Solution to first homework problem.

Physics 524, 14 April 2015

Problem: Derive Equation 25.80

$$J_a^\mu = -\bar{\psi}_i \gamma^\mu T_{ij}^a \psi_j + f^{abc} A_\nu^b F_c^{\mu\nu}$$

Solution: The conserved currents are

$$J_a^\mu = \sum_n \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_n\right)} \frac{\delta \phi_n}{\delta \alpha_a}$$

summed over the fields ϕ_n of the action density

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + g f^{abc} A_\mu^b A_\nu^c\right)^2 + \bar{\psi}_i\left(i\delta_{ij}\gamma^\mu\partial_\mu + g\gamma^\mu A_\mu^a T_{ij}^a - m\delta_{ij}\right)\psi_j$$

which is invariant under the global transformation

$$\begin{aligned} \psi_i &\to \psi_i + i\alpha^a T_{ij}^a \psi_j \\ A_\mu^a &\to A_\mu^a - f^{abc}\alpha^b A_\mu^c. \end{aligned}$$

One sums over both the matter fields $\phi_n = \psi_i$ and the gauge fields $\phi_n = A_\mu^a$.

First one sums over the fields $\phi_n = \psi_i$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi_i\right)} \frac{\delta \psi_i}{\delta \alpha_a} &= \frac{\partial}{\partial\left(\partial_\mu \psi_i\right)}\left[\bar{\psi}_i\left(i\delta_{ij}\gamma^\mu\partial_\mu + g\gamma^\mu A_\mu^b T_{ij}^b - m\delta_{ij}\right)\psi_j\right]\frac{\delta \psi_i}{\delta \alpha_a} \\ &= \frac{\partial}{\partial\left(\partial_\mu \psi_i\right)}\left[i\bar{\psi}_i\delta_{ij}\gamma^\mu\partial_\mu\psi_j + \bar{\psi}_i g\gamma^\mu A_\mu^b T_{ij}^b\psi_j - \bar{\psi}_i m\delta_{ij}\psi_j\right]\frac{\delta \psi_i}{\delta \alpha_a} \\ &= i\bar{\psi}_i\gamma^\mu\frac{\delta \psi_i}{\delta \alpha_a}. \end{aligned}$$

Then one uses

$$\frac{\delta \psi_i}{\delta \alpha_a} = \frac{\partial}{\delta \alpha_a}\left(\psi_i + i\alpha^b T_{ij}^b \psi_j\right) = iT_{ij}^a \psi_j$$

to get

$$\boxed{\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi_i\right)} \frac{\delta \psi_i}{\delta \alpha_a} = -\bar{\psi}_i \gamma^\mu T_{ij}^a \psi_j.} \tag{1}$$

Now one sums over the gauge fields, $\phi_n = A_\nu^b$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu^b\right)} \frac{\delta A_\nu^b}{\delta \alpha_a} &= \frac{\partial}{\partial\left(\partial_\mu A_\nu^b\right)}\left[-\frac{1}{4}\left(\partial_\mu A_\nu^b - \partial_\nu A_\mu^b + g f^{bcd} A_\mu^c A_\nu^d\right) F_b^{\mu\nu}\right]\frac{\delta A_\nu^b}{\delta \alpha_a} \\ &= -F_b^{\mu\nu}\frac{\delta A_\nu^b}{\delta \alpha_a}. \end{aligned}$$

The gauge fields vary as

$$\frac{\delta A^b_\nu}{\delta \alpha_a} = \frac{\delta}{\delta \alpha_a}\left(A^b_\nu - f^{bcd}\alpha^c A^d_\nu\right) = -f^{bad}A^d_\nu.$$

Thus one gets

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A^b_\nu\right)}\frac{\delta A^b_\nu}{\delta \alpha_a} = -F^{\mu\nu}_b(-f^{bad}A^d_\nu) = f^{adb}A^d_\nu F^{\mu\nu}_b. \tag{2}$$

Equations 1 and 2 now give Equation 25.80 from Schwartz

$$\boxed{J^\mu = -\bar{\psi}_i\gamma^\mu T^a_{ij}\psi_j + f^{abc}A^b_\nu F^{\mu\nu}_c.}$$