Cross-sections

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1 Fermi's Golden Rule

There are two standard ways of normalizing states. We have been using continuum normalization

$$\langle p'|p\rangle = \delta^{(3)}(\boldsymbol{p'} - \boldsymbol{p}). \tag{1}$$

The other way is"box" normalization. We put the universe in a space-time box of duration T and width L. Our possible momenta are

$$\boldsymbol{p} = \frac{2\pi}{L}(n_1, n_2 n_3) \tag{2}$$

so that the wave function is single valued when we identify points on opposite sides of the box. So our delta functions become

$$\delta_V^{(3)}(\boldsymbol{p'} - \boldsymbol{p}) = \int_V e^{i(\boldsymbol{p'}-\boldsymbol{p})\cdot\boldsymbol{x}}\,\frac{d^3x}{(2\pi)^3} = \frac{V}{(2\pi)^3}\,\delta_{\boldsymbol{p'},\boldsymbol{p}} \quad \text{and} \quad \delta_T(E'-E) = \int_T e^{i(E'-E)\,t}\,\frac{dt}{2\pi}. \tag{3}$$

We can normalize our box states

$$|p\rangle_V = \sqrt{\frac{(2\pi)^3}{V}}\;|p\rangle \tag{4}$$

so that now

$$\langle p'|p\rangle_V = \delta_{\boldsymbol{p'},\boldsymbol{p}}. \tag{5}$$

Similarly, a two-particle box state is

$$|p_1, p_2\rangle_V = \frac{(2\pi)^3}{V}\ |p_1, p_2\rangle, \tag{6}$$

and (if the particles are different)

$$\langle p_1', p_2'|p_1, p_2\rangle_V = \delta_{\boldsymbol{p_1'},\boldsymbol{p_1}}\delta_{\boldsymbol{p_2'},\boldsymbol{p_2}}. \tag{7}$$

If they are the same, then

$$\langle p_1', p_2'|p_1, p_2\rangle_V = \delta_{\boldsymbol{p_1'},\boldsymbol{p_1}}\delta_{\boldsymbol{p_2'},\boldsymbol{p_2}} \pm \delta_{\boldsymbol{p_1'},\boldsymbol{p_2}}\delta_{\boldsymbol{p_2'},\boldsymbol{p_1}} \tag{8}$$

with $\pm = (-1)^{2j}$ where j is the spin of the particles.

Following Weinberg (section 3.4 of TQTOF1), the probability of a scattering process in which N_i incoming particles become N_o outgoing particles is

$$P(i \to o) = |S_{oiV}|^2 \tag{9}$$

in which the S-matrix element S_{oiV} in box normalization is related to the one in continuum normalization by

$$S_{oiV} = \left(\frac{(2\pi)^3}{V}\right)^{(N_o+N_i)/2} S_{oi}. \tag{10}$$

So

$$P(i \to o) = |S_{oiV}|^2 = \left(\frac{(2\pi)^3}{V}\right)^{N_o+N_i} |S_{oi}|^2. \tag{11}$$

It follows from our formula (2) for the possible momenta that the number of outgoing states in

$$do \equiv d^3p_1' \ldots d^3p_{N_o}' \tag{12}$$

is

$$dN_o = \left(\frac{V}{(2\pi)^3}\right)^{N_o} do. \tag{13}$$

So the probability of the process $i \to do$ is

$$dP(i \to o) = P(i \to o)\, dN_o = \left(\frac{(2\pi)^3}{V}\right)^{N_o+N_i} |S_{oi}|^2 \left(\frac{V}{(2\pi)^3}\right)^{N_o} do = \left(\frac{(2\pi)^3}{V}\right)^{N_i} |S_{oi}|^2\, do \tag{14}$$

which is where we stopped on Tuesday.

It is conventional to write the S-matrix element as

$$S_{oi} \equiv -2i\pi\, \delta^3_V(\boldsymbol{p'_o} - \boldsymbol{p_i})\, \delta_T(E'_o - E_i)\, M_{oi} \tag{15}$$

in which the matrix element M_{oi} is free of delta functions (we are ignoring processes that are disconnected). The squares of our box delta functions are

$$\begin{aligned} \left[\delta^3_V(\boldsymbol{p'_o} - \boldsymbol{p_i})\right]^2 &= \delta^3_V(\boldsymbol{p'_o} - \boldsymbol{p_i})\, \delta^3_V(\boldsymbol{0}) = \delta^3_V(\boldsymbol{p'_o} - \boldsymbol{p_i})\, V/(2\pi)^3 \\ \left[\delta_T(E'_o - E_i)\right]^2 &= \delta_T(E'_o - E_i)\, \delta_T(0) = \delta_T(E'_o - E_i)\, T/2\pi. \end{aligned} \tag{16}$$

So $dP(i \to o)$ is

$$\begin{aligned} dP(i \to o) = & \left(\frac{(2\pi)^3}{V}\right)^{N_i} |S_{oi}|^2\, do = \left(\frac{(2\pi)^3}{V}\right)^{N_i} \left[2\pi\, \delta^3_V(\boldsymbol{p'_o} - \boldsymbol{p_i})\, \delta_T(E'_o - E_i)\, M_{oi}\right]^2 do \\ & = (2\pi)^2 \left(\frac{(2\pi)^3}{V}\right)^{N_i-1} \frac{T}{2\pi}\, |M_{oi}|^2\, \delta^3_V(\boldsymbol{p'_o} - \boldsymbol{p_i})\, \delta_T(E'_o - E_i)\, do. \end{aligned} \tag{17}$$

To get the transition rate, we divide by the time T

$$d\Gamma(i \to o) \equiv dP(i \to o)/T = (2\pi)^{3N_i-2} V^{1-N_i}\, |M_{oi}|^2\, \delta^4(p'_o - p_i)\, do \tag{18}$$

in which we have

$$S_{oi} \equiv -2i\pi\, \delta^4(p'_o - p_i)\, M_{oi} \tag{19}$$

as we let the box swell to infinity. These last two equations are the keys to cross-section formulas.

Two cases are of special importance. The case $N_i = 1$ applies to a single particle that decays into N_o particles. Here the transition rate is

$$d\Gamma(i \to o) = 2\pi \, |M_{oi}|^2 \, \delta^4(p'_o - p_i) \, do. \tag{20}$$

The case $N_i = 2$ describes the standard scattering process in which two incoming particles turn into N_o outgoing particles. The rate is

$$d\Gamma(i \to o) = (2\pi)^4 \, V^{-1} \, |M_{oi}|^2 \, \delta^4(p'_o - p_i) \, do, \tag{21}$$

but we must divide by the flux F of incoming particles. The flux at particle 2 due to a density $1/V$ of particle 1 coming at speed v is

$$F = \frac{v}{V}. \tag{22}$$

The rate divided by the flux is the differential cross-section

$$\begin{aligned} d\sigma &= d\Gamma(i \to o)/F = (2\pi)^4 \, V^{-1} \, |M_{oi}|^2 \, \delta^4(p'_o - p_i) \, do/F = (2\pi)^4 \, V^{-1} \, |M_{oi}|^2 \, \delta^4(p'_o - p_i) \, do \, V/v \\ &= (2\pi)^4 \, v^{-1} \, |M_{oi}|^2 \, \delta^4(p'_o - p_i) \, do. \end{aligned} \tag{23}$$

In a general Lorentz frame, the relative speed is

$$v = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}. \tag{24}$$

Homework problem 1: Find a simple expression for v when $p_2 = (m_2, \mathbf{0})$.

Homework problem 2: Show that a simple expression for v for the case in which $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \mathbf{0}$ is

$$v = \frac{kE}{E_1 E_2} \tag{25}$$

in which $k = |\boldsymbol{p}_1|$ and $E = E_1 + E_2$. This “speed” can exceed unity and can approach 2.

The product $\delta^4(p'_o - p_i)\, do$ is called the phase-space factor. We will find it easiest to compute in the "center-of-mass" frame $\boldsymbol{p}_i = \mathbf{0}$. There the delta function $\delta^3(\boldsymbol{p'_o} - \boldsymbol{p_i}) = \delta^3(\boldsymbol{p'_o})$ lets us drop $d^3p'_k$ for one of the N_o outgoing particles as long as we remember to set

$$\boldsymbol{p'_k} = -(\boldsymbol{p}'_1 + \cdots + \boldsymbol{p}'_{k-1} + \boldsymbol{p}'_{k+1} + \cdots + \boldsymbol{p}'_{N_o}). \tag{26}$$

The simplest case is $N_o = 2$. We set

$$\boldsymbol{p}'_1 = -\boldsymbol{p}'_2 \tag{27}$$

and find that

$$\delta^4(p'_o - p_i)\, do \to \delta(E'_1 + E'_2 - E)\, d^3p'_2. \tag{28}$$

To evaluate this expression, we use the delta-function identities

$$\delta(ax) = \frac{1}{|a|} \quad \text{and} \quad \delta(f(x)) = \sum_{x'} \frac{1}{|f'(x')|}\delta(x - x') \tag{29}$$

in which the sum is over the zeros x' of $f(x)$. Since $|\boldsymbol{p}'_1| = |\boldsymbol{p}'_2| \equiv k'$, we must solve the equation

$$E = E'_1 + E'_2 = \sqrt{k'^2 + m'^2_1} + \sqrt{k'^2 + m'^2_2}. \tag{30}$$

It has only one root

$$k' = \frac{\sqrt{(E^2 - m'^2_1 - m'^2_2)^2 - 4m'^2_1 m'^2_2}}{2E} \tag{31}$$

or if the masses are equal, $m'_1 = m'_2 \equiv m'$, simply

$$k' = \frac{1}{2}\sqrt{E^2 - 4m'^2}. \tag{32}$$

Homework problem 3: Derive this formula (31) for the root k'.

So the derivative we need is

$$\left| \frac{d}{dk'} \left(\sqrt{k'^2 + m'^2_1} + \sqrt{k'^2 + m'^2_2} \right) \right| = \frac{k'}{E'_1} + \frac{k'}{E'_2} = \frac{k'E}{E'_1 E'_2} \tag{33}$$

and so with $d^3k' = k'^2 dk' d\Omega$ we have

$$\delta^4(p'_o - p_i)\, do \to \delta(E'_1 + E'_2 - E)\, d^3k'_2 = \frac{E'_1 E'_2}{k' E}\, k'^2\, d\Omega = \frac{E'_1 E'_2}{E}\, k'\, d\Omega. \tag{34}$$

In these formulas, k' is given by (31) or (32).

So the two-particle decay rate into solid angle $d\Omega$ is from (20)

$$d\Gamma(i \to o) = 2\pi\, |M_{oi}|^2\, \delta^4(p'_o - p_i)\, do = 2\pi\, |M_{oi}|^2\, \frac{E'_1 E'_2}{E}\, k'\, d\Omega. \tag{35}$$

Similarly, the cross-section for $2 \to 2$ scattering in the center-of-mass frame is from (23) and the result (25) of problem 2

$$d\sigma = (2\pi)^4\, v^{-1}\, |M_{oi}|^2\, \delta^4(p'_o - p_i)\, do = (2\pi)^4\, \frac{E_1 E_2}{kE}\, |M_{oi}|^2\, \frac{E'_1 E'_2}{E}\, k'\, d\Omega \tag{36}$$

or

$$\frac{d\sigma}{d\Omega} = (2\pi)^4\, \frac{k'\, E'_1 E'_2\, E_1 E_2}{k\, E^2}\, |M_{oi}|^2. \tag{37}$$

2 Muon Pair Production

As an example of how to compute $|M_{oi}|^2$, let's recall the amplitude for the process $e^+ + e^- \to \mu^+ + \mu^-$

$$\begin{aligned}\langle p', s'; q', t'|U|p, s; q, t\rangle &= \frac{-ie^2}{(2\pi)^2}\overline{u}_\mu(p', s')\gamma^a v_\mu(q', t')\overline{v}_e(q, t)\gamma_a u_e(p, s)\frac{\delta(p' + q' - p - q)}{(p + q)^2 - i\epsilon} \\ &= -2\pi i\delta(p' + q' - p - q)\frac{e^2}{(2\pi)^3}\overline{u}_\mu(p', s')\gamma^a v_\mu(q', t')\overline{v}_e(q, t)\gamma_a u_e(p, s)\frac{1}{(p + q)^2 - i\epsilon}\end{aligned} \tag{38}$$

which tells us that

$$M_{oi} = \frac{e^2}{(2\pi)^3}\overline{u}_\mu(p', s')\gamma^a v_\mu(q', t')\overline{v}_e(q, t)\gamma_a u_e(p, s)\frac{1}{(p + q)^2 - i\epsilon}. \tag{39}$$

The x-section is

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 \, \frac{k'\, E_1' E_2' \, E_1 E_2}{k\, E^2} \, |M_{oi}|^2. \tag{40}$$

This formula becomes much simpler if we average over the initial spins and sum over the final spins

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \sum_{s,t,s't'} (2\pi)^4 \, \frac{k'\, E_1' E_2' \, E_1 E_2}{k\, E^2} \, |M_{oi}|^2. \tag{41}$$

which makes sense because we rarely know the initial spins and rarely measure the final spins.

We have carried out the spin sums in section 4 of the notes on perturbation theory. There we found that

$$\begin{aligned} \frac{1}{4} \sum_{s,t,s't'} |M_{oi}|^2 &= \frac{1}{4} \sum_{s,t,s't'} \frac{e^4}{(2\pi)^6} \, |\overline{u}_\mu(p',s') \gamma^a v_\mu(q',t') \overline{v}_e(q,t) \gamma_a u_e(p,s)|^2 \, \frac{1}{[(p+q)^2]^2} \\ &= \frac{1}{4} \frac{e^4}{(2\pi)^6} \frac{\mathrm{Tr}\left[(-i\not{q}' - m_\mu)\gamma^a(-i\not{p}' + m_\mu)\gamma^b\right] \mathrm{Tr}\left[(-i\not{q} - m_e)\gamma_a(-i\not{p} + m_e)\gamma_b\right]}{2p^0 2q^0 2p'^0 2q'^0 (p+q)^4}. \end{aligned} \tag{42}$$

So combining the last two equations, we get

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{4} \sum_{s,t,s't'} (2\pi)^4 \, \frac{k'\, E_1' E_2' \, E_1 E_2}{k\, E^2} \, |M_{oi}|^2 \\ &= \frac{1}{4} (2\pi)^4 \frac{k'\, E_1' E_2' \, E_1 E_2}{k\, E^2} \, \frac{e^4}{(2\pi)^6} \frac{\mathrm{Tr}\left[(-i\not{q}' - m_\mu)\gamma^a(-i\not{p}' + m_\mu)\gamma^b\right] \mathrm{Tr}\left[(-i\not{q} - m_e)\gamma_a(-i\not{p} + m_e)\gamma_b\right]}{2p^0 2q^0 2p'^0 2q'^0 (p+q)^4} \\ &= \frac{e^4}{(16\pi)^2} \, \frac{p'}{p\, E^2} \, \frac{\mathrm{Tr}\left[(-i\not{q}' - m_\mu)\gamma^a(-i\not{p}' + m_\mu)\gamma^b\right] \mathrm{Tr}\left[(-i\not{q} - m_e)\gamma_a(-i\not{p} + m_e)\gamma_b\right]}{(p+q)^4}. \end{aligned} \tag{43}$$

Now in the center-of-mass frame $p + q = p^0 + q^0 = E$, so we have

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{(16\pi)^2} \, \frac{k'}{k\, E^6} \, \mathrm{Tr}\left[(-i\not{q}' - m_\mu)\gamma^a(-i\not{p}' + m_\mu)\gamma^b\right] \mathrm{Tr}\left[(-i\not{q} - m_e)\gamma_a(-i\not{p} + m_e)\gamma_b\right]. \tag{44}$$

3 Traces

The gamma matrices in Weinberg's notation are

$$\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \vec{\gamma} = -i \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \tag{45}$$

and they satisfy the anticommutation relation

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{46}$$

The fifth gamma matrix is

$$\gamma^5 = \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{47}$$

and it anticommutes with the other four gamma matrices

$$\{\gamma_5, \gamma^a\} = 0. \tag{48}$$

Since $\gamma_5^2 = 1$, we may think of it as a fourth spatial gamma matrix and write $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$ for $\alpha, \beta = 0, 1, 2, 3, 4$ with $\eta^{00} = -1$, $\eta^{kk} = 1$ for $k = 1 - 4$, and η zero otherwise.

Homework problem 4: Verify equations (46 & 48).

We can use these properties of the gamma matrices to show that the trace of an odd number of gamma matrices vanishes

$$\mathrm{Tr}\left[\gamma^{a_1} \dots \gamma^{a_{2n+1}}\right] = 0. \tag{49}$$

To see why, we note that since γ_5 anticommutes with the other four gamma matrices (48), we have

$$\gamma^a = -\gamma_5\gamma^a\gamma_5. \tag{50}$$

Thus, if we substitute this expression for every gamma matrix in the trace (49), then we find that all the γ_5's cancel and so that

$$\mathrm{Tr}\left[\gamma^{a_1} \dots \gamma^{a_{2n+1}}\right] = \mathrm{Tr}\left[(-\gamma_5\gamma^{a_1}\gamma_5) \dots (-\gamma_5\gamma^{a_{2n+1}}\gamma_5)\right] = (-1)^{2n+1}\mathrm{Tr}\left[\gamma^{a_1} \dots \gamma^{a_{2n+1}}\right] \tag{51}$$

which shows that the trace of an odd number of gamma matrices vanishes. Since $\mathrm{Tr}\gamma^a = 0$, this result is obviously true for $n = 0$.

Physics would be so much simpler if the trace of an even number of gamma matrices also vanished. Instead, we find

$$\mathrm{Tr}\left(\left\{\gamma^a, \gamma^b\right\}\right) = 2\mathrm{Tr}\left(\gamma^a \gamma^b\right) = \mathrm{Tr}\,\left(2\eta^{ab}\right) = 8\,\eta^{ab}, \tag{52}$$

so the trace of two gamma matrices is

$$\mathrm{Tr}\left(\gamma^a\,\gamma^b\right) = 4\eta^{ab}. \tag{53}$$

The trace of four gamma matrices, all different, is proportional to the trace of γ_5 which vanishes. So the trace of four gamma matrices is nonzero only if two of their indices are the same. But then since $\mathrm{Tr}\left(\gamma^a\,\gamma^b\right) = 4\eta^{ab}$, the trace also will vanish unless the other two indices also are equal. Thus, for instance, since $(\gamma^c)^2 = \eta^{cc}$, we have

$$\mathrm{Tr}\left(\gamma^a\,\gamma^a\,\gamma^b\,\gamma^b\,\right) = \mathrm{Tr}\eta^{aa}\eta^{bb} = 4\,\eta^{aa}\,\eta^{bb}. \tag{54}$$

So if a and b are the same, then

$$\mathrm{Tr}\,(\gamma^a\,\gamma^a\,\gamma^a\,\gamma^a\,) = \mathrm{Tr}\eta^{aa}\eta^{aa} = 4\,\eta^{aa}\,\eta^{aa} = 4. \tag{55}$$

If instead, $a \neq b$, then $\gamma^a\,\gamma^b = -\gamma^b\,\gamma^a$, and so

$$\mathrm{Tr}\left(\gamma^a\,\gamma^b\,\gamma^a\,\gamma^b\right) = -\mathrm{Tr}\left(\gamma^a\,\gamma^a\,\gamma^b\,\gamma^b\right) = -4\,\eta^{aa}\,\eta^{bb} \tag{56}$$

while

$$\mathrm{Tr}\left(\gamma^a\,\gamma^b\,\gamma^b\,\gamma^a\right) = -\mathrm{Tr}\left(\gamma^a\,\gamma^b\,\gamma^a\,\gamma^b\right) = \mathrm{Tr}\left(\gamma^a\,\gamma^a\,\gamma^b\,\gamma^b\right) = 4\,\eta^{aa}\,\eta^{bb}. \tag{57}$$

So we have

$$\mathrm{Tr}\left(\gamma^a\,\gamma^b\,\gamma^c\,\gamma^d\right) = 4\left(\eta^{ab}\,\eta^{cd} - \eta^{ac}\,\eta^{bd} + \eta^{ad}\,\eta^{bc}\right). \tag{58}$$

In general, the trace of an even number of gamma matrices will vanish unless all the indices match in pairs in which case it is ± 4. Computers programs exist that compute such traces.

4 Cross-section for Muon Pair Production

We must still evaluate the traces

$$T_1 T_2 = \mathrm{Tr}\left[(-i\not{q}' - m_\mu)\gamma^a(-i\not{p}' + m_\mu)\gamma^b\right] \mathrm{Tr}\left[(-i\not{q} - m_e)\gamma_a(-i\not{p} + m_e)\gamma_b\right] . \tag{59}$$

The first trace is

$$\begin{aligned} T_1 &= \mathrm{Tr}\left[(-i\not{q}' - m_\mu)\gamma^a(-i\not{p}' + m_\mu)\gamma^b\right] = -\mathrm{Tr}\left[\not{q}'\gamma^a\not{p}'\gamma^b\right] - m_\mu^2 \mathrm{Tr}\gamma^a\gamma^b \\ &= -q'_c p'_d \,\mathrm{Tr}\left[\gamma^c\gamma^a\gamma^d\gamma^b\right] - 4m_\mu^2\eta^{ab} = -4q'_c p'_d \left(\eta^{ca}\eta^{db} - \eta^{cd}\eta^{ab} + \eta^{bc}\eta^{ad}\right) - 4m_\mu^2\eta^{ab} \\ &= -4\left(q'^a p'^b - q'\cdot p'\eta^{ab} + q'^b p'^a + m_\mu^2\eta^{ab}\right) . \end{aligned} \tag{60}$$

And the second one is

$$\begin{aligned} T_2 &= \mathrm{Tr}\left[(-i\not{q} - m_e)\gamma_a(-i\not{p} + m_e)\gamma_b\right] = -\mathrm{Tr}\left[\not{q}\gamma_a\not{p}\gamma_b\right] - m_e^2 \mathrm{Tr}\gamma_a\gamma_b \\ &= -q^e p^f \,\mathrm{Tr}\left[\gamma_e\gamma_a\gamma_f\gamma_b\right] - 4m_e^2\eta_{ab} = -4q^e p^f \left(\eta_{ea}\eta_{fb} - \eta_{ef}\eta_{ab} + \eta_{be}\eta_{af}\right) - 4m_e^2\eta_{ab} \\ &= -4\left(q_a p_b - q\cdot p\,\eta_{ab} + q_b p_a + m_e^2\eta_{ab}\right) . \end{aligned} \tag{61}$$

So the product of the traces is

$$\begin{aligned} T_1 T_2 &= 16\left(q'^a p'^b - q'\cdot p'\eta^{ab} + q'^b p'^a + m_\mu^2\eta^{ab}\right)\left(q_a p_b - q\cdot p\,\eta_{ab} + q_b p_a + m_e^2\eta_{ab}\right) \\ &= 32\left[(q'\cdot q)(p'\cdot p) + (q'\cdot p)(p'\cdot q) - m_\mu^2 (q\cdot p) - m_e^2 (q'\cdot p') + 2m_\mu^2 m_e^2\right] . \end{aligned} \tag{62}$$

So the differential x-section in the c-o-m frame is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{e^4}{(16\pi)^2}\frac{k'}{k\,E^6} \mathrm{Tr}\left[(-i\not{q}' - m_\mu)\gamma^a(-i\not{p}' + m_\mu)\gamma^b\right] \mathrm{Tr}\left[(-i\not{q} - m_e)\gamma_a(-i\not{p} + m_e)\gamma_b\right] \\ &= \frac{e^4}{(16\pi)^2}\frac{k'}{k\,E^6}\, 32\left[(q'\cdot q)(p'\cdot p) + (q'\cdot p)(p'\cdot q) - m_\mu^2 (q\cdot p) - m_e^2 (q'\cdot p') + 2m_\mu^2 m_e^2\right] \\ &= \frac{e^4}{8\pi^2}\frac{k'}{k\,E^6}\left[(q'\cdot q)(p'\cdot p) + (q'\cdot p)(p'\cdot q) - m_\mu^2 (q\cdot p) - m_e^2 (q'\cdot p') + 2m_\mu^2 m_e^2\right] \end{aligned} \tag{63}$$

which has the right dimensions of area because the dimension of energy is inverse length in natural units.

In the c-o-m frame, with $p = (E_1, \boldsymbol{p})$, $q = (E_2, \boldsymbol{q}) = (E_1, -\boldsymbol{p})$, $p' = (E_1', \boldsymbol{p'})$, $q' = (E_2', \boldsymbol{q'}) = (E_1', -\boldsymbol{p'})$, $\boldsymbol{p} \cdot \boldsymbol{p'} = \cos\theta\, kk'$, $\boldsymbol{q} \cdot \boldsymbol{q'} = \cos\theta\, kk'$, $E_1 = E_2$, and $E_1' = E_2'$, we have

$$\begin{aligned} q' \cdot q &= kk' \cos\theta - E_1 E_1' \\ p' \cdot p &= \boldsymbol{p} \cdot \boldsymbol{p'} - E_1 E_1' = kk' \cos\theta - E_1 E_1' = q' \cdot q \\ q' \cdot p &= \boldsymbol{q'} \cdot \boldsymbol{p} - E_2' E_1 = -kk' \cos\theta - E_1 E_1' \\ p' \cdot q &= -kk' \cos\theta - E_1 E_1' = q' \cdot p \\ q' \cdot p' &= -k'^2 - E_1'^2 \\ q \cdot p &= -k^2 - E_1^2. \end{aligned} \tag{64}$$

So the differential x-section in the c-o-m frame is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{e^4}{8\pi^2} \frac{k'}{k\, E^6} \left[(kk' \cos\theta - E_1 E_1')^2 + (-kk' \cos\theta - E_1 E_1')^2 - m_\mu^2 \left(-k^2 - E_1^2 \right) - m_e^2 \left(-k'^2 - E_1'^2 \right) + 2 m_\mu^2 m_e^2 \right] \\ &= \frac{e^4}{8\pi^2} \frac{k'}{k\, E^6} \left[2k^2 k'^2 \cos^2\theta + 2 E_1^2 E_1'^2 + m_\mu^2 \left(k^2 + E_1^2 \right) + m_e^2 \left(k'^2 + E_1'^2 \right) + 2 m_\mu^2 m_e^2 \right]. \end{aligned} \tag{65}$$

In the limit in which $m_e/m_\mu \approx 1/200 \to 0$, we have $E_1 = k$ and $E = 2k$, so

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{e^4}{8\pi^2} \frac{k'}{k\, E^6} \left[2k^2 k'^2 \cos^2\theta + 2 E_1^2 E_1'^2 + m_\mu^2 \left(k^2 + E_1^2 \right) \right] \\ &= \frac{e^4}{8\pi^2} \frac{k'}{k\, (2k)^6} \left[2k^2 k'^2 \cos^2\theta + 2k^2 E_1'^2 + 2 m_\mu^2 k^2 \right] \\ &= \frac{e^4}{2^8 \pi^2} \frac{k'}{k^5} \left[k'^2 \cos^2\theta + E_1'^2 + m_\mu^2 \right] = \frac{e^4}{2^8 \pi^2} \frac{k'}{k^5} \left[k'^2 \cos^2\theta + k'^2 + 2 m_\mu^2 \right] \\ &= \frac{e^4}{2^8 \pi^2} \frac{k'}{k^5} \left[k'^2 \left(1 + \cos^2\theta \right) + 2 m_\mu^2 \right]. \end{aligned} \tag{66}$$

Now $E_1' = E_1 = k$, and $k' = \sqrt{k^2 - m_\mu^2}$, so

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{e^4}{2^8\pi^2}\,\frac{k'}{k^5}\,\left[\left(k^2 - m_\mu^2\right)\left(1+\cos^2\theta\right) + 2m_\mu^2\right] \\
&= \frac{e^4}{2^8\pi^2}\,\frac{\sqrt{k^2 - m_\mu^2}}{k^5}\,\left[\left(k^2 - m_\mu^2\right)\left(1+\cos^2\theta\right) + 2m_\mu^2\right] \\
&= \frac{e^4}{2^8\pi^2}\,\frac{\sqrt{1 - m_\mu^2/k^2}}{k^4}\,\left[\left(k^2 - m_\mu^2\right)\left(1+\cos^2\theta\right) + 2m_\mu^2\right] \\
&= \frac{e^4}{2^8\pi^2}\,\frac{\sqrt{1 - m_\mu^2/k^2}}{k^4}\,\left[k^2\left(1+\cos^2\theta\right) + m_\mu^2\left(1-\cos^2\theta\right)\right] \\
&= \frac{e^4}{2^8\pi^2}\,\frac{\sqrt{1 - m_\mu^2/k^2}}{k^2}\,\left[\left(1+\cos^2\theta\right) + \left(m_\mu^2/k^2\right)\left(1-\cos^2\theta\right)\right] \\
&= \frac{e^4}{2^8\pi^2 k^2}\sqrt{1 - \frac{m_\mu^2}{k^2}}\,\left[\left(1 + \frac{m_\mu^2}{k^2}\right) + \left(1 - \frac{m_\mu^2}{k^2}\right)\cos^2\theta\right].
\end{aligned} \tag{67}$$

Somewhat more simply, with $E \equiv E_1 = k$ and $E_{\mathrm{cm}} = 2k$, we have

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2^6\pi^2 E_{\mathrm{cm}}^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\,\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]. \tag{68}$$

With $\alpha \equiv e^2/4\pi \approx 1/137.04$, this is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\mathrm{cm}}^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\,\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right] \tag{69}$$

which is (5.12) of Peskin & Schroeder. Integrating over $d\Omega$, we get for the total x-section

$$\sigma = \frac{4\pi\alpha^2}{3E_{\mathrm{cm}}^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left(1 + \frac{1}{2}\frac{m_\mu^2}{E^2}\right) \tag{70}$$

which is their (5.13).

5 Tau Pairs

We can immediately deduce from these formulas those that apply to the process $e^+ + e^- \to \tau^+ + \tau^-$. Since the tau and muon are just heavy copies of the electron, all we need do is to replace m_e by m_τ:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\mathrm{cm}}^2}\sqrt{1-\frac{m_\tau^2}{E^2}}\left[\left(1+\frac{m_\tau^2}{E^2}\right)+\left(1-\frac{m_\tau^2}{E^2}\right)\cos^2\theta\right] \tag{71}$$

and

$$\sigma = \frac{4\pi\alpha^2}{3E_{\mathrm{cm}}^2}\sqrt{1-\frac{m_\tau^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\tau^2}{E^2}\right). \tag{72}$$

Measurements of this x-section in 1978 gave $m_\tau = 1782^{+2}_{-7}$ MeV.

6 Gauge Invariance

The reason we could generalize our formulas for muon pair production to tau pair production is that all the charged leptons are coupled to the photon in the same way. Although electrodynamics is an abelian gauge theory, we might as well consider the general case of a nonabelian gauge theory.

The action density of a Yang-Mills theory is unchanged when a space-time dependent unitary matrix $U(x)$ changes a vector $\psi(x)$ of matter fields to $\psi'(x) = U(x)\psi(x)$. Terms like $\psi^\dagger\psi$ are invariant because $\psi^\dagger(x)U^\dagger(x)U(x)\psi(x) = \psi^\dagger(x)\psi(x)$, but how can kinetic terms like $\partial_i\psi^\dagger\,\partial^i\psi$ be made invariant? Yang and Mills introduced matrices A_i of gauge fields, replaced ordinary derivatives ∂_i by **covariant derivatives** $D_i \equiv \partial_i + A_i$, and required that $D_i'\psi' = UD_i\psi$ or that

$$\left(\partial_i + A_i'\right)U = \partial_i U + U\partial_i + A_i'U = U\left(\partial_i + A_i\right). \tag{73}$$

Their nonabelian gauge transformation is

$$\begin{aligned}\psi'(x) &= U(x)\psi(x)\\ A_i'(x) &= U(x)A_i(x)U^\dagger(x) - \left(\partial_i U(x)\right)U^\dagger(x).\end{aligned} \tag{74}$$

One often writes the unitary matrix as $U(x) = \exp(-ig\,\theta_a(x)\,t_a)$ in which g is a coupling constant, the functions $\theta_a(x)$ parametrize the gauge transformation, and the generators t_a belong to the representation that acts on the vector $\psi(x)$ of matter fields.

In the case of electrodynamics, the unitary matrix is a member of the group $U(1)$; it is just a phase factor $U(x) = \exp(-ie\,\theta_a(x))$. The abelian gauge transformation is

$$\begin{aligned} \psi'(x) &= U(x)\psi(x) = e^{-ie\,\theta(x)}\psi(x) \\ A_i'(x) &= U(x)A_i(x)U^\dagger(x) - (\partial_i U(x))\,U^\dagger(x) = A_i(x) + ie\partial_i\theta(x). \end{aligned} \tag{75}$$

I have been using a notation in which A_i is antihermitian to simplify the algebra.