2 \\ &= 4m^2 + m_v^2 - 4(m^2 + m_v^2) + 3m_v^2 \\ &= 0, \end{aligned} \quad (426)$$

which is an example of the remarkable formula of S. Ferrara, L. Girardello, and F. Palumbo [7].

The previous case in which $m^2 > \xi q > 0$ provides a second example of the vanishing of the super-trace of the squared masses. In this case the masses are: $m_v = m_\lambda = 0$, $m_e = m$, $m_{\bar{e}_R}^2 = m^2 + \xi q$, and $m_{\bar{e}_L}^2 = m^2 - \xi q$. Thus the supertrace again vanishes:

$$\begin{aligned} \sum_i (-1)^{2j_i} (2j_i + 1) m_i^2 &= 2m_{\bar{e}_R}^2 + 2m_{\bar{e}_L}^2 - 4m_e^2 \\ &= 2(m^2 + \xi q) + 2(m^2 - \xi q) - 4m^2 \\ &= 0. \end{aligned} \quad (427)$$

12 Non-Abelian Susy Gauge Theories

12.1 The Non-Abelian Vector Multiplet

For an arbitrary non-abelian gauge group, the action density of the supersymmetric pure gauge theory is

$$\mathcal{L} = -\frac{1}{4}v_{mn}^a v_a^{mn} - i\bar{\lambda}^a \bar{\sigma}^m \mathcal{D}_m^{ab} \lambda^b + \frac{1}{2}D^a D^a \quad (428)$$

in which the indices a, b , and c here represent Yang-Mills indices in the adjoint representation of the gauge group. The term “pure” means that the gauge fields v_m^a and λ^a are not coupled to “matter” fields, and that there are no matter fields in the theory. The Yang-Mills field strength v_{mn}^a is

$$v_{mn}^a = \partial_m v_n^a - \partial_n v_m^a + gt^{abc} v_m^b v_n^c, \quad (429)$$

and the covariant derivative of the gaugino field λ^a is

$$\mathcal{D}_m^{ab} \lambda^b = \partial_m \delta_{ab} \lambda^b + gt^{acb} v_m^c \lambda^b, \quad (430)$$

in which the real numbers t^{abc} are the structure constants of the gauge group and therefore also the generators of the adjoint representation.

The action density (428) changes by at most a total derivative under the augmented supersymmetry transformation

$$\delta v_m^a = -i\bar{\lambda}^a \bar{\sigma}^m \xi + i\bar{\xi} \bar{\sigma}^m \lambda^a \quad (431)$$

$$\delta \lambda^a = \sigma^{mn} \xi v_{mn}^a + i\xi D^a \quad (432)$$

$$\delta D^a = -\xi \sigma^m \mathcal{D}_m^{ab} \bar{\lambda}^b - \mathcal{D}_m^{ab} \lambda^b \sigma^m \bar{\xi}. \quad (433)$$

The constraints imposed on the auxiliary fields D^a by the action density (428) are

$$D^a = 0. \quad (434)$$

12.2 Adding Matter Multiplets

One may add to the action density (428) terms for a right-handed chiral multiplet

$$\mathcal{L}_R = -\overline{\mathcal{D}_m^{ij} A_j} \mathcal{D}^{mik} A_k - i\bar{\psi}_i \bar{\sigma}^m \mathcal{D}_m^{ij} \psi_j + \bar{F}_i F_i$$

$$-i\sqrt{2}g\left(\bar{A}_i T_{ij}^a \psi_j \lambda^a - \bar{\lambda}^a \bar{\psi}_i T_{ij}^a A_j\right) - gD^a \bar{A}_i T_{ij}^a A_j \quad (435)$$

in which the covariant derivatives are:

$$\mathcal{D}_m^{ij} A_j = \delta_{ij} \partial_m A_j - igv_m^a T_{ij}^a A_j \quad (436)$$

$$\mathcal{D}_m^{ij} \psi_j = \delta_{ij} \partial_m \psi_j - igv_m^a T_{ij}^a \psi_j, \quad (437)$$

and the hermitian matrices T_{ij}^a are the generators of the gauge group in the representation according to which the fields ψ_j and A_i transform.

From these formulae for the right-handed multiplet (A_i, ψ_i, F_i) , we may infer the terms for a left-handed multiplet $(B_i, \bar{\chi}_i, G_i)$. By integrating by parts and using the identity (49), we see that the action density

$$-i\bar{\psi}_i \bar{\sigma}^m (\delta_{ij} \partial_m - igv_m^a T_{ij}^a) \psi_j \quad (438)$$

is equivalent to

$$-i\psi_j \sigma^m (\delta_{ij} \partial_m + igv_m^a T_{ij}^a) \bar{\psi}_i. \quad (439)$$

Thus if we wish to define the covariant derivatives of the fields B_i and $\bar{\chi}_i$ of the left-handed multiplet as

$$\mathcal{D}_m^{ij} B_j = \delta_{ij} \partial_m B_j - igv_m^a T_{ij}^a B_j \quad (440)$$

$$\mathcal{D}_m^{ij} \bar{\chi}_j = \delta_{ij} \partial_m \bar{\chi}_j - igv_m^a T_{ij}^a \bar{\chi}_j \quad (441)$$

which is similar to the definitions (436–437) for the covariant derivatives of the fields of the right-handed multiplet, then we must replace T_{ij}^a by $-T_{ji}^a = -T_{ij}^{a*}$ throughout the terms that refer to the left-handed multiplet $(B, \bar{\chi}, G)$. This is permissible since the generators T^a and $-T^{a*}$ have the same structure constants. The action density for the left-handed multiplet $(B, \bar{\chi}, G)$ is thus:

$$\begin{aligned} \mathcal{L}_L = & -\overline{\mathcal{D}_m^{ij} B_j} \mathcal{D}^{mik} B_k - i\bar{\chi}_i \sigma^m \mathcal{D}_m^{ij} \bar{\chi}_j + \bar{G}_i G_i \\ & -i\sqrt{2}g\left(\bar{\lambda}^a \bar{B}_i T_{ij}^a \bar{\chi}_j - \chi_i T_{ij}^a B_j \lambda^a\right) + gD^a \bar{B}_i T_{ij}^a B_j. \end{aligned} \quad (442)$$

12.3 The General Non-Abelian Gauge Theory

After field shifts to omit the terms linear in the complex scalar fields and after some integrations by parts, the most general renormalizable, supersymmetric,

gauge-invariant Lagrange density is

$$\mathcal{L} = -\frac{1}{4}v_{mn}^a v_a^{mn} \quad (443)$$

$$-i\bar{\lambda}^a \bar{\sigma}^m \mathcal{D}_m^{ab} \lambda^b \quad (444)$$

$$+\frac{1}{2}D^a D^a \quad (445)$$

$$+\xi_a D^a \quad (446)$$

$$-\overline{\mathcal{D}_m^{ij} A_j} \mathcal{D}^{mik} A^k \quad (447)$$

$$-i\bar{\psi}^i \bar{\sigma}^m \mathcal{D}_m^{ij} \psi^j \quad (448)$$

$$+\bar{F}_i F_i \quad (449)$$

$$+\mu_i F_i + \bar{\mu}_i \bar{F}_i \quad (450)$$

$$-i\sqrt{2}g \left(\bar{A}_i T_{ij}^a \psi_j \lambda^a - \bar{\lambda}^a \bar{\psi}_i T_{ij}^a A_j \right) \quad (451)$$

$$-gD^a \bar{A}_i T_{ij}^a A_j \quad (452)$$

$$+m_{ij} \left(A_i F_j - \frac{1}{2}\psi_i \psi_j \right) + \bar{m}_{ij} \left(\bar{A}_i \bar{F}_j - \frac{1}{2}\bar{\psi}_i \bar{\psi}_j \right) \quad (453)$$

$$+g_{ijk} (F_i A_j A_k - \psi_i \psi_j A_k) + \bar{g}_{ijk} (\bar{F}_i \bar{A}_j \bar{A}_k - \bar{\psi}_i \bar{\psi}_j \bar{A}_k), \quad (454)$$

in which the indices a, b , and c here are Yang-Mills indices in the adjoint representation of the gauge group and the indices i, j , and k here are Yang-Mills indices in an arbitrary representation of the gauge group. In the Fayet-Iliopoulos term (446), the sum must be restricted to those generators of the gauge group that commute with all the generators of the gauge group. (Such generators are said to form invariant abelian subalgebras.) In the O’Raifeartaigh term (450), the sum must be restricted to those chiral fields that are singlets under all gauge transformations, *i.e.*, to fields that are completely neutral and interact only with gravity or with other fields not in the present model. The symmetric tensors m_{ij} and g_{ijk} must be invariant under the action of the gauge group.

The Yang-Mills terms are in vector notation:

$$\mathcal{D}_m A = \partial_m A - igv_m^a T^a A \quad (455)$$

$$\mathcal{D}_m \psi = \partial_m \psi - igv_m^a T^a \psi \quad (456)$$

$$\mathcal{D}_m^{ab} \lambda^b = \partial_m \lambda^a + gt^{abc} v_m^b \lambda^c \quad (457)$$

$$v_{mn}^a = \partial_m v_n^a - \partial_n v_m^a + gt^{abc} v_m^b v_n^c \quad (458)$$

in which v_m^a is the gauge field, λ^a is the gaugino field, ψ is the chiral Fermi field, A is the chiral scalar field, T^a is a generator of the gauge group in the arbitrary representation, the t^{abc} are the structure constants of the gauge group (and therefore also the generators of the adjoint representation), and g is the coupling constant of the gauge group, assumed here to be simple. If the gauge group is not simple, then it can have a different coupling constant for each invariant subalgebra. Thus if the group is $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$, then instead of (455), for instance, one might write

$$(\mathcal{D}_m A)_i = \partial_m A_i - ig_3 \sum_{a=1}^8 v_m^a (T_3^a)_{ij} A_j - ig_2 \sum_{b=1}^3 v_m^b (T_2^b)_{ij} A_j - ig_1 v_m^Y (T_1^Y)_{ij} A_j. \quad (459)$$

For the case of the up and down squark fields, for example, the matrix T_3^8 would be

$$T_3^8 = \frac{1}{2} \begin{pmatrix} \lambda_8 & 0 \\ 0 & \lambda_8 \end{pmatrix}, \quad (460)$$

in which λ_8 is the eighth 3×3 Gell-Mann matrix and 0 is the 3×3 matrix of zeros; and the matrix T_2^1 would be

$$T_2^1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad (461)$$

The first term (443) is the gauge-field action. The second term (444) is the gauged kinetic action of the gauginos. The third term (445) is the auxiliary fields of the vector multiplet. The fourth term (447) is the gauged kinetic action of the scalar fields (squarks and sleptons and Higgs). The fifth term (448) is the gauged kinetic action of the chiral Fermi fields (quarks and leptons and higgsinos). The sixth term (449) is the auxiliary fields of the chiral multiplet. The seventh term (451) is a Yukawa interaction of the chiral scalar, chiral Fermi, and gaugino fields. The eighth term (452) couples the auxiliary fields of the vector multiplet with the chiral scalar fields. The ninth term (453) is a hermitian combination of mass terms that couple the chiral scalar and auxiliary fields and the chiral Fermi fields. The last term (454) is a hermitian combination of Yukawa interactions among the chiral scalar and auxiliary fields and the chiral Fermi and scalar fields.

Under the augmented supersymmetry transformation

$$\delta A^i = \sqrt{2}\xi\psi^i \quad (462)$$

$$\delta\psi^i = i\sqrt{2}\sigma^m\bar{\xi}\mathcal{D}_m^{ij}A^j + \sqrt{2}\xi F^i \quad (463)$$

$$\delta F^i = i\sqrt{2}\bar{\xi}\bar{\sigma}^m\mathcal{D}_m^{ij}\psi^j - 2igT_{ij}^a A^j \bar{\xi}\bar{\lambda}^a \quad (464)$$

$$\delta v_m^a = -i\bar{\lambda}^a\bar{\sigma}^m\xi + i\bar{\xi}\bar{\sigma}^m\lambda^a \quad (465)$$

$$\delta\lambda^a = \sigma^{mn}\xi v_{mn}^a + i\xi D^a \quad (466)$$

$$\delta D^a = -\xi\sigma^m\mathcal{D}_m^{ab}\bar{\lambda}^b - \mathcal{D}_m^{ab}\lambda^b\sigma^m\bar{\xi}, \quad (467)$$

the Lagrange density (443–454) changes only by a total derivative. The Yang-Mills group indices a, b, c, i, j, k are placed where they fit, and no distinction is made between raised and lowered Yang-Mills indices.

The constraints are

$$0 = \frac{\partial\mathcal{L}}{\partial D^a} = \xi_a + D^a - \sum g\bar{A}_i T_{ij}^a A_j \quad (468)$$

in which the sum is over all the matter multiplets that couple to D^a as well as over i and j ,

$$0 = \frac{\partial\mathcal{L}}{\partial F_i} = \mu_i + \bar{F}_i + m_{ij}A_j + g_{ijk}A_j A_k, \quad (469)$$

and

$$0 = \frac{\partial\mathcal{L}}{\partial \bar{F}_i} = \bar{\mu}_i + F_i + \bar{m}_{ij}\bar{A}_j + \bar{g}_{ijk}\bar{A}_j\bar{A}_k. \quad (470)$$

By implementing these constraints, we may remove the auxiliary fields D^a , F_i , and \bar{F}_i from the action density (443–454). The resulting expression is the most general supersymmetric, gauge-invariant Lagrange density into which we must fit the minimal supersymmetric standard model apart from terms that explicitly break supersymmetry:

$$\mathcal{L} = -\frac{1}{4}v_{mn}^a v_a^{mn} - i\bar{\lambda}^a\bar{\sigma}^m\mathcal{D}_m^{ab}\lambda^b - \overline{\mathcal{D}_m^{ij}A^j} \mathcal{D}^{mik}A^k - i\bar{\psi}^i\bar{\sigma}^m\mathcal{D}_m^{ij}\psi^j \quad (471)$$

$$- \frac{1}{2}\left(\xi_a - \sum g\bar{A}_i T_{ij}^a A_j\right)^2 - i\sqrt{2}g\left(\bar{A}_i T_{ij}^a \psi_j \lambda^a - \bar{\lambda}^a \bar{\psi}_i T_{ij}^a A_j\right) \quad (472)$$

$$- \frac{1}{2}m_{ij}\psi_i\psi_j - \frac{1}{2}\bar{m}_{ij}\bar{\psi}_i\bar{\psi}_j - g_{ijk}\psi_i\psi_j A_k - \bar{g}_{ijk}\bar{\psi}_i\bar{\psi}_j\bar{A}_k \quad (473)$$

$$- |\mu_i + m_{ij}A_j + g_{ijk}A_j A_k|^2. \quad (474)$$

The first line (471) consists of the kinetic action densities of the gauge, gaugino, and matter fields. These terms will be present in any SSM. The only arbitrariness in them is the choice of the gauge group and of the representations into which we fit the known fields. In the MSSM, the gauge group is $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$, and the particles are the usual three generations of 15 suspects, plus the gauge fields, plus two Higgs doublets, plus all the superpartners. The second line (472) contains the quartic and Yukawa interaction terms that are required by susy and by the first line (471), as well as the Fayet-Iliopoulos term (proportional to ξ_a), which is absent when the gauge group is semi-simple (*i.e.*, when its Lie group has no invariant abelian subalgebras). The third line (473) contains completely arbitrary fermion mass terms and Yukawa interactions. The fourth line (474) consists of mass terms for the scalar matter fields and of cubic and quartic self-interactions among these fields; these terms are dependent upon the preceding line (473) which is itself wholly arbitrary, and upon the O’Raifeartaigh term (proportional to μ_i), which occurs only when there is a chiral field that is completely neutral. Yet the gauge group of the standard model is not semi-simple; and if neutrinos are Dirac fermions, then the right-handed neutrino is a chiral field that is completely neutral.

One of the principal benefits of supersymmetry is the cancellation of quartic and quadratic divergences. The most obvious of the obviated divergences are the first two terms of the zero-point energy

$$E_0 = \sum_i (-1)^{2j} (2j + 1) \int d^3k \frac{1}{2} \sqrt{k^2 + m_i^2} \quad (475)$$

in which m_i is the mass of particle i of spin j . If

$$\sum_i (-1)^{2j} (2j + 1) = 0, \quad (476)$$

then the quartic divergence cancels; if

$$\sum_i (-1)^{2j} (2j + 1) m_i^2 = 0, \quad (477)$$

then the quadratic divergence also cancels. When supersymmetry is a symmetry of the action but not a symmetry of the vacuum, the cancellations (476) and (477) in general persist at least at tree level as shown by Ferrara, Girardello, and Palumbo [7].

Other sum rules similar to (477) but specific to a single chiral supermultiplet also hold when supersymmetry is broken spontaneously. These more specific sum rules may be incompatible with observed masses, and theories that evade them — such as ones with a Fayet–Iliopoulos D -term, *e.g.* (374), — often have their own problems. But by introducing mirror fermions, so that every left-handed fermion has a right-handed partner with the same behavior under gauge transformations, one may be able to construct a theory that respects the sum rules (476) and (477) and still has an acceptable phenomenology.

Most phenomenologists, however, have resorted to the use of terms that explicitly break supersymmetry but in ways that do not give rise to new, field-dependent quadratic divergences. The possible terms that can be added to break supersymmetry softly are Majorana mass terms for the gauginos

$$\delta\mathcal{L}_m = -\frac{1}{2}m\lambda\lambda - \frac{1}{2}\bar{m}\bar{\lambda}\bar{\lambda}, \quad (478)$$

cubic analytic polynomials in the scalar fields

$$\delta\mathcal{L}_A = c_0 + c_i A_i + c_{ij} A_i A_j + c_{ijk} A_i A_j A_k + \bar{c}_0 + \bar{c}_i \bar{A}_i + \bar{c}_{ij} \bar{A}_i \bar{A}_j + \bar{c}_{ijk} \bar{A}_i \bar{A}_j \bar{A}_k, \quad (479)$$

and quadratic forms in the scalar fields

$$\delta\mathcal{L}_{A^*A} = -m_{ij} \bar{A}_i A_j. \quad (480)$$

Such terms can arise in the low-energy limit of a theory in which local supersymmetry, *i.e.*, supergravity, is spontaneously broken at high energy.

Some dimension-three terms do generate awkward quadratic divergences: explicit mass terms for matter fermions and cubic forms that mix scalar fields and their hermitian conjugates are not soft [8].

It should be noted, however, that the explicit breaking of supersymmetry entails a quadratic divergence in the zero-point energy.

13 Superfield Notation

13.1 Superfields

Superfields are functions of space-time and Grassmann coordinates $x, \theta, \bar{\theta}$. They provide an efficient way of finding action densities and of computing Feynman diagrams. By using superfield notation, one may symbolize compactly the structural parts of the action densities that were described in the long formulas of the preceding section. But the superfield formalism is a very technical subject, and so I shall focus on its use as notation.

Because Grassmann coordinates are anti-commuting numbers, a general superfield $F(x, \theta, \bar{\theta})$ may be expressed as a polynomial in the Grassmann coordinates with highest term $\theta\theta\bar{\theta}\bar{\theta}$

$$\begin{aligned} F(x, \theta, \bar{\theta}) = & f(x) + \theta \phi(x) + \bar{\theta} \bar{\chi}(x) \\ & + \theta\theta m(x) + \bar{\theta}\bar{\theta} n(x) + \theta\sigma^m\bar{\theta} v_m(x) \\ & + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta} d(x). \end{aligned} \quad (481)$$

This long, reducible expression involves nine different fields.

13.2 Chiral Superfields

A simpler superfield is the right-handed chiral superfield Φ which is a function of $x^m + i\theta\sigma^m\bar{\theta}$ and θ

$$\begin{aligned} \Phi(x^m + i\theta\sigma^m\bar{\theta}, \theta) = & A(x) + i\theta\sigma^m\bar{\theta} \partial_m A(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta} \square A(x) \\ & + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta \partial_m \psi(x) \sigma^m \bar{\theta} + \theta\theta F(x). \end{aligned} \quad (482)$$

We recognize the fields of the right-handed chiral multiplet (128). In fact if we let $y^m = x^m + i\theta\sigma^m\bar{\theta}$, then we may write the superfield Φ in the form

$$\Phi(y, \theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \quad (483)$$

in which the argument of the auxiliary field F could as well be x .

The adjoint superfield Φ^\dagger is a left-handed superfield which is a function of $x^m - i\theta\sigma^m\bar{\theta}$ and $\bar{\theta}$

$$\begin{aligned}\Phi^\dagger(x^m - i\theta\sigma^m\bar{\theta}, \bar{\theta}) &= A^\dagger(x) - i\theta\sigma^m\bar{\theta}\partial_m A^\dagger(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square A^\dagger(x) \\ &\quad + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^m\partial_m\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^\dagger(x).\end{aligned}\quad (484)$$

With $z^m = x^m - i\theta\sigma^m\bar{\theta}$, the adjoint superfield Φ^\dagger may be written as

$$\Phi^\dagger(z, \bar{\theta}) = A^\dagger(z) + \sqrt{2}\bar{\theta}\bar{\psi}(z) + \bar{\theta}\bar{\theta}F^\dagger(z).\quad (485)$$

One of the neat things about superfields is that the supersymmetry transformations (133–138), *etc.*, can be thought of as translations in superspace

$$\begin{aligned}x'^m &= x^m + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta} \\ \theta' &= \theta + \xi \\ \bar{\theta}' &= \bar{\theta} + \bar{\xi}.\end{aligned}\quad (486)$$

For instance after this translation, the part of Φ that is independent of θ and of $\bar{\theta}$ is

$$A(x) + \delta A(x) = A(x) + \sqrt{2}\xi\psi(x);\quad (487)$$

the part that depends linearly upon θ is

$$\sqrt{2}\theta(\psi(x) + \delta\psi(x)) = \sqrt{2}\theta\left(\psi(x) + i\sqrt{2}\sigma^m\bar{\xi}\partial_m A(x) + \sqrt{2}\xi F(x)\right),\quad (488)$$

in which the second term of $\delta\psi$ has arisen both from the translation of the argument x of $A(x)$ and from the translation of $\bar{\theta}$; and the part that depends quadratically upon θ is

$$\theta\theta(F(x) + \delta F(x)) = \theta\theta\left(F(x) + i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m\psi\right),\quad (489)$$

in which the change $\delta F(x)$ has arisen both from the translation of the argument x of $\psi(x)$ and from the translation of $\bar{\theta}$.

Under a susy transformation, the auxiliary field $F(x)$ of a chiral superfield changes only by a total derivative, and its space-time integral is a susy invariant. Such a term, stripped of Grassmann variables, may therefore be used as an invariant part of a supersymmetric action density. Many susy invariants are of this form.

The product of two right-handed chiral superfields

$$\begin{aligned}\Phi_1(y, \theta)\Phi_2(y, \theta) &= \left[A_1(y) + \sqrt{2}\theta\psi_1(y) + \theta\theta F_1(y) \right] \\ &\times \left[A_2(y) + \sqrt{2}\theta\psi_2(y) + \theta\theta F_2(y) \right]\end{aligned}\quad (490)$$

is a right-handed chiral superfield:

$$\begin{aligned}\Phi_1(y, \theta)\Phi_2(y, \theta) &= A_1(y)A_2(y) + \sqrt{2}\theta [A_1(y)\psi_2(y) + A_2(y)\psi_1(y)] \\ &+ \theta\theta [A_1(y)F_2(y) + A_2(y)F_1(y) - \psi_1(y)\psi_2(y)]\end{aligned}\quad (491)$$

in which the identity (31) was used to write $2\theta\psi_1\theta\psi_2$ as $-\theta\theta\psi_1\psi_2$. Similarly the product of two left-handed chiral superfields

$$\begin{aligned}\Phi_1^\dagger(z, \bar{\theta})\Phi_2^\dagger(z, \bar{\theta}) &= \left[A_1^\dagger(z) + \sqrt{2}\bar{\theta}\psi_1^\dagger(z) + \bar{\theta}\bar{\theta}F_1^\dagger(z) \right] \\ &\times \left[A_2^\dagger(z) + \sqrt{2}\bar{\theta}\psi_2^\dagger(z) + \bar{\theta}\bar{\theta}F_2^\dagger(z) \right]\end{aligned}\quad (492)$$

is a left-handed chiral superfield:

$$\begin{aligned}\Phi_1^\dagger(z, \bar{\theta})\Phi_2^\dagger(z, \bar{\theta}) &= A_1^\dagger(z)A_2^\dagger(z) + \sqrt{2}\bar{\theta} [A_1^\dagger(z)\psi_2^\dagger(z) + A_2^\dagger(z)\psi_1^\dagger(z)] \\ &+ \bar{\theta}\bar{\theta} [A_1^\dagger(z)F_2^\dagger(z) + A_2^\dagger(z)F_1^\dagger(z) - \psi_1^\dagger(z)\psi_2^\dagger(z)].\end{aligned}\quad (493)$$

Thus under a supersymmetry transformation, the $\theta\theta$ component of the product of two chiral superfields

$$\Phi_i\Phi_j|_{\theta\theta} = A_iF_j + A_jF_i - \psi_i\psi_j = A_iF_j + A_jF_i - \frac{1}{2}(\psi_i\psi_j + \psi_j\psi_i)\quad (494)$$

changes only by a total derivative, and its space-time integral therefore is a susy invariant, one encountered earlier (453). Similarly the $\theta\theta$ component of the product of three chiral superfields is a susy invariant:

$$\Phi_i\Phi_j\Phi_k|_{\theta\theta} = F_iA_jA_k + F_jA_kA_i + F_kA_iA_j - \psi_i\psi_jA_k - \psi_j\psi_kA_i - \psi_k\psi_iA_j, \quad (495)$$

which occurs in (454). Such $\theta\theta$ components sometimes are called ‘‘F terms’’ because the $\theta\theta$ component of the superfield Φ is F , as shown by (482).

Chiral superfields commute, and so the expressions (494) and (495) for $\Phi_i\Phi_j$ and $\Phi_i\Phi_j\Phi_k$ are symmetric in i, j and in i, j, k , respectively. This symmetry is the reason why the matrices m and g the superpotential (207) are

symmetric. It follows that combinations like $\varepsilon^{ij}\Phi_i\Phi_j$ vanish. But one may construct “anti-symmetric” forms like

$$\varepsilon^{ij}\Phi_i\Psi_j = \Phi_1\Psi_2 - \Phi_2\Psi_1 \quad (496)$$

that do not vanish. At the risk of overemphasizing this point, which I find confusing, let us distinguish the two superfields Φ and Ψ by indices and write the preceding expression as

$$\varepsilon^{ij}\Phi_i\Psi_j = \varepsilon^{ij}\Phi_{1i}\Phi_{2j}. \quad (497)$$

Then the AF part of this “anti-symmetric” form is

$$A_{11}F_{22} + F_{11}A_{22} - A_{12}F_{21} - F_{12}A_{21} = A_{11}F_{22} + A_{22}F_{11} - A_{12}F_{21} - A_{21}F_{12} \quad (498)$$

which is anti-symmetric under the interchange of the first indices and of the second indices but is symmetric under the interchange of the double indices.

The kinetic part of the chiral action density (209) apart from total derivatives is the $\theta\theta\bar{\theta}\bar{\theta}$ component of the product $\Phi_i^\dagger\Phi_i$

$$\Phi_i^\dagger\Phi_i|_{\theta\theta\bar{\theta}\bar{\theta}} = \frac{i}{2}\partial_n\bar{\psi}_i\bar{\sigma}^n\psi_i - \frac{i}{2}\bar{\psi}_i\bar{\sigma}^n\partial_n\psi_i - \partial_n\bar{A}_i\partial^n A_i + \bar{F}_i F_i. \quad (499)$$

Such $\theta\theta\bar{\theta}\bar{\theta}$ components often are called “D terms” because the $\theta\theta\bar{\theta}\bar{\theta}$ component of the vector superfield V in the Wess-Zumino gauge is D , as shown by (500).

13.3 Vector Superfields

In what is called the Wess-Zumino gauge, the vector superfield is

$$V = \theta\sigma^m\bar{\theta}v_m(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x), \quad (500)$$

in which both v_m and D are hermitian fields so that the superfield V is also hermitian.

Superfield notation is less transparent for gauge fields than for chiral fields. One first defines the differential operators

$$D_a = \frac{\partial}{\partial\theta^a} + i\sigma_{a\dot{a}}^m\bar{\theta}^{\dot{a}}\frac{\partial}{\partial x^m} \quad \text{and} \quad \bar{D}_{\dot{a}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{a}}} - i\theta^a\sigma_{a\dot{a}}^m\frac{\partial}{\partial x^m}, \quad (501)$$

in which the differentiation with respect to θ^a is from the left, *e.g.*,

$$\frac{\partial}{\partial\theta^a}\theta\theta = \frac{\partial}{\partial\theta^a}\theta^b\theta_b = \theta_a. \quad (502)$$

In terms of the real superfield V , one then defines the chiral superfields

$$W_a = -\frac{1}{4}\bar{D}\bar{D}D_aV \quad \text{and} \quad \bar{W}_{\dot{a}} = -\frac{1}{4}DD\bar{D}_{\dot{a}}V. \quad (503)$$

The $\theta\theta$ component of WW is

$$WW|_{\theta\theta} = -\frac{1}{2}v^{mn}v_{mn} - 2i\lambda\sigma^m\partial_m\bar{\lambda} + D^2 + \frac{i}{4}v^{mn}v^{lk}\varepsilon_{mnlk}, \quad (504)$$

in which D is the auxiliary field, not the differential operator, and ε_{mnlk} is the totally anti-symmetric tensor in four indices with $\varepsilon_{0123} = -1$. The meager notational payoff for these spectacularly elaborate definitions is that we may now write the abelian action density (366) in the form

$$\frac{1}{4}\left(W^aW_a|_{\theta\theta} + \bar{W}_{\dot{a}}\bar{W}^{\dot{a}}|_{\bar{\theta}\bar{\theta}}\right) = -\frac{1}{4}v^{mn}v_{mn} - i\lambda\sigma^m\partial_m\bar{\lambda} + \frac{1}{2}D^2. \quad (505)$$

14 The Minimal Supersymmetric Standard Model

The gauge group of the standard model is $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ which is spontaneously broken to $SU(3)_c \otimes U(1)_{em}$. There are three families or generations of quarks and leptons, each with 15 or 16 particles depending on whether there are right-handed neutrinos. The first family consists of 6 left-handed quarks which form a triplet under $SU(3)_c$ and a doublet under $SU(2)_L$ and carry hypercharge $Y = 1/3$, quantum numbers which may be written as $(3, 2, 1/3)$. The hypercharge Y is chosen so as to satisfy the Gell-Mann–Nishijima relation

$$Q = I_3 + \frac{1}{2}Y. \quad (506)$$

The other 6 first-family quarks form a right-handed color triplet u_R that is a singlet under $SU(2)_L$ and has $Y = 4/3$ and a second right-handed color triplet d_R that is also a singlet under $SU(2)_L$ and has $Y = -2/3$. The leptons

are colorless and form a left-handed doublet under $SU(2)_L$ with $Y = -1$, a right-handed $SU(2)_L$ singlet e_R with $Y = -2$, and possibly a right-handed $SU(2)_L$ singlet ν_R with $Y = 0$. These quantum numbers may be restated as

$$\begin{array}{ccc} \begin{pmatrix} \vec{u}_L \\ \vec{d}_L \end{pmatrix} & \bar{u}_R & \vec{d}_R \\ (3, 2, \frac{1}{3}) & (3, 1, \frac{4}{3}) & (3, 1, -\frac{2}{3}) \end{array} \quad (507)$$

for the quarks and as

$$\begin{array}{ccc} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} & \nu_R & e_R \\ (1, 2, -1) & (1, 1, 0) & (1, 1, -2) \end{array} \quad (508)$$

for the leptons.

The standard model agrees with all experiments, except possibly in the neutrino sector where it is easy to add right-handed neutrino spinors to make mass terms. So in building the minimal supersymmetric standard model (MSSM), one tries to rock this peaceful boat as little as possible. It would be nice if we could put the known particles into supersymmetric multiplets without adding any super particles. Alas, exactly the opposite is true: nobody has identified a single pair of particles that are exchanged by supersymmetry. So supersymmetry doubles the number of particles, as well as the number of typos. Thus to every particle, we attach a super particle. For instance to the left-handed quark field q_L we associate a “left-handed” squark field \tilde{q}_L , and to the right-handed quark field d_R , we associate a “right-handed” field \tilde{d}_R . Actually the situation is slightly worse: we must add a second Higgs doublet and its super partners.

14.1 Why Two Higgs Superfields?

There are three reasons why one must have two Higgs doublets and the spartners. The simplest reason is that the spartner of the Higgs doublet (h^+, h^0) of the standard model is a doublet of fermions $(\tilde{h}^+, \tilde{h}^0)$. These higgsinos have $Y = 1$ and would generate a $U(1)$ anomaly since the trace over all fermions

would now be non zero,

$$\text{Tr } Y^3 = 2. \quad (509)$$

One may solve this problem by adding a second Higgs superfield with spin-zero fields (h^0, h^-) and higgsinos $(\tilde{h}^0, \tilde{h}^-)$ with $Y = -1$. Now we are back to

$$\text{Tr } Y^3 = 0 \quad (510)$$

as in the standard model.

The second reason why two Higgs fields are needed is to give mass to the down quarks, d , s , and b . The quark mass terms of the standard model are of the form

$$c_u \varepsilon_{ij} q_{Li} h_j u_R^\dagger + c_d h_i^\dagger q_{Li} d_R^\dagger + \text{h.c.}, \quad (511)$$

in which h is the Higgs doublet $(\tilde{h}^+, \tilde{h}^0)$, q_L is the doublet of left-handed quark fields (u_L, d_L) , and the family indices have been suppressed. The matrix ε_{ij} which is proportional to the second Pauli matrix σ_2 ensures that under the $SU(2)_L$ transformation g the vector $\sigma_2 h$ transforms as $(\sigma_2 h)' = \sigma_2 g h = g^{-1\top} \sigma_2 h$ which makes the first term invariant. If we promote all the fields of the first term to superfields, using the conventional notation $H_2 = (H_2^+, H_2^0)$ and $Q_L = (U_L, D_L)$, then there is no problem with the mass term for the up quarks

$$c_u \varepsilon_{ij} Q_{Li} H_{2j} U_R^\dagger \Big|_{\bar{\theta}\bar{\theta}} \quad (512)$$

which is the product of three left-handed chiral superfields. The problem is with the mass term for the down quarks. A product of three superfields is invariant under supersymmetry only if all three superfields are of the same chirality. Now Q_{Li} and D_R^\dagger are left handed, but H_{2i}^\dagger is right handed. So one introduces a second left-handed doublet of Higgs superfields $H_1 = (H_1^0, H_1^-)$ with $Y = -1$ and writes the down-quark mass term as

$$c_d \varepsilon_{ij} Q_{Li} H_{1j} D_R^\dagger \Big|_{\bar{\theta}\bar{\theta}}. \quad (513)$$

The third reason why one needs two Higgs superfield doublets, H_1 and H_2 , is that the electroweak gauge bosons W_i and B of $SU(2)_L \otimes U(1)_Y$ are the spin-one fields of the superfields V_i and V . Three of these four massless gauge bosons become the massive vector bosons W^+ , W^- , and Z by absorbing three massless spin-zero bosons; three of their massless spin-one-half superpartners

\tilde{W}_i and \tilde{B} must therefore become massive by absorbing three massless spin-one-half fields. Were there only one Higgs superfield doublet, there would only be two massless spin-one-half fields as candidates for absorption. With a second Higgs superfield doublet, there is one massless spin-one-half field to spare.

After the spontaneous breakdown of $SU(2)_L \otimes U(1)_Y$ to $U(1)_{EM}$, the physical fields are: the massive charged W^\pm 's, the massive Z , and the massless photon, all vector bosons; three massive Dirac fields \tilde{W}^\pm , \tilde{Z} , and one massless neutral chiral spinor, all spin-one-half fields; and two charged and three neutral bosons, all of spin zero. Of course, these spin-one-half and spin-zero fields may mix with other fields of the MSSM.

14.2 The Electroweak Superpotential

Since the rotation group, $SU(3)_c$, and $U(1)_{EM}$ are unbroken, the fields that might assume non-zero mean values in the vacuum are likely to be the spin-zero, colorless, and neutral. The left-handed purely electroweak chiral superfields of the MSSM are the two Higgs doublets H_1 and H_2 , the lepton doublet $L = (N^0, E^-)$, and the adjoint E_R^\dagger of the superfield of the right-handed electron. In view of the neutrino experiments which are suggestive of neutrino masses, it makes sense to include the adjoint N_R^\dagger of the superfield of the right-handed neutrino. Then avoiding the letter W and again suppressing family indices, we may write the electroweak superpotential as

$$\begin{aligned}
P = & c_1 \varepsilon_{ij} H_{1i} H_{2j} + c_2 \varepsilon_{ij} L_i H_{2j} + c_3 \varepsilon_{ij} L_i H_{1j} E_R^\dagger \\
& + c_4 N_R^\dagger + c_5 N_R^\dagger N_R^\dagger + c_6 N_R^\dagger N_R^\dagger N_R^\dagger \\
& + c_7 \varepsilon_{ij} H_{1i} H_{2j} N_R^\dagger + c_8 \varepsilon_{ij} L_i H_{2j} N_R^\dagger.
\end{aligned} \tag{514}$$

Keeping in mind that $\varepsilon_{12} = -1$, one may expand this gauge-invariant and susy-invariant superpotential in terms of the components of the several doublets

$$\begin{aligned}
P = & c_1 (H_1^- H_2^+ - H_1^0 H_2^0) + c_2 (E_L H_2^+ - N_L H_2^0) \\
& + c_3 (E_L H_1^0 - N_L H_1^-) E_R^\dagger + c_4 N_R^\dagger + c_5 N_R^\dagger N_R^\dagger \\
& + c_6 N_R^\dagger N_R^\dagger N_R^\dagger + c_7 (H_1^- H_2^+ - H_1^0 H_2^0) N_R^\dagger
\end{aligned}$$

$$+ c_8 (E_L H_2^+ - N_L H_2^0) N_R^\dagger. \quad (515)$$

We may add to the action density the $\bar{\theta}\bar{\theta}$ component of P and its hermitian adjoint.

One conventionally distinguishes superpartners by tildes, writing the selectron as \tilde{e} . Let us extend this notation to auxiliary fields, distinguishing them with a grave accent, \grave{e} , so that the superfield of the left-handed electron takes the form

$$E_L = \tilde{e}_L + \sqrt{2}\bar{\theta}e_L + \bar{\theta}\bar{\theta}\grave{e}_L. \quad (516)$$

Then the bosonic part of the $\bar{\theta}\bar{\theta}$ component of P is

$$\begin{aligned} P|_{\bar{\theta}\bar{\theta}b} = & c_1 (h_1^- \grave{h}_2^+ + \grave{h}_1^- h_2^+ - h_1^0 \grave{h}_2^0 - \grave{h}_1^0 h_2^0) \\ & + c_2 (\tilde{e}_L \grave{h}_2^+ + \grave{e}_L h_2^+ - \tilde{\nu}_L \grave{h}_2^0 - \grave{\nu}_L h_2^0) \\ & + c_3 [(\tilde{e}_L \grave{h}_1^0 + \grave{e}_L h_1^0 - \tilde{\nu}_L \grave{h}_1^- - \grave{\nu}_L h_1^-) \tilde{e}_R^\dagger + (\tilde{e}_L h_1^0 - \tilde{\nu}_L h_1^-) \grave{e}_R^\dagger] \\ & + c_4 \grave{\nu}_R^\dagger + 2c_5 \tilde{\nu}_R^\dagger \grave{\nu}_R^\dagger + 3c_6 \tilde{\nu}_R^\dagger \tilde{\nu}_R^\dagger \grave{\nu}_R^\dagger \\ & + c_7 [(h_1^- \grave{h}_2^+ + \grave{h}_1^- h_2^+ - h_1^0 \grave{h}_2^0 - \grave{h}_1^0 h_2^0) \tilde{\nu}_R^\dagger \\ & \quad + (h_1^- h_2^+ - h_1^0 h_2^0) \grave{\nu}_R^\dagger] \\ & + c_8 [(\tilde{e}_L \grave{h}_2^+ + \grave{e}_L h_2^+ - \tilde{\nu}_L \grave{h}_2^0 - \grave{\nu}_L h_2^0) \tilde{\nu}_R^\dagger \\ & \quad + (\tilde{e}_L h_2^+ - \tilde{\nu}_L h_2^0) \grave{\nu}_R^\dagger]. \end{aligned} \quad (517)$$

Let us collect the terms of the action density that involve the auxiliary field $\grave{\nu}_R$ of the right-handed neutrino:

$$\begin{aligned} \mathcal{L}_{\grave{\nu}_R} = & \grave{\nu}_R^\dagger \grave{\nu}_R + c_4 \grave{\nu}_R^\dagger + \bar{c}_4 \grave{\nu}_R + 2c_5 \tilde{\nu}_R^\dagger \grave{\nu}_R^\dagger + 2\bar{c}_5 \tilde{\nu}_R \grave{\nu}_R \\ & + 3c_6 \tilde{\nu}_R^\dagger \tilde{\nu}_R^\dagger \grave{\nu}_R^\dagger + 3\bar{c}_6 \tilde{\nu}_R \tilde{\nu}_R \grave{\nu}_R \\ & + c_7 (h_1^- h_2^+ - h_1^0 h_2^0) \grave{\nu}_R^\dagger + \bar{c}_7 (h_1^+ h_2^- - h_1^{0\dagger} h_2^{0\dagger}) \grave{\nu}_R \\ & + c_8 (\tilde{e}_L h_2^+ - \tilde{\nu}_L h_2^0) \grave{\nu}_R^\dagger + \bar{c}_8 (\tilde{e}_L^\dagger h_2^- - \tilde{\nu}_L^\dagger h_2^{0\dagger}) \grave{\nu}_R. \end{aligned} \quad (518)$$

The constraint on $\grave{\nu}_R$ is

$$\grave{\nu}_R = -c_4 - 2c_5 \tilde{\nu}_R^\dagger - 3c_6 (\tilde{\nu}_R^\dagger)^2 - c_7 (h_1^- h_2^+ - h_1^0 h_2^0) - c_8 (\tilde{e}_L h_2^+ - \tilde{\nu}_L h_2^0). \quad (519)$$

So the action density $\mathcal{L}_{\tilde{\nu}_R}$ becomes

$$\mathcal{L}_{\tilde{\nu}_R} = - \left| c_4 + 2c_5 \tilde{\nu}_R^\dagger + 3c_6 \left(\tilde{\nu}_R^\dagger \right)^2 + c_7 \left(h_1^- h_2^+ - h_1^0 h_2^0 \right) + c_8 \left(\tilde{e}_L h_2^+ - \tilde{\nu}_L h_2^0 \right) \right|^2. \quad (520)$$

The part of the action density that involves the auxiliary field \hat{h}_1^0 is

$$\begin{aligned} \mathcal{L}_{\hat{h}_1^0} &= \hat{h}_1^{0\dagger} \hat{h}_1^0 - c_1 \hat{h}_1^0 h_2^0 - \bar{c}_1 \hat{h}_1^{0\dagger} h_2^{0\dagger} + c_3 \tilde{e}_L \tilde{e}_R^\dagger \hat{h}_1^0 + \bar{c}_3 \tilde{e}_L^\dagger \tilde{e}_R \hat{h}_1^{0\dagger} \\ &\quad - c_7 \hat{h}_1^0 h_2^0 \tilde{\nu}_R^\dagger - \bar{c}_7 \hat{h}_1^{0\dagger} h_2^{0\dagger} \tilde{\nu}_R. \end{aligned} \quad (521)$$

So \hat{h}_1^0 must be

$$\hat{h}_1^0 = \bar{c}_1 h_2^{0\dagger} - \bar{c}_3 \tilde{e}_L^\dagger \tilde{e}_R + \bar{c}_7 h_2^{0\dagger} \tilde{\nu}_R, \quad (522)$$

and $\mathcal{L}_{\hat{h}_1^0}$ becomes

$$\mathcal{L}_{\hat{h}_1^0} = - \left| \bar{c}_1 h_2^{0\dagger} - \bar{c}_3 \tilde{e}_L^\dagger \tilde{e}_R + \bar{c}_7 h_2^{0\dagger} \tilde{\nu}_R \right|^2. \quad (523)$$

The action density for the auxiliary field \hat{h}_2^0 is

$$\begin{aligned} \mathcal{L}_{\hat{h}_2^0} &= \hat{h}_2^{0\dagger} \hat{h}_2^0 - c_1 h_1^0 \hat{h}_2^0 - \bar{c}_1 h_1^{0\dagger} \hat{h}_2^{0\dagger} - c_2 \tilde{\nu}_L \hat{h}_2^0 - \bar{c}_2 \tilde{\nu}_L^\dagger \hat{h}_2^{0\dagger} \\ &\quad - c_7 h_1^0 \hat{h}_2^0 \tilde{\nu}_R^\dagger - \bar{c}_7 h_1^{0\dagger} \hat{h}_2^{0\dagger} \tilde{\nu}_R - c_8 \tilde{\nu}_L \tilde{\nu}_R^\dagger \hat{h}_2^0 - \bar{c}_8 \tilde{\nu}_L^\dagger \tilde{\nu}_R \hat{h}_2^{0\dagger}. \end{aligned} \quad (524)$$

So \hat{h}_2^0 must be

$$\hat{h}_2^0 = \bar{c}_1 h_1^{0\dagger} + \bar{c}_2 \tilde{\nu}_L^\dagger + \bar{c}_7 h_1^{0\dagger} \tilde{\nu}_R + \bar{c}_8 \tilde{\nu}_L^\dagger \tilde{\nu}_R, \quad (525)$$

and $\mathcal{L}_{\hat{h}_2^0}$ must be

$$\mathcal{L}_{\hat{h}_2^0} = - \left| \bar{c}_1 h_1^{0\dagger} + \bar{c}_2 \tilde{\nu}_L^\dagger + \bar{c}_7 h_1^{0\dagger} \tilde{\nu}_R + \bar{c}_8 \tilde{\nu}_L^\dagger \tilde{\nu}_R \right|^2. \quad (526)$$

The action density for the remaining neutral chiral auxiliary field $\hat{\nu}_L$ is

$$\mathcal{L}_{\hat{\nu}_L} = \hat{\nu}_L^\dagger \hat{\nu}_L - c_2 \hat{\nu}_L h_2^0 - \bar{c}_2 \hat{\nu}_L^\dagger h_2^{0\dagger} - c_3 \hat{\nu}_L h_1^- \tilde{e}_R^\dagger - \bar{c}_3 \hat{\nu}_L^\dagger h_1^+ \tilde{e}_R - c_8 \hat{\nu}_L \tilde{\nu}_R^\dagger h_2^0 - \bar{c}_8 \hat{\nu}_L^\dagger \tilde{\nu}_R h_2^{0\dagger}. \quad (527)$$

The constraint on $\hat{\nu}_L$ is

$$\hat{\nu}_L = \bar{c}_2 h_2^{0\dagger} + \bar{c}_3 h_1^+ \tilde{e}_R + \bar{c}_8 \tilde{\nu}_R h_2^{0\dagger}. \quad (528)$$

So the action density $\mathcal{L}_{\tilde{\nu}_L}$ is

$$\mathcal{L}_{\tilde{\nu}_L} = - \left| \bar{c}_2 h_2^{0\dagger} + \bar{c}_3 h_1^+ \tilde{e}_R + \bar{c}_8 \tilde{\nu}_R h_2^{0\dagger} \right|^2. \quad (529)$$

Since charge is conserved, we shall be interested mainly in the purely neutral part of the above contributions to the scalar potential and in other purely neutral terms that arise from the $SU(2)_L$ auxiliary fields \vec{D}

$$\mathcal{L}_{\vec{D}} = -\frac{1}{2} \left(\sum g_2 \bar{A}_i \frac{1}{2} \vec{\sigma}_{ij} A_j \right)^2 \quad (530)$$

and from the $U(1)_Y$ auxiliary field D

$$\mathcal{L}_D = -\frac{1}{2} \left(-\xi + \sum g_1 \frac{1}{2} y_i |A_i|^2 \right)^2. \quad (531)$$

The purely neutral part of $\mathcal{L}_{\vec{D}}$ is

$$\mathcal{L}_{\vec{D}}^0 = -\frac{1}{8} g_2^2 \left(|h_1^0|^2 - |h_2^0|^2 + |\tilde{\nu}_L|^2 \right)^2, \quad (532)$$

while that of \mathcal{L}_D is

$$\mathcal{L}_D^0 = -\frac{1}{2} \left[-\xi + \frac{g_1}{2} \left(-|h_1^0|^2 + |h_2^0|^2 - |\tilde{\nu}_L|^2 \right) \right]^2. \quad (533)$$

If we now gather all these purely neutral terms, then we may write the purely neutral scalar potential as

$$\begin{aligned} V_0 = & \left| c_4 + 2c_5 \tilde{\nu}_R^\dagger + 3c_6 \left(\tilde{\nu}_R^\dagger \right)^2 - c_7 h_1^0 h_2^0 - c_8 \tilde{\nu}_L h_2^0 \right|^2 + |\bar{c}_1 + \bar{c}_7 \tilde{\nu}_R|^2 |h_2^0|^2 \\ & + \left| \bar{c}_1 h_1^{0\dagger} + \bar{c}_2 \tilde{\nu}_L^\dagger + \bar{c}_7 h_1^{0\dagger} \tilde{\nu}_R + \bar{c}_8 \tilde{\nu}_L^\dagger \tilde{\nu}_R \right|^2 + |\bar{c}_2 + \bar{c}_8 \tilde{\nu}_R|^2 |h_2^0|^2 \\ & + \frac{1}{2} \left[\xi + \frac{g_1}{2} \left(|h_1^0|^2 - |h_2^0|^2 + |\tilde{\nu}_L|^2 \right) \right]^2 \\ & + \frac{1}{8} g_2^2 \left(|h_1^0|^2 - |h_2^0|^2 + |\tilde{\nu}_L|^2 \right)^2. \end{aligned} \quad (534)$$

The last two terms in this expression for V_0 are positive and can not both be zero unless $\xi = 0$. Thus if $\xi \neq 0$, then $V_0 > 0$, and supersymmetry is broken spontaneously. It is also clear that if $|\xi|$ is sufficiently large, then the minimum of the potential will be drawn toward

$$|h_1^0|^2 - |h_2^0|^2 + |\tilde{\nu}_L|^2 = -2 \frac{\xi}{g_1}, \quad (535)$$

so that gauge symmetry will also be broken spontaneously.

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