Feynman's Fermion Propagator

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1 The field

We expand a spin-one-half field as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{s=-}^{+} \left[u(p,s) a(p,s) e^{-ipx} + v(p,s) b^\dagger(p,s) e^{ipx} \right]. \tag{1}$$

Because spin-one-half fields anti commute, we define the time-ordered product with an extra minus sign

$$\langle 0|T[\psi(0)\bar{\psi}(x)]|0\rangle = \theta(-t)\langle 0|\psi(0)\bar{\psi}(x)|0\rangle - \theta(t)\langle 0|\bar{\psi}(x)\psi(0)|0\rangle. \tag{2}$$

The first term has

$$\begin{aligned}\langle 0|\psi(0)_j\bar{\psi}_\ell(x)|0\rangle &= \langle 0| \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_q}} \sum_{s,s'=-}^{+} \left[u_j(p,s) a(p,s) + v_j(p,s) b^\dagger(p,s) \right] \\ &\quad \times \left[\bar{u}_\ell(q,s') a^\dagger(q,s') e^{iqx} + \bar{v}_\ell(q,s') b(q,s') e^{-iqx} \right] |0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_q}} \sum_{s,s'=-}^{+} \langle 0| u_j(p,s) a(p,s) \bar{u}_\ell(q,s') a^\dagger(q,s') e^{iqx} |0\rangle\end{aligned} \tag{3}$$

since $\langle 0|b^\dagger(p,s) = 0$ and $b(q,s)|0\rangle = 0$. Using the anticommutation relation

$$\{a(p,s), a^\dagger(q,s')\} = (2\pi)^3 \delta(\vec{p} - \vec{q}) \delta_{s,s'}, \tag{4}$$

we get

$$\langle 0|\psi_j(0)\bar{\psi}_\ell(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\sum_{s=-}^{+} u_j(p,s)\bar{u}_\ell(q,s')e^{iqx}(2\pi)^3\delta(\vec{p}-\vec{q})\delta_{s,s'}$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\sum_{s=-}^{+} u_j(p,s)\bar{u}_\ell(p,s)e^{ipx} = \int \frac{d^3p}{(2\pi)^3}\frac{(\not{p}+m)_{j,\ell}}{2\omega_p}e^{ipx}. \tag{5}$$

The second term is

$$\langle 0|\bar{\psi}_\ell(x)\psi_j(0)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\sum_{s,s'=-}^{+} \langle 0|\left[\bar{u}_\ell(q,s')a^\dagger(q,s')e^{iqx}+\bar{v}_\ell(q,s')b(q,s')e^{-iqx}\right]$$
$$\times\left[u_j(p,s)a(p,s)+v_j(p,s)b^\dagger(p,s)\right]|0\rangle$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\sum_{s,s'=-}^{+} \langle 0|\bar{v}_\ell(q,s')b(q,s')e^{-iqx}v_j(p,s)b^\dagger(p,s)|0\rangle$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\sum_{s=-}^{+} \bar{v}_\ell(p,s)e^{-ipx}v_j(p,s) = \int \frac{d^3p}{(2\pi)^3}\frac{(\not{p}-m)_{j,\ell}}{2\omega_p}e^{-ipx}. \tag{6}$$

2 Contour integrals for Heaviside functions

We will use the integral formulas

$$e^{i\omega_p t}\theta(-t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{ip^0t}\frac{1}{p^0-(\omega_p-i\epsilon)}dp^0$$
$$-e^{-i\omega_p t}\theta(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{ip^0t}\frac{1}{p^0-(-\omega_p+i\epsilon)}dp^0. \tag{7}$$

3 The propagator

We now insert the two terms (5 & 6) into the definition (2) of the fermionic time-ordered product:

$$\langle 0|T[\psi_j(0)\bar{\psi}_\ell(x)]|0\rangle = \theta(-t)\langle 0|\psi_j(0)\bar{\psi}_\ell(x)|0\rangle - \theta(t)\langle 0|\bar{\psi}_\ell(x)\psi_j(0)|0\rangle$$
$$= \theta(-t)\int \frac{d^3p}{(2\pi)^3}\frac{(\not{p}+m)_{j,\ell}}{2\omega_p}e^{ipx} - \theta(t)\int \frac{d^3p}{(2\pi)^3}\frac{(\not{p}-m)_{j,\ell}}{2\omega_p}e^{-ipx} \tag{8}$$

and recognize that in this formula

$$\not{p} = \omega_p \gamma^0 - \vec{p}\cdot\vec{\gamma}. \tag{9}$$

Using this expression, we have

$$\begin{aligned} \langle 0|T[\psi_j(0)\bar{\psi}_\ell(x)]|0\rangle = \theta(-t)\int \frac{d^3p}{(2\pi)^3}\frac{(\omega_p\gamma^0 - \vec{p}\cdot\vec{\gamma}+m)_{j,\ell}}{2\omega_p}e^{i\omega t - i\vec{p}\cdot\vec{x}} \\ - \theta(t)\int \frac{d^3p}{(2\pi)^3}\frac{(\omega_p\gamma^0 - \vec{p}\cdot\vec{\gamma}-m)_{j,\ell}}{2\omega_p}e^{-i\omega t + i\vec{p}\cdot\vec{x}}. \end{aligned} \tag{10}$$

We now use the contour-integral formulas (7) for the Heaviside functions

$$\begin{aligned} \langle 0|T[\psi_j(0)\bar{\psi}_\ell(x)]|0\rangle = &\int \frac{d^3p}{(2\pi)^3}\frac{(\omega_p\gamma^0 - \vec{p}\cdot\vec{\gamma}+m)_{j,\ell}}{2\omega_p}e^{-i\vec{p}\cdot\vec{x}}\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{ip^0t}\frac{1}{p^0-(\omega_p - i\epsilon)}dp^0 \\ &+ \int \frac{d^3p}{(2\pi)^3}\frac{(\omega_p\gamma^0 - \vec{p}\cdot\vec{\gamma}-m)_{j,\ell}}{2\omega_p}e^{i\vec{p}\cdot\vec{x}}\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{ip^0t}\frac{1}{p^0-(-\omega_p + i\epsilon)}dp^0. \end{aligned} \tag{11}$$

In the second integral, we flip the sign of $\vec{p}$

$$\begin{aligned} \langle 0|T[\psi_j(0)\bar{\psi}_\ell(x)]|0\rangle = &\int \frac{d^3p}{(2\pi)^3}\frac{(\omega_p\gamma^0 - \vec{p}\cdot\vec{\gamma}+m)_{j,\ell}}{2\omega_p}e^{-i\vec{p}\cdot\vec{x}}\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{ip^0t}\frac{1}{p^0-(\omega_p - i\epsilon)}dp^0 \\ &+ \int \frac{d^3p}{(2\pi)^3}\frac{(\omega_p\gamma^0 + \vec{p}\cdot\vec{\gamma}-m)_{j,\ell}}{2\omega_p}e^{-i\vec{p}\cdot\vec{x}}\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{ip^0t}\frac{1}{p^0-(-\omega_p + i\epsilon)}dp^0 \end{aligned} \tag{12}$$

and combine the momentum integrations

$$\begin{aligned} \langle 0|T[\psi_j(0)\bar{\psi}_\ell(x)]|0\rangle = &\int \frac{d^4p}{(2\pi)^4}\frac{(\omega_p\gamma^0 - \vec{p}\cdot\vec{\gamma}+m)_{j,\ell}}{2\omega_p}e^{ipx}\frac{i}{p^0-(\omega_p - i\epsilon)} \\ &+ \int \frac{d^4p}{(2\pi)^4}\frac{(\omega_p\gamma^0 + \vec{p}\cdot\vec{\gamma}-m)_{j,\ell}}{2\omega_p}e^{ipx}\frac{i}{p^0-(-\omega_p + i\epsilon)}. \end{aligned} \tag{13}$$

High-school algebra (always the hard part of any calculation) now gives

$$\begin{aligned} \frac{1}{p^0-(\omega_p - i\epsilon)} + \frac{1}{p^0-(-\omega_p + i\epsilon)} &= \frac{p^0-(-\omega_p+i\epsilon)+p^0-(\omega_p-i\epsilon)}{[p^0-(\omega_p-i\epsilon)][p^0-(-\omega_p+i\epsilon)]} \\ &= \frac{2p^0}{(p^0)^2-(\omega_p-i\epsilon)^2} = \frac{2p^0}{p^2-m^2+i\epsilon} \end{aligned} \tag{14}$$

and

$$\frac{1}{p^0 - (\omega_p - i\epsilon)} - \frac{1}{p^0 - (-\omega_p + i\epsilon)} = \frac{p^0 - (-\omega_p + i\epsilon) - p^0 + (\omega_p - i\epsilon)}{[p^0 - (\omega_p - i\epsilon)][p^0 - (-\omega_p + i\epsilon)]}$$
$$= \frac{2\omega_p}{(p^0)^2 - (\omega_p - i\epsilon)^2} = \frac{2\omega_p}{p^2 - m^2 + i\epsilon}. \tag{15}$$

Using the first high-school formula for the $\omega_p \gamma^0$ term and the second for the $-\vec{p}\cdot\vec{\gamma}+m$ term, we may write the propagator as

$$\langle 0|T[\psi_j(0)\bar{\psi}_\ell(x)]|0\rangle = i\int \frac{d^4p}{(2\pi)^4} e^{ipx} \left[\frac{\omega_p\gamma^0}{2\omega_p}\frac{2p^0}{p^2 - m^2 + i\epsilon} + \frac{-\vec{p}\cdot\vec{\gamma} + m}{2\omega_p}\frac{2\omega_p}{p^2 - m^2 + i\epsilon}\right]_{j,\ell}$$
$$= i\int \frac{d^4p}{(2\pi)^4}\, e^{ipx}\, \frac{(p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)_{j,\ell}}{p^2 - m^2 + i\epsilon}$$
$$= i\int \frac{d^4p}{(2\pi)^4}\, e^{ipx}\, \frac{(p_\mu\gamma^\mu + m)_{j,\ell}}{p^2 - m^2 + i\epsilon}$$
$$= i\int \frac{d^4p}{(2\pi)^4}\, e^{ipx}\, \frac{(\not{p} + m)_{j,\ell}}{p^2 - m^2 + i\epsilon}$$
$$= (\partial_\mu\gamma^\mu + im)_{j,\ell}\int \frac{d^4p}{(2\pi)^4}\, e^{ipx}\, \frac{1}{p^2 - m^2 + i\epsilon}. \tag{16}$$

Replacing x by $y - x$ and translating by x gives

$$\langle 0|T[\psi_j(x)\bar{\psi}_\ell(y)]|0\rangle = \int \frac{d^4p}{(2\pi)^4}\, e^{ip(y-x)}\, \frac{i(\not{p} + m)_{j,\ell}}{p^2 - m^2 + i\epsilon}$$
$$= (\partial_\mu\gamma^\mu + im)_{j,\ell}\int \frac{d^4p}{(2\pi)^4}\, e^{ip(y-x)}\, \frac{1}{p^2 - m^2 + i\epsilon}. \tag{17}$$