and using $f''(w) = \rho\, e^{i\phi}$ and $2\theta + \phi = \pi$ to show that

$$\rho\, e^{-2i\theta} = \rho\, e^{i\phi - i\pi} = -\rho\, e^{i\phi} = -f''(w) \tag{5.300}$$

we get our formula for the saddle-point integral (5.294)

$$I(x) \approx \left(\frac{2\pi}{-x f''(w)} \right)^{1/2} h(w)\, e^{x f(w)}. \tag{5.301}$$

If there are n saddle points w_j for $j = 1, \ldots, n$, then the integral $I(x)$ is the sum

$$I(x) \approx \sum_{j=1}^{N} \left(\frac{2\pi}{-x f''(w_j)} \right)^{1/2} h(w_j)\, e^{x f(w_j)}. \tag{5.302}$$

5.23 The Abel-Plana Formula and the Casimir Effect

This section is optional on a first reading.

Suppose the function $f(z)$ is analytic and bounded for $n_1 \le \operatorname{Re} z \le n_2$. Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be two contours that respectively run counter-clockwise along the rectangles with vertices $n_1, n_2, n_2 + i\infty, n_1 + i\infty$ and $n_1, n_2, n_2 - i\infty, n_1 - i\infty$ indented with tiny semi-circles and quarter-circles so as to avoid the integers $z = n_1,\ n_1 + 1,\ n_1 + 2,\ \ldots,\ n_2$ while keeping $\mathrm{Im} z > 0$ in the upper rectangle and $\mathrm{Im} z < 0$ in the lower one (and $n_1 < \operatorname{Re} z < n_2$). Then the contour integrals

$$\mathcal{I}_\pm = \int_{C_\pm} \frac{f(z)}{e^{\mp 2\pi i z} - 1}\, dz = 0 \tag{5.303}$$

vanish by Cauchy's theorem (5.22) since the poles of the integrand lie outside the indented rectangles.

The absolute value of the exponential $\exp(-2\pi i z)$ is arbitrarily large on the top of the upper rectangle $\mathcal{C}_+$ where $\mathrm{Im} z = \infty$, and so that leg of the contour integral $\mathcal{I}_+$ vanishes. Similarly, the bottom leg of the contour integral $\mathcal{I}_-$ vanishes. Thus we can separate the difference $\mathcal{I}_+ - \mathcal{I}_-$ into a term T_x due to the integrals near the x-axis between n_1 and n_2, a term T_1 involving integrals between n_1 and $n_1 \pm i\infty$, and a term T_2 involving integrals between n_2 and $n_2 \pm i\infty$, that is, $0 = \mathcal{I}_+ - \mathcal{I}_- = T_x + T_1 + T_2$.

The term $T_x = I_x + S$ consists of the integrals I_x along the segments of the x-axis from n_1 to n_2 and a sum S over the tiny integrals along the semi-circles and quarter circles that avoid the integers from n_1 to n_2. Elementary algebra simplifies the integral I_x to

$$I_x = \int_{n_1}^{n_2} f(x) \left[\frac{1}{e^{-2\pi i x} - 1} + \frac{1}{e^{+2\pi i x} - 1} \right] dx = - \int_{n_1}^{n_2} f(x)\, dx. \qquad (5.304)$$

The sum S is over the semi-circles that avoid $n_1 + 1, \ldots, n_2 - 1$ and over the quarter-circles that avoid n_1 and n_2. For any integer $n_1 < n < n_2$, the integral along the semi-circle of $\mathcal{C}_{n+}$ minus that along the semi-circle of $\mathcal{C}_{n-}$, both around n, contributes to S the quantity

$$\begin{aligned} S_n &= \int_{SC_{n+}} \frac{f(z)}{e^{-2\pi i z} - 1}\, dz - \int_{SC_{n-}} \frac{f(z)}{e^{2\pi i z} - 1}\, dz \\ &= \int_{SC_{n+}} \frac{f(z)}{e^{-2\pi i (z-n)} - 1}\, dz - \int_{SC_{n-}} \frac{f(z)}{e^{2\pi i (z-n)} - 1}\, dz \qquad (5.305) \end{aligned}$$

since $\exp(\pm 2\pi i n) = 1$. The first integral is clockwise in the upper half plane, the second clockwise in the lower half plane. So if we make both integrals counter-clockwise, inserting minus signs, we find as the radii of these semi-circles shrink to zero

$$S_n = \oint \frac{f(z)}{2\pi i (z-n)} dz = f(n). \qquad (5.306)$$

One may show (exercise 5.39) that the quarter-circles around n_1 and n_2 contribute $(f(n_1) + f(n_2))/2$ to the sum S. Thus the term T_x is

$$T_x = \frac{1}{2} f(n_1) + \sum_{n=n_1+1}^{n_2-1} f(n) + \frac{1}{2} f(n_2) \; - \int_{n_1}^{n_2} f(x)\, dx. \qquad (5.307)$$

Since $\exp(-2\pi i n_1) = 1$, the difference between the integrals along the imaginary axes above and below n_1 is (exercise 5.40)

$$\begin{aligned} T_1 &= \int_{n_1 + i\infty}^{n_1} \frac{f(z)}{e^{-2\pi i z} - 1}\, dz - \int_{n_1}^{n_1 - i\infty} \frac{f(z)}{e^{2\pi i z} - 1}\, dz \qquad (5.308) \\ &= -i \int_0^\infty \frac{f(n_1 + iy) - f(n_1 - iy)}{e^{2\pi y} - 1}\, dy. \qquad (5.309) \end{aligned}$$

Similarly, the difference between the integrals along the imaginary axes above and below n_2 is (exercise 5.41)

$$\begin{aligned} T_2 &= \int_{n_2}^{n_2 + i\infty} \frac{f(z)}{e^{-2\pi i z} - 1}\, dz - \int_{n_2 - i\infty}^{n_2} \frac{f(z)}{e^{2\pi i z} - 1}\, dz \qquad (5.310) \\ &= i \int_0^\infty \frac{f(n_2 + iy) - f(n_2 - iy)}{e^{2\pi y} - 1}\, dy. \qquad (5.311) \end{aligned}$$

Since $\mathcal{I}_+ - \mathcal{I}_- = T_x + T_1 + T_2 = 0$, we can use (5.307) and (5.309–5.311) to build the Abel-Plana formula (Whittaker and Watson, 1927, p. 145)

$$\frac{1}{2}f(n_1) + \sum_{n=n_1+1}^{n_2-1} f(n) + \frac{1}{2}f(n_2) \; - \int_{n_1}^{n_2} f(x)\,dx \tag{5.312}$$
$$= i\int_0^\infty \frac{f(n_1+iy) - f(n_1-iy) - f(n_2+iy) + f(n_2-iy)}{e^{2\pi y}-1}\,dy$$

(Niels Abel 1802–1829 and Giovanni Plana 1781–1864).

In particular, if $f(z) = z$, the integral over y vanishes, and the Abel-Plana formula (5.312) gives

$$\frac{1}{2}n_1 \; + \sum_{n=n_1+1}^{n_2-1} n \; + \frac{1}{2}n_2 \; = \int_{n_1}^{n_2} x\,dx \tag{5.313}$$

which is an example of the trapezoidal rule.

Example 5.37 (The Casimir Effect) The Abel-Plana formula provides one of the clearer formulations of the Casimir effect. We will assume that the hamiltonian for the electromagnetic field in empty space is a sum over two polarizations and an integral over all momenta of a symmetric product

$$H_0 = \sum_{s=1}^{2} \int \hbar\,\omega(\boldsymbol{k})\,\frac{1}{2}\left[a_s^\dagger(\boldsymbol{k})a_s(\boldsymbol{k}) + a_s(\boldsymbol{k})a_s^\dagger(\boldsymbol{k})\right]\,d^3k \tag{5.314}$$

of the annihilation and creation operators $a_s(\boldsymbol{k})$ and $a_s^\dagger(\boldsymbol{k})$ which satisfy the commutation relations

$$[a_s(\boldsymbol{k}), a_{s'}^\dagger(\boldsymbol{k}')] = \delta_{ss'}\,\delta(\boldsymbol{k}-\boldsymbol{k}') \;\text{ and }\; [a_s(\boldsymbol{k}), a_{s'}(\boldsymbol{k}')] = 0 = [a_s^\dagger(\boldsymbol{k}), a_{s'}^\dagger(\boldsymbol{k}')]. \tag{5.315}$$

The vacuum state $|0\rangle$ has no photons, and so on it $a_s(\boldsymbol{k})|0\rangle = 0$ (and $\langle 0|a_s^\dagger(\boldsymbol{k}) = 0$). But because the operators in H_0 are symmetrically ordered, the energy E_0 of the vacuum as given by (5.314) is not zero; instead it is quarticly divergent

$$E_0 = \langle 0|H_0|0\rangle = \sum_{s=1}^{2}\int \hbar\,\omega(\boldsymbol{k})\,\frac{1}{2}\delta(\boldsymbol{0})\,d^3k = V\int \hbar\,\omega(\boldsymbol{k})\,\frac{d^3k}{(2\pi)^3} \tag{5.316}$$

in which we used the delta-function formula

$$\delta(\boldsymbol{k}-\boldsymbol{k}') = \int e^{\pm i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}\,\frac{d^3x}{(2\pi)^3} \tag{5.317}$$

to identify $\delta(\boldsymbol{0})$ as the volume V of empty space divided by $(2\pi)^3$. Since the photon has no mass, its (angular) frequency $\omega(\boldsymbol{k})$ is $c|\boldsymbol{k}|$, and so the energy density E_0/V is

$$\frac{E_0}{V} = \hbar c \int k^3 \frac{dk}{2\pi^2} = \hbar c \frac{K^4}{8\pi^2} = \frac{\hbar c}{8\pi^2} \frac{1}{d^4} \tag{5.318}$$

in which we cut off the integral at some short distance $d = K^{-1}$ below which the hamiltonian (5.314) and the commutation relations (5.315) are no longer valid. But the energy density of empty space is

$$\Omega_\Lambda \rho_c = \Omega_\Lambda \frac{3H_0^2}{8\pi G} \approx \frac{\hbar c}{8\pi^2} \frac{1}{(2.8 \times 10^{-5}\,\mathrm{m})^4} \tag{5.319}$$

which corresponds to a distance scale d of 28 micrometers. Since quantum electrodynamics works well down to about 10^{-18} m, this distance scale is too big by thirteen orders of magnitude.

If the universe were inside an enormous, perfectly conducting, metal cube of side L, then the tangential electric and normal magnetic fields would vanish on the surface of the cube $\boldsymbol{E}_t(\boldsymbol{r},t) = 0 = \boldsymbol{B}_n(\boldsymbol{r},t)$. The available wave-numbers of the electromagnetic field inside the cube then would be $k_{\boldsymbol{n}} = 2\pi(n_1, n_2, n_3)/L$, and the energy density would be

$$\frac{E_0}{V} = \frac{2\pi\hbar c}{L^4} \sum_{\boldsymbol{n}} \sqrt{\boldsymbol{n}^2}. \tag{5.320}$$

The Casimir effect exploits the difference between the continuous (5.318) and discrete (5.320) energy densities for the case of two metal plates of area A separated by a short distance $\ell \ll \sqrt{A}$.

If the plates are good conductors, then at low frequencies the boundary conditions $\boldsymbol{E}_t(\boldsymbol{r},t) = 0 = \boldsymbol{B}_n(\boldsymbol{r},t)$ hold, and the tangential electric and normal magnetic field vanish on the surfaces of the metal plates. At high frequencies, above the plasma frequency ω_p of the metal, these boundary conditions fail because the relative electric permittivity of the metal

$$\epsilon(\omega) \approx 1 - \frac{\omega_p^2}{\omega^2}\left(1 - \frac{i}{\omega\tau}\right) \tag{5.321}$$

has a positive real part. Here τ is the mean time between electron collisions.

The modes that satisfy the low-frequency boundary conditions $\boldsymbol{E}_t(\boldsymbol{r},t) = 0 = \boldsymbol{B}_n(\boldsymbol{r},t)$ are (Bordag et al., 2009, p. 30)

$$\omega(\boldsymbol{k}_\perp, n) \equiv c\sqrt{\boldsymbol{k}_\perp^2 + \left(\frac{\pi n}{\ell}\right)^2} \tag{5.322}$$

where $\boldsymbol{n} \cdot \boldsymbol{k}_\perp = 0$. The difference between the zero-point energies of these modes and those of the continuous modes in the absence of the two plates per unit area would be

$$\frac{E(\ell)}{A} = \frac{\pi\hbar c}{\ell}\int_0^\infty \frac{k_\perp dk_\perp}{2\pi}\left[\sum_{n=0}^{\infty}\sqrt{\frac{\ell^2 k_\perp^2}{\pi^2}+n^2} - \int_0^\infty\sqrt{\frac{\ell^2 k_\perp^2}{\pi^2}+x^2}\,dx - \frac{\ell k_\perp}{2\pi}\right] \tag{5.323}$$

if the boundary conditions held at all frequencies. With $p = \ell k_\perp/\pi$, we will represent the failure of these boundary conditions at the plasma frequency ω_p by means of a cutoff function like $c(n) = (1+n/n_p)^{-4}$ where $n_p = \omega_p \ell/\pi c$. In terms of such a cutoff function, the energy difference per unit area is

$$\frac{E(\ell)}{A} = \frac{\pi^2\hbar c}{2\ell^3}\int_0^\infty p\,dp\left[\sum_{n=0}^{\infty} c(n)\sqrt{p^2+n^2} - \int_0^\infty c(x)\sqrt{p^2+x^2}\,dx - \frac{p}{2}\right]. \tag{5.324}$$

Since $c(n)$ falls off as $(n_p/n)^4$ for $n \gg n_p$, we may neglect terms in the sum and integral beyond some integer M that is much larger than n_p

$$\frac{E(\ell)}{A} = \frac{\pi^2\hbar c}{2\ell^3}\int_0^\infty p\,dp\left[\sum_{n=0}^{M} c(n)\sqrt{p^2+n^2} - \int_0^M c(x)\sqrt{p^2+x^2}\,dx - \frac{p}{2}\right]. \tag{5.325}$$

The function

$$f(z) = c(z)\sqrt{p^2+z^2} = \frac{\sqrt{p^2+z^2}}{(1+z/n_p)^4} \tag{5.326}$$

is analytic in the right half-plane $\mathrm{Re}\, z = x > 0$ (exercise 5.42) and tends to zero $\lim_{x\to\infty}|f(x+iy)| \to 0$ as $\mathrm{Re}\, z = x \to \infty$. So we can apply the Abel-Plana formula (5.312) with $n_1 = 0$ and $n_2 = M$ to the term in the square brackets in (5.325) and get

$$\begin{aligned}\frac{E(\ell)}{A} = \frac{\pi^2\hbar c}{2\ell^3}\int_0^\infty p\,dp\Bigg\{&\frac{c(M)}{2}\sqrt{p^2+M^2} \\ +i\int_0^\infty\Big[&c(iy)\sqrt{p^2+(\epsilon+iy)^2} - c(-iy)\sqrt{p^2+(\epsilon-iy)^2} \\ &- c(M+iy)\sqrt{p^2+(M+iy)^2} \\ &+c(M-iy)\sqrt{p^2+(M-iy)^2}\Big]\frac{dy}{e^{2\pi y}-1}\Bigg\}\end{aligned} \tag{5.327}$$

in which the infinitesimal ϵ reminds us that the contour lies inside the right half-plane.

We now take advantage of the properties of the cutoff function $c(z)$. Since $M \gg n_p$, we can neglect the term $c(M)\sqrt{p^2+M^2}/2$. The denominator

$\exp(2\pi y)-1$ also allows us to neglect the terms $\mp c(M \mp iy)\sqrt{p^2+(M\mp iy)^2}$. We are left with

$$\frac{E(\ell)}{A} = \frac{\pi^2 \hbar c}{2\ell^3}\int_0^\infty p\,dp \times i\int_0^\infty \left[c(iy)\sqrt{p^2+(\epsilon+iy)^2} - c(-iy)\sqrt{p^2+(\epsilon-iy)^2}\right]\frac{dy}{e^{2\pi y}-1}. \tag{5.328}$$

Since the y integration involves the factor $1/(\exp(2\pi y)-1)$, we can neglect the detailed behavior of the cutoff functions $c(iy)$ and $c(-iy)$ for $y > n_p$ where they differ appreciably from unity. The energy now is

$$\frac{E(\ell)}{A} = \frac{\pi^2 \hbar c}{2\ell^3}\int_0^\infty p\,dp\int_0^\infty i\frac{\sqrt{p^2+(\epsilon+iy)^2}-\sqrt{p^2+(\epsilon-iy)^2}}{e^{2\pi y}-1}dy. \tag{5.329}$$

When $y < p$, the square-roots with the ϵ's cancel. But for $y > p$, they are

$$\sqrt{p^2-y^2\pm 2i\epsilon y} = \pm i\sqrt{y^2-p^2}. \tag{5.330}$$

Their difference is $2i\sqrt{y^2-p^2}$, and so $E(\ell)$ is

$$\frac{E(\ell)}{A} = \frac{\pi^2 \hbar c}{2\ell^3}\int_0^\infty p\,dp\int_0^\infty \frac{-2\sqrt{y^2-p^2}\,\theta(y-p)}{e^{2\pi y}-1}dy \tag{5.331}$$

in which the Heaviside step function $\theta(x) \equiv (x+|x|)/(2|x|)$ keeps $y > p$

$$\frac{E(\ell)}{A} = -\frac{\pi^2 \hbar c}{\ell^3}\int_0^y p\,dp\int_0^\infty \frac{\sqrt{y^2-p^2}}{e^{2\pi y}-1}dy. \tag{5.332}$$

The p-integration is elementary, and so the energy difference is

$$\frac{E(\ell)}{A} = -\frac{\pi^2 \hbar c}{3\ell^3}\int_0^\infty \frac{y^3\,dy}{e^{2\pi y}-1} = -\frac{\pi^2 \hbar c}{3\ell^3}\frac{B_2}{8} = -\frac{\pi^2 \hbar c}{720\,\ell^3} \tag{5.333}$$

in which B_2 is the second Bernoulli number (4.109). The pressure pushing the plates together then is

$$p = -\frac{1}{A}\frac{\partial E(\ell)}{\partial \ell} = -\frac{\pi^2 \hbar c}{240\,\ell^4} \tag{5.334}$$

a result due to Casimir (Hendrik Brugt Gerhard Casimir, 1909–2000).

Although the Casimir effect is very attractive because of its direct connection with the symmetric ordering of the creation and annihilation operators in the hamiltonian (5.314), the reader should keep in mind that neutral atoms are mutually attractive, which is why most gases are diatomic, and that Lifshitz explained the effect in terms of the mutual attraction of the atoms in the metal plates (Lifshitz, 1956; Milonni and Shih, 1992) (Evgeny Mikhailovich Lifshitz, 1915–1985). □