2.9 Dirac Notation

Vectors $|j\rangle$ that are orthonormal $\langle k|j\rangle = \delta_{k,j}$ span a vector space and express the identity operator I of the space as (1.132)

$$I = \sum_{j=1}^{N} |j\rangle\langle j|. \tag{2.99}$$

Multiplying from the right by any vector $|g\rangle$ in the space, we get

$$|g\rangle = I|g\rangle = \sum_{j=1}^{N} |j\rangle\langle j|g\rangle \tag{2.100}$$

which says that every vector $|g\rangle$ in the space has an expansion (1.133) in terms of the N orthonormal basis vectors $|j\rangle$. The coefficients $\langle j|g\rangle$ of the expansion are inner products of the vector $|g\rangle$ with the basis vectors $|j\rangle$.

These properties of finite-dimensional vector spaces also are true of infinite-dimensional vector spaces of functions. We may use as basis vectors the phases $\exp(inx)/\sqrt{2\pi}$. They are orthonormal with inner product (2.1)

$$(m, n) = \int_0^{2\pi} \left(\frac{e^{imx}}{\sqrt{2\pi}}\right)^* \frac{e^{inx}}{\sqrt{2\pi}}\, dx = \int_0^{2\pi} \frac{e^{i(n-m)x}}{2\pi}\, dx = \delta_{m,n} \tag{2.101}$$

which in Dirac notation with $\langle x|n\rangle = \exp(inx)/\sqrt{2\pi}$ and $\langle m|x\rangle = \langle x|m\rangle^*$ is

$$\langle m|n\rangle = \int_0^{2\pi} \langle m|x\rangle\langle x|n\rangle\, dx = \int_0^{2\pi} \frac{e^{i(n-m)x}}{2\pi}\, dx = \delta_{m,n}. \tag{2.102}$$

The identity operator for Fourier's space of functions is

$$I = \sum_{n=-\infty}^{\infty} |n\rangle\langle n|. \tag{2.103}$$

So we have

$$|f\rangle = I|f\rangle = \sum_{n=-\infty}^{\infty} |n\rangle\langle n|f\rangle \tag{2.104}$$

and

$$\langle x|f\rangle = \langle x|I|f\rangle = \sum_{n=-\infty}^{\infty} \langle x|n\rangle\langle n|f\rangle = \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{\sqrt{2\pi}}\,\langle n|f\rangle \tag{2.105}$$

which with $\langle n|f\rangle = f_n$ is the Fourier series (2.2). The coefficients $\langle n|f\rangle = f_n$

are the inner products (2.3)

$$\langle n|f\rangle = \int_0^{2\pi} \langle n|x\rangle\langle x|f\rangle\,dx = \int_0^{2\pi} \frac{e^{-inx}}{\sqrt{2\pi}}\,\langle x|f\rangle\,dx = \int_0^{2\pi} \frac{e^{-inx}}{\sqrt{2\pi}}\,f(x)\,dx. \tag{2.106}$$

2.10 Dirac's Delta Function

A Dirac delta function is a (continuous, linear) map from a space of functions into the real or complex numbers. It is a **functional** that associates a number with each function in the function space. Thus $\delta(x-y)$ associates the number $f(y)$ with the function $f(x)$. We may write this association as

$$f(y) = \int f(x)\,\delta(x-y)\,dx. \tag{2.107}$$

Delta functions pop up all over physics. The inner product of two of the kets $|x\rangle$ that appear in the Fourier-series formulas (2.105) and (2.106) is a delta function, $\langle x|y\rangle = \delta(x-y)$. The formula (2.106) for the coefficient $\langle n|f\rangle$ becomes obvious if we write the identity operator for functions defined on the interval $[0, 2\pi]$ as

$$I = \int_0^{2\pi} |x\rangle\langle x|\,dx \tag{2.108}$$

for then

$$\langle n|f\rangle = \langle n|I|f\rangle = \int_0^{2\pi} \langle n|x\rangle\langle x|f\rangle\,dx = \int_0^{2\pi} \frac{e^{-inx}}{\sqrt{2\pi}}\,\langle x|f\rangle\,dx. \tag{2.109}$$

The equation $|y\rangle = I|y\rangle$ with the identity operator (2.108) gives

$$|y\rangle = I|y\rangle = \int_0^{2\pi} |x\rangle\langle x|y\rangle\,dx. \tag{2.110}$$

Multiplying (2.108) from the right by $|f\rangle$ and from the left by $\langle y|$, we get

$$f(y) = \langle y|I|f\rangle = \int_0^{2\pi} \langle y|x\rangle\langle x|f\rangle\,dx = \int_0^{2\pi} \langle y|x\rangle f(x)\,dx. \tag{2.111}$$

These relations (2.110) and (2.111) say that the inner product $\langle y|x\rangle$ is a **delta function**, $\langle y|x\rangle = \langle x|y\rangle = \delta(x-y)$.

The Fourier-series formulas (2.105) and (2.106) lead to a statement about the completeness of the phases $\exp(inx)/\sqrt{2\pi}$

$$f(x) = \sum_{n=-\infty}^{\infty} f_n\,\frac{e^{inx}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} \frac{e^{-iny}}{\sqrt{2\pi}}\,f(y)\frac{e^{inx}}{\sqrt{2\pi}}\,dy. \tag{2.112}$$

Interchanging and rearranging, we have

$$f(x) = \int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} \frac{e^{in(x-y)}}{2\pi} \right) f(y)\, dy. \tag{2.113}$$

But $f(x)$ and the phases e^{inx} are **periodic** with period 2π, so we also have

$$f(x + 2\pi\ell) = \int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} \frac{e^{in(x-y)}}{2\pi} \right) f(y)\, dy. \tag{2.114}$$

Thus we arrive at the **Dirac comb**

$$\sum_{n=-\infty}^{\infty} \frac{e^{in(x-y)}}{2\pi} = \sum_{\ell=-\infty}^{\infty} \delta(x - y - 2\pi\ell) \tag{2.115}$$

or more simply

$$\sum_{n=-\infty}^{\infty} \frac{e^{inx}}{2\pi} = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty} \cos(nx)\right] = \sum_{\ell=-\infty}^{\infty} \delta(x - 2\pi\ell). \tag{2.116}$$

Example 2.13 (Dirac's Comb) The sum of the first 100,000 terms of this cosine series (2.116) for the Dirac comb is plotted for the interval $(-15, 15)$ in Fig. 2.11. Gibbs overshoots appear at the discontinuities. The integral of the first 100,000 terms from -15 to 15 is 5.0000. □

The stretched Dirac comb is

$$\sum_{n=-\infty}^{\infty} \frac{e^{2\pi in(x-y)/L}}{L} = \sum_{\ell=-\infty}^{\infty} \delta(x - y - \ell L). \tag{2.117}$$

Example 2.14 (Parseval's Identity) Using our formula (2.35) for the Fourier coefficients of a stretched interval, we can relate a sum of products $f_n^* \, g_n$ of the Fourier coefficients of the functions $f(x)$ and $g(x)$ to an integral of the product $f^*(x)\, g(x)$

$$\sum_{n=-\infty}^{\infty} f_n^* \, g_n = \sum_{n=-\infty}^{\infty} \int_0^L dx \, \frac{e^{i2\pi nx/L}}{\sqrt{L}} \, f^*(x) \int_0^L dy \, \frac{e^{-i2\pi ny/L}}{\sqrt{L}} \, g(y). \tag{2.118}$$

This sum contains Dirac's comb (2.117) and so

$$\begin{aligned} \sum_{n=-\infty}^{\infty} f_n^* \, g_n &= \int_0^L dx \int_0^L dy \, f^*(x)\, g(y) \, \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{i2\pi n(x-y)/L} \\ &= \int_0^L dx \int_0^L dy \, f^*(x)\, g(y) \sum_{\ell=-\infty}^{\infty} \delta(x - y - \ell L). \end{aligned} \tag{2.119}$$

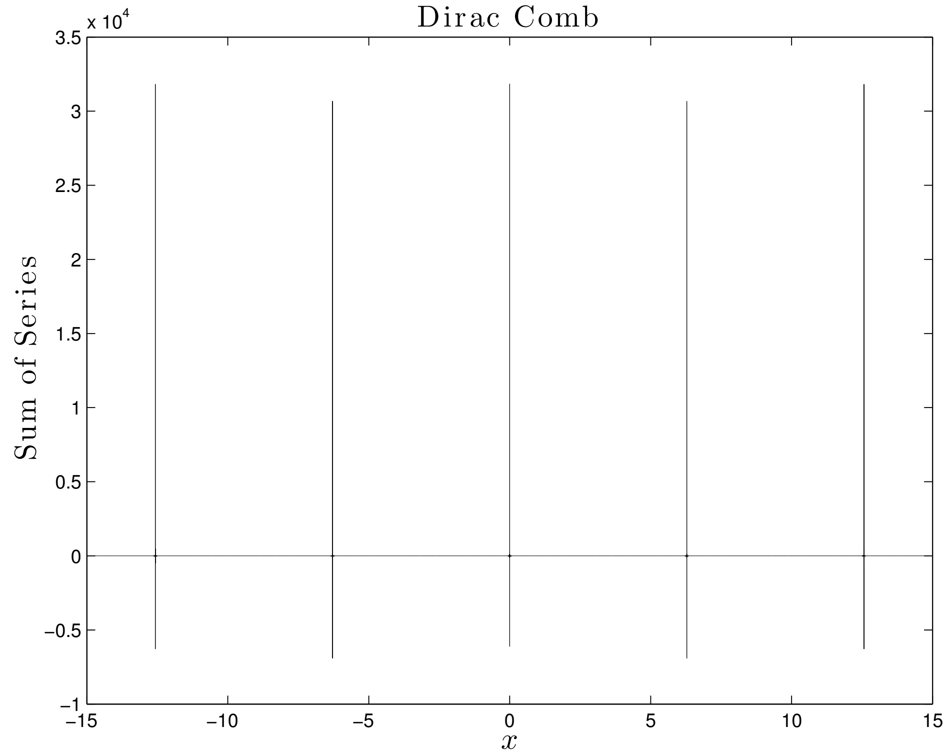

Figure 2.11 The sum of the first 100,000 terms of the series (2.116) for the Dirac comb is plotted for $-15 \le x \le 15$. Both Dirac spikes and Gibbs overshoots are visible.

But because only the $\ell = 0$ tooth of the comb lies in the interval $[0, L]$, we have more simply

$$\sum_{n=-\infty}^{\infty} f_n^* \, g_n = \int_0^L dx \int_0^L dy \, f^*(x) \, g(y) \, \delta(x-y) = \int_0^L dx \, f^*(x) \, g(x). \quad (2.120)$$

In particular, if the two functions are the same, then

$$\sum_{n=-\infty}^{\infty} |f_n|^2 = \int_0^L dx \, |f(x)|^2 \quad (2.121)$$

which is **Parseval's identity**. Thus if a function is **square integrable** on an interval, then the sum of the squares of the absolute values of its Fourier coefficients is the integral of the square of its absolute value. □

Example 2.15 (Derivatives of Delta Functions) Delta functions and other generalized functions or distributions map smooth functions that vanish at infinity into numbers in ways that are linear and continuous. Derivatives of delta functions are defined so as to allow integrations by parts. Thus the nth derivative of the delta function $\delta^{(n)}(x-y)$ maps the function $f(x)$ to $(-1)^n$ times its nth derivative $f^{(n)}(y)$ at y

$$\int \delta^{(n)}(x-y)\, f(x)\, dx = \int \delta(x-y)\, (-1)^n\, f^{(n)}(x)\, dx = (-1)^n\, f^{(n)}(y) \quad (2.122)$$

with no surface term. □

Example 2.16 (The Equation $xf(x) = a$) Dirac's delta function sometimes appears unexpectedly. For instance, the general solution to the equation $x\, f(x) = a$ is $f(x) = a/x + b\, \delta(x)$ where b is an arbitrary constant (Dirac, 1967, sec. 15), (Waxman and Peck, 1998). Similarly, the general solution to the equation $x^2\, f(x) = a$ is $f(x) = a/x^2 + b\, \delta(x)/x + c\, \delta(x) + d\, \delta'(x)$ in which $\delta'(x)$ is the derivative of the delta function, and b, c, and d are arbitrary constants. □

2.11 The Harmonic Oscillator

The hamiltonian for the harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2. \quad (2.123)$$

The commutation relation $[q,p] \equiv qp - pq = i\hbar$ implies that the **lowering** and **raising** operators

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(q + \frac{ip}{m\omega} \right) \quad \text{and} \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(q - \frac{ip}{m\omega} \right) \quad (2.124)$$

obey the commutation relation $[a, a^\dagger] = 1$. In terms of a and $a^\dagger$, which also are called the **annihilation** and **creation** operators, the hamiltonian H has the simple form

$$H = \hbar\omega \left(a^\dagger a + \tfrac{1}{2} \right). \quad (2.125)$$

There is a unique state $|0\rangle$ that is annihilated by the operator a, as may be seen by solving the differential equation

$$\langle q'|a|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\, \langle q'| \left(q + \frac{ip}{m\omega} \right) |0\rangle = 0. \quad (2.126)$$

Since $\langle q'|q = q'\langle q'|$ and

$$\langle q'|p|0\rangle = \frac{\hbar}{i}\frac{d\langle q'|0\rangle}{dq'} \tag{2.127}$$

the resulting differential equation is

$$\frac{d\langle q'|0\rangle}{dq'} = -\frac{m\omega}{\hbar}q'\langle q'|0\rangle. \tag{2.128}$$

Its suitably normalized solution is the wave function for the ground state of the harmonic oscillator

$$\langle q'|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega q'^2}{2\hbar}\right). \tag{2.129}$$

For $n = 0, 1, 2, \ldots$, the nth eigenstate of the hamiltonian H is

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n |0\rangle \tag{2.130}$$

where $n! \equiv n(n-1)\ldots 1$ is n-**factorial** and $0! = 1$. Its energy is

$$H|n\rangle = \hbar\omega\big(n + \tfrac{1}{2}\big)|n\rangle. \tag{2.131}$$

The identity operator is

$$I = \sum_{n=0}^{\infty} |n\rangle\langle n|. \tag{2.132}$$

An arbitrary state $|\psi\rangle$ has an expansion in terms of the eigenstates $|n\rangle$

$$|\psi\rangle = I|\psi\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\psi\rangle \tag{2.133}$$

and evolves in time like a Fourier series

$$|\psi, t\rangle = e^{-iHt/\hbar}|\psi\rangle = e^{-iHt/\hbar}\sum_{n=0}^{\infty} |n\rangle\langle n|\psi\rangle = e^{-i\omega t/2}\sum_{n=0}^{\infty} e^{-in\omega t}|n\rangle\langle n|\psi\rangle \tag{2.134}$$

with wave function

$$\psi(q, t) = \langle q|\psi, t\rangle = e^{-i\omega t/2}\sum_{n=0}^{\infty} e^{-in\omega t}\langle q|n\rangle\langle n|\psi\rangle. \tag{2.135}$$

The wave functions $\langle q|n\rangle$ of the energy eigenstates are related to the Hermite polynomials (example 8.6)

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \tag{2.136}$$

by a change of variables $x = \sqrt{m\omega/\hbar}\, q \equiv sq$ and a normalization factor

$$\langle q|n\rangle = \frac{\sqrt{s}\, e^{-(sq)^2/2}}{\sqrt{2^n n! \sqrt{\pi}}}\, H_n(sq) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{e^{-m\omega q^2/2\hbar}}{\sqrt{2^n n!}}\, H_n\left(\left(\frac{m\omega}{\hbar}\right)^{1/2} q\right). \tag{2.137}$$

The **coherent state** $|\alpha\rangle$

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger}|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{2.138}$$

is an eigenstate $a|\alpha\rangle = \alpha|\alpha\rangle$ of the lowering (or annihilation) operator a with eigenvalue α. Its time evolution is simply

$$|\alpha, t\rangle = e^{-i\omega t/2} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\left(\alpha e^{-i\omega t}\right)^n}{\sqrt{n!}} |n\rangle = e^{-i\omega t/2}\, |\alpha e^{-i\omega t}\rangle. \tag{2.139}$$

2.12 Nonrelativistic Strings

If we clamp the ends of a nonrelativistic string at $x = 0$ and $x = L$, then the amplitude $y(x,t)$ will obey the boundary conditions

$$y(0,t) = y(L,t) = 0 \tag{2.140}$$

and the wave equation

$$v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{2.141}$$

as long as $y(x,t)$ remains small. The functions

$$y_n(x,t) = \sin\frac{n\pi x}{L}\left(f_n \sin\frac{n\pi v t}{L} + d_n \cos\frac{n\pi v t}{L}\right) \tag{2.142}$$

satisfy this wave equation (2.141) and the boundary conditions (2.140). They represent waves traveling along the x-axis with speed v.

The space S_L of functions $f(x)$ that satisfy the boundary condition (2.140) is spanned by the functions $\sin(n\pi x/L)$. One may use the integral formula

$$\int_0^L \sin\frac{n\pi x}{L}\, \sin\frac{m\pi x}{L}\, dx = \frac{L}{2}\, \delta_{nm} \tag{2.143}$$

to derive for any function $f \in S_L$ the Fourier series

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\frac{n\pi x}{L} \tag{2.144}$$