16
Path Integrals

16.1 Path Integrals and Classical Physics

Since Richard Feynman invented them over 60 years ago, path integrals have been used with increasing frequency in high-energy and condensed-matter physics, in finance, and in biophysics (Kleinert, 2009). Feynman used them to express matrix elements of the time-evolution operator $\exp(-itH/\hbar)$ in terms of the classical action. Others have used them to compute matrix elements of the Boltzmann operator $\exp(-H/kT)$ which in the limit of zero temperature projects out the ground state $|E_0\rangle$ of the system

$$\lim_{T\to 0} e^{-(H-E_0)/kT} = \lim_{T\to 0} \sum_{n=0}^{\infty} |E_n\rangle e^{-(E_n-E_0)/kT} \langle E_n| = |E_0\rangle\langle E_0|. \tag{16.1}$$

They are the basis of lattice gauge theory.

They magically express the quantum-mechanical probability amplitude for a process as a sum of exponentials $\exp(iS/\hbar)$ of the classical action S of the various ways that process might occur.

16.2 Gaussian Integrals

The path integrals we can do are gaussian integrals of infinite order. So we begin by recalling the basic integral formula (5.166)

$$\int_{-\infty}^{\infty} \exp\left[-ia\left(x - \frac{b}{2a}\right)^2 \right] dx = \sqrt{\frac{\pi}{ia}} \tag{16.2}$$

which holds for real a and b, and also the one (5.167)

$$\int_{-\infty}^{\infty} \exp\left[-r\left(x-\frac{c}{2r}\right)^2\right] dx = \sqrt{\frac{\pi}{r}} \tag{16.3}$$

which is true for positive r and complex c. Equivalent formulas for real a and b, positive r, and complex c are

$$\int_{-\infty}^{\infty} \exp\left(-iax^2+ibx\right)\, dx = \sqrt{\frac{\pi}{ia}}\, \exp\left(i\frac{b^2}{4a}\right) \tag{16.4}$$

$$\int_{-\infty}^{\infty} \exp\left(-rx^2+cx\right)\, dx = \sqrt{\frac{\pi}{r}}\, \exp\left(\frac{c^2}{4r}\right). \tag{16.5}$$

This last formula will be useful with $x = p$, $r = \epsilon/(2m)$, and $c = i\epsilon\dot{q}$

$$\int_{-\infty}^{\infty} \exp\left(-\epsilon\,\frac{p^2}{2m}+i\epsilon\,\dot{q}\,p\right) dp = \sqrt{\frac{2\pi m}{\epsilon}}\;\exp\left(-\epsilon\,\frac{1}{2}m\dot{q}^2\right) \tag{16.6}$$

as will (16.2) with $x = p$, $a = \epsilon/(2m)$, and $b = \epsilon\dot{q}$

$$\int_{-\infty}^{\infty} \exp\left(-i\epsilon\,\frac{p^2}{2m}+i\epsilon\,\dot{q}\,p\right) dp = \sqrt{\frac{2\pi m}{i\epsilon}}\;\exp\left(i\epsilon\,\frac{1}{2}m\dot{q}^2\right). \tag{16.7}$$

Doable path integrals are multiple gaussian integrals. One may show (exercise 16.1) that for positive $r_1, \dots, r_N$ and complex $c_1, \dots, c_N$, the integral (16.5) leads to

$$\int_{-\infty}^{\infty} \exp\left(\sum_i -r_i x_i^2 + c_i x_i\right) \prod_{i=1}^{N} dx_i = \left(\prod_{i=1}^{N}\sqrt{\frac{\pi}{r_i}}\right) \exp\left(\frac{1}{4}\sum_i \frac{c_i^2}{r_i}\right). \tag{16.8}$$

If R is the $N \times N$ diagonal matrix with positive entries $\{r_1, r_2, \dots, r_N\}$, and X and C are N-vectors with real $\{x_i\}$ and complex $\{c_i\}$ entries, then this formula (16.8) in matrix notation is

$$\int_{-\infty}^{\infty} \exp\left(-X^{\mathsf{T}}RX + C^{\mathsf{T}}X\right) \prod_{i=1}^{N} dx_i = \sqrt{\frac{\pi^N}{\det(R)}}\, \exp\left(\frac{1}{4}\,C^{\mathsf{T}}R^{-1}C\right). \tag{16.9}$$

Now every positive symmetric matrix S is of the form $S = ORO^{\mathsf{T}}$ for some positive diagonal matrix R. So inserting $R = O^{\mathsf{T}}SO$ into the previous equation (16.9) and using the invariance of determinants under orthogonal transformations, we find

$$\int_{-\infty}^{\infty} \exp\big(-X^{\mathsf{T}}O^{\mathsf{T}}SOX + C^{\mathsf{T}}X\big) \prod_{i=1}^{N} dx_i = \sqrt{\frac{\pi^N}{\det(S)}}\, \exp\left[\frac{1}{4}\,C^{\mathsf{T}}O^{\mathsf{T}}S^{-1}OC\right]. \tag{16.10}$$

The jacobian of the orthogonal transformation $Y = OX$ and $D = OC$ is unity, and so

$$\int_{-\infty}^{\infty} \exp\big(- Y^{\mathsf{T}} S Y + D^{\mathsf{T}} Y \big) \prod_{i=1}^{N} dy_i = \sqrt{\frac{\pi^N}{\det(S)}}\, \exp\left(\frac{1}{4}\, D^{\mathsf{T}} S^{-1} D\right) \quad (16.11)$$

in which S is a positive symmetric matrix, and D is a complex vector.

The other basic gaussian integral (16.4) leads for real S and D to (exercise 16.2)

$$\int_{-\infty}^{\infty} \exp\big(- iY^{\mathsf{T}} S Y + iD^{\mathsf{T}} Y \big) \prod_{i=1}^{N} dy_i = \sqrt{\frac{\pi^N}{\det(iS)}}\, \exp\left(\frac{i}{4}\, D^{\mathsf{T}} S^{-1} D\right). \quad (16.12)$$

The vector $\overline{Y}$ that makes the argument $-\, iY^{\mathsf{T}} S Y + iD^{\mathsf{T}} Y$ of the exponential of this multiple gaussian integral (16.12) stationary is (exercise 16.3)

$$\overline{Y} = \frac{1}{2} S^{-1} D. \quad (16.13)$$

The exponential of that integral evaluated at its stationary point $\overline{Y}$ is

$$\exp\big(- i\overline{Y}^{\mathsf{T}} S \overline{Y} + iD^{\mathsf{T}} \overline{Y} \big) = \exp\left(\frac{i}{4}\, D^{\mathsf{T}} S^{-1} D\right). \quad (16.14)$$

Thus, the multiple gaussian integral (16.12) is equal to its exponential evaluated at at its stationary point $\overline{Y}$, apart from a prefactor involving the determinant $\det iS$.

Similarly, the vector $\overline{Y}$ that makes the argument $-Y^{\mathsf{T}} S Y + D^{\mathsf{T}} Y$ of the exponential of the multiple gaussian integral (16.11) stationary is $\overline{Y} = S^{-1} D/2$, and that exponential evaluated at $\overline{Y}$ is

$$\exp\big(- Y^{\mathsf{T}} S Y + D^{\mathsf{T}} Y \big) = \exp\left(\frac{1}{4}\, D^{\mathsf{T}} S^{-1} D\right). \quad (16.15)$$

Once again, a multiple gaussian integral is simply its exponential evaluated at its stationary point $\overline{Y}$, apart from a prefactor involving the determinant $\det S$.

16.3 Path Integrals in Imaginary Time

At the imaginary times $t = -i\beta\hbar$, the time-evolution operator $\exp(-itH/\hbar)$ is $\exp(-\beta H)$ in which the inverse temperature $\beta = 1/kT$ is the reciprocal of Boltzmann's constant $k = 8.617 \times 10^{-5}$ eV/K times the absolute temperature T. In the low-temperature limit, $\exp(-\beta H)$ is a projection operator (16.1) on the ground state of the system. These path integrals in imaginary time also are called **euclidean** path integrals.

Let us consider a quantum-mechanical system with hamiltonian

$$H = \frac{p^2}{2m} + V(q) \tag{16.16}$$

in which the commutator of the position q and momentum p operators is $[q, p] = i$ in units in which $\hbar = 1$. For tiny ϵ, the corrections to the approximation

$$\exp\left[-\epsilon\left(\frac{p^2}{2m} + V(q)\right)\right] \approx \exp\left(-\epsilon\,\frac{p^2}{2m}\right)\exp\left(-\epsilon\,V(q)\right) + \mathcal{O}(\epsilon^2) \tag{16.17}$$

are of second order in ϵ.

To evaluate the matrix element $\langle q''|\exp(-\epsilon H)|q'\rangle$, we insert the identity operator I in the form of an integral over the momentum eigenstates

$$I = \int_{-\infty}^{\infty} |p'\rangle\langle p'|dp' \tag{16.18}$$

and use the inner product $\langle q''|p'\rangle = \exp(iq''p')/\sqrt{2\pi}$ so as to get as $\epsilon \to 0$

$$\begin{aligned}\langle q''|\exp(-\epsilon H)|q'\rangle &= \int_{-\infty}^{\infty}\langle q''|\exp\left(-\epsilon\,\frac{p^2}{2m}\right)|p'\rangle\langle p'|\exp\left(-\epsilon\,V(q)\right)|q'\rangle\,dp' \\ &= e^{-\epsilon\,V(q')}\int_{-\infty}^{\infty}\exp\left[-\epsilon\,\frac{p'^2}{2m} + i\,p'\,(q''-q')\right]\frac{dp'}{2\pi}. \end{aligned} \tag{16.19}$$

We now adopt the suggestive notation

$$\frac{q''-q'}{\epsilon} = \dot{q}' \tag{16.20}$$

and use the integral formula (16.6) so as to obtain

$$\begin{aligned}\langle q''|\exp(-\epsilon\,H)|q'\rangle &= \frac{1}{2\pi}\,e^{-\epsilon\,V(q')}\int_{-\infty}^{\infty}\exp\left(-\epsilon\,\frac{p'^2}{2m} + i\,\epsilon\,p'\,\dot{q}'\right)dp' \\ &= \left(\frac{m}{2\pi\epsilon}\right)^{1/2}\exp\left\{-\epsilon\left[\tfrac{1}{2}\,m\,\dot{q}'^2 + V(q')\right]\right\} \end{aligned} \tag{16.21}$$

in which q'' enters through the notation (16.20).

The next step is to link two of these matrix elements together

$$\begin{aligned}\langle q'''|e^{-2\epsilon H}|q'\rangle &= \int_{-\infty}^{\infty}\langle q'''|e^{-\epsilon H}|q''\rangle\langle q''|e^{-\epsilon H}|q'\rangle dq'' \qquad (16.22) \\ &= \frac{m}{2\pi\epsilon}\int_{-\infty}^{\infty}\exp\left\{-\epsilon\left[\tfrac{1}{2}\,m\,\dot{q}''^2 + V(q'') + \tfrac{1}{2}\,m\,\dot{q}'^2 + V(q')\right]\right\}dq''.\end{aligned}$$

Linking three of these matrix elements together and using subscripts instead of primes, we have

$$\begin{aligned}\langle q_3|e^{-3\epsilon H}|q_0\rangle &= \iint_{-\infty}^{\infty} \langle q_3|e^{-\epsilon H}|q_2\rangle\langle q_2|e^{-\epsilon H}|q_1\rangle\langle q_1|e^{-\epsilon H}|q_0\rangle \, dq_1 dq_2 \\ &= \left(\frac{m}{2\pi\epsilon}\right)^{3/2} \iint_{-\infty}^{\infty} \exp\left\{ -\epsilon \sum_{j=0}^{2} \left[\tfrac{1}{2}\, m\, \dot{q}_j^2 + V(q_j) \right] \right\} dq_1 dq_2.\end{aligned} \tag{16.23}$$

Boldly passing from 3 to n and suppressing some integral signs, we get

$$\begin{aligned}\langle q_n|e^{-n\epsilon H}|q_0\rangle &= \iiint_{-\infty}^{\infty} \langle q_n|e^{-\epsilon H}|q_{n-1}\rangle \ldots \langle q_1|e^{-\epsilon H}|q_0\rangle \, dq_1 \ldots dq_{n-1} \\ &= \left(\frac{m}{2\pi\epsilon}\right)^{n/2} \iiint_{-\infty}^{\infty} \exp\left\{ -\epsilon \sum_{j=0}^{n-1} \left[\tfrac{1}{2}\, m\, \dot{q}_j^2 + V(q_j) \right] \right\} dq_1 \ldots dq_{n-1}.\end{aligned} \tag{16.24}$$

Writing dt for ϵ and taking the limits $\epsilon \to 0$ and $n \equiv \beta/\epsilon \to \infty$, we find that the matrix element $\langle q_\beta|e^{-\beta H}|q_0\rangle$ is a path integral of the exponential of the average energy multiplied by $-\beta$

$$\langle q_\beta|e^{-\beta H}|q_0\rangle = \int \exp\left[-\int_0^\beta \tfrac{1}{2} m \dot{q}^2(t) + V(q(t)) \, dt \right] Dq \tag{16.25}$$

in which $Dq \equiv (n\, m/2\pi\beta)^{n/2}\, dq_1 dq_2 \cdots dq_{n-1}$ as $n \to \infty$. We sum over all paths $q(t)$ that go from $q(0) = q_0$ at inverse temperature $\beta = 0$ to $q(\beta) = q_\beta$ at inverse temperature β.

In the limit $\beta \to \infty$, the operator $\exp(-\beta H)$ becomes proportional to a projection operator (16.1) on the ground state of the theory.

Path integrals in imaginary time are called *euclidean* mainly to distinguish them from *Minkowski* path integrals, which represent matrix elements of the time-evolution operator $\exp(-itH)$ in real time.

In three-dimensional space, $\boldsymbol{q}(t)$ replaces $q(t)$ in equation (16.25)

$$\langle \boldsymbol{q}_\beta|e^{-\beta H}|\boldsymbol{q}_0\rangle = \int \exp\left[-\int_0^\beta \tfrac{1}{2} m \dot{\boldsymbol{q}}^2 + V(\boldsymbol{q}) \, dt \right] D\boldsymbol{q}. \tag{16.26}$$

16.4 Path Integrals in Real Time

Path integrals in real time represent the time-evolution operator $\exp(-itH)$. Using the integral formula (16.7), we find in the limit $\epsilon \to 0$

$$
\begin{aligned}
\langle q''|e^{-i\,\epsilon\,H}|q'\rangle &= \int_{-\infty}^{\infty} \langle q''|\exp\left[-\,i\,\epsilon\,\frac{p^2}{2m}\right]|p'\rangle\langle p'|\exp\left[-\,i\,\epsilon\,V(q)\right]|q'\rangle\,dp' \\
&= \frac{1}{2\pi}\,e^{-i\,\epsilon\,V(q')}\int_{-\infty}^{\infty}\exp\left[-\,i\,\epsilon\,\frac{p'^2}{2m}+i\,p'\,(q''-q')\right]\,dp' \\
&= \frac{1}{2\pi}\,e^{-i\,\epsilon\,V(q')}\int_{-\infty}^{\infty}\exp\left[-\,i\,\epsilon\,\frac{p'^2}{2m}+i\,\epsilon\,p'\dot{q}'\right]\,dp' \\
&= \left(\frac{m}{2\pi i\epsilon}\right)^{1/2}\exp\left[i\,\epsilon\,\left(\frac{m\,\dot{q}'^2}{2}-V(q')\right)\right]. \qquad (16.27)
\end{aligned}
$$

When we link together n of these matrix elements, we get the real-time version of (16.25)

$$
\langle q_n|e^{-in\epsilon H}|q_0\rangle = \left(\frac{m}{2\pi i\epsilon}\right)^{n/2}\iiint_{-\infty}^{\infty}\exp\left\{i\epsilon\sum_{j=0}^{n-1}\left[\tfrac{1}{2}\,m\,\dot{q}_j^2 - V(q_j)\right]\right\}dq_1\ldots dq_{n-1}. \qquad (16.28)
$$

Writing dt for ϵ and taking the limits $\epsilon \to 0$ and $n\epsilon \to t$, we find that the amplitude $\langle q_t|e^{-itH}|q_0\rangle$ is the path integral

$$
\langle q_t|e^{-itH}|q_0\rangle = \int \exp\left[i\int_0^t \tfrac{1}{2}\,m\,\dot{q}^2 - V(q)\,dt'\right]Dq \qquad (16.29)
$$

in which Dq differs from the one that appears in euclidian path integrals by the substitution $\beta \to it$

$$
Dq = \lim_{n\to\infty}\left(\frac{nm}{2\pi it}\right)^{n/2} dq_1 dq_2\cdots dq_{n-1}. \qquad (16.30)
$$

The integral in the exponent is the **classical action**

$$
S[q] = \int_0^t \tfrac{1}{2}\,m\,\dot{q}^2 - V(q)\,dt' \qquad (16.31)
$$

for a process $q(t')$ that runs from $q(0) = q_0$ to $q(t) = q_t$. We sum over all such processes.

In three-dimensional space

$$
\langle \boldsymbol{q}_t|e^{-itH}|\boldsymbol{q}_0\rangle = \int \exp\left[i\int_0^t \tfrac{1}{2}\,m\,\dot{\boldsymbol{q}}^2 - V(\boldsymbol{q})\,dt'\right]D\boldsymbol{q} \qquad (16.32)
$$

replaces (16.29).

The units of action are energy × time, and the argument of the exponential must be dimensionless, so in ordinary units the amplitude (16.29) is

$$\langle q_t|e^{-itH/\hbar}|q_0\rangle = \int e^{iS[q]/\hbar} Dq. \tag{16.33}$$

When is this amplitude big? When is it tiny? Suppose there is a process $q(t) = q_c(t)$ that goes from $q_c(0) = q_0$ to $q_c(t) = q_t$ in time t and that obeys the classical equation of motion (15.14–15.15)

$$\frac{\delta S[q_c]}{\delta q_c} = m\ddot{q}_c + V'(q_c) = 0. \tag{16.34}$$

The action of such a classical process is stationary, that is, $S[q_c + dq]$ differs from $S[q_c]$ only by terms of second order in δq. So there are infinitely many other processes that have the same action to within a fraction of $\hbar$. These processes add with nearly the same phase to the path integral (16.33) and so make a huge contribution to the amplitude $\langle q_t|e^{-itH}|q_0\rangle$.

But if no classical process goes from q_0 to q_t in time t, then the nonclassical processes from q_0 to q_t in time t have actions that differ among themselves by large multiples of $\hbar$. Their amplitudes cancel each other, and so the resulting amplitude that is tiny. Thus **the real-time path integral explains the principle of stationary action** (section 11.37).

Does this path integral satisfy Schrödinger's equation? To see that it does, we'll use (16.27) in the more explicit form

$$\langle q''|e^{-i\epsilon H}|q'\rangle = \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \exp\left[i\,\frac{m(q''-q')^2}{2\epsilon} - i\,\epsilon\,V(q')\right] \tag{16.35}$$

to write $\psi(q'', t+\epsilon) = \langle q''|\psi, t+\epsilon\rangle$ as an integral of $\psi(q', t) = \langle q'|\psi, t\rangle$

$$\begin{aligned}\langle q''|\psi, t+\epsilon\rangle &= \int \langle q''|e^{-i\epsilon H}|q'\rangle\langle q'|\psi, t\rangle\, dq' \qquad (16.36)\\ &= \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \int \exp\left[i\,\frac{m(q''-q')^2}{2\epsilon} - i\,\epsilon\,V(q')\right]\langle q'|\psi, t\rangle\, dq'.\end{aligned}$$

Keeping only leading terms in ϵ, we have

$$\psi(q'', t+\epsilon) = \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} e^{-i\epsilon V(q'')} \int_{-\infty}^{\infty} \exp\left[i\frac{m(q''-q')^2}{2\epsilon}\right]\psi(q', t)\, dq'. \tag{16.37}$$

Letting $x = q' - q''$ and $q' = q'' + x$, we have

$$\psi(q'', t+\epsilon) = \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} e^{-i\epsilon V(q'')} \int \exp\left[i\frac{mx^2}{2\epsilon}\right]\psi(q''+x, t)\, dx. \tag{16.38}$$

We now expand $\psi(q'' + x, t)$ as

$$\psi(q'' + x, t) = \psi(q'', t) + x\psi'(q'', t) + \frac{1}{2}x^2\psi''(q'', t) + \dots \tag{16.39}$$

and $\psi(q'', t + \epsilon)$ as

$$\psi(q'', t + \epsilon) = \psi(q'', t) + \epsilon\dot{\psi}(q'', t) + \dots . \tag{16.40}$$

The integral formula (16.2) implies

$$\int_{-\infty}^{\infty} e^{imx^2/2\epsilon}\, dx = \left(\frac{2\pi i\epsilon}{m}\right)^{1/2} \tag{16.41}$$

and its derivative with respect to $im/2\epsilon$ gives

$$\int_{-\infty}^{\infty} x^2\, e^{imx^2/2\epsilon}\, dx = \frac{i\epsilon}{m}\left(\frac{2\pi i\epsilon}{m}\right)^{1/2} \quad \text{while} \quad \int_{-\infty}^{\infty} x\, e^{imx^2/2\epsilon}\, dx = 0. \tag{16.42}$$

Substituting the expansions (16.39 & 16.40) for $\psi(q'' + x, t)$ and $\psi(q'', t + \epsilon)$ into the integral (16.38) and using the integral formulas (16.41 & 16.41), we get

$$\psi(q'', t) + \epsilon\dot{\psi}(q'', t) = \left[1 - i\epsilon V(q'')\right]\left[\psi(q'', t) + \frac{i\epsilon}{m}\,\psi''(q'', t)\right] \tag{16.43}$$

which is Schrödinger's equation

$$i\,\dot{\psi} = -\frac{1}{2m}\,\psi'' + V\,\psi \tag{16.44}$$

in natural units or $i\hbar\,\dot{\psi} = -\hbar^2\,\psi''/2m + V\psi$ in arbitrary units.

16.5 Path Integral for a Free Particle

The amplitide for a free nonrelativistic particle to go from the origin to the point $\boldsymbol{q}$ in time t is the path integral (16.32)

$$\langle \boldsymbol{q}|e^{-itH}|\boldsymbol{q} = \boldsymbol{0}\rangle = \int e^{iS_0[\boldsymbol{q}]}\, D\boldsymbol{q} = \int \exp\left(i\int_0^t \tfrac{1}{2}\, m\,\dot{\boldsymbol{q}}^2(t')\, dt'\right) D\boldsymbol{q}. \tag{16.45}$$

The classical path that goes from $\boldsymbol{0}$ to $\boldsymbol{q}$ in time t is $\boldsymbol{q}_c(t') = (t'/t)\,\boldsymbol{q}$. The general path $\boldsymbol{q}(t')$ over which we integrate is $\boldsymbol{q}(t') = \boldsymbol{q}_c(t') + \boldsymbol{\delta q}(t')$. Since both $\boldsymbol{q}(t')$ and $\boldsymbol{q}_c(t')$ go from $\boldsymbol{0}$ to $\boldsymbol{q}$ in time t, the otherwise arbitrary path $\boldsymbol{\delta q}(t')$ must be a loop that goes from $\boldsymbol{\delta q}(0) = 0$ to $\boldsymbol{\delta q}(t) = 0$ in time t. The velocity $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_c + \dot{\boldsymbol{\delta q}}$ is the sum of the constant classical velocity $\dot{\boldsymbol{q}}_c = \boldsymbol{q}/t$ and the loop velocity $\dot{\boldsymbol{\delta q}}$. The first-order change vanishes

$$m \int_{t_1}^{t_2} \dot{\boldsymbol{q}}_c \cdot \frac{d\boldsymbol{\delta q}}{dt}\, dt = m\, \dot{\boldsymbol{q}}_c \cdot \int_{t_1}^{t_2} \frac{d\boldsymbol{\delta q}}{dt}\, dt = m\, \dot{\boldsymbol{q}}_c \cdot [\boldsymbol{\delta q}(t_2) - \boldsymbol{\delta q}(t_1)] = 0 \quad (16.46)$$

and so the action $S_0[\boldsymbol{q}]$ is the classical action plus the loop action

$$S_0[\boldsymbol{q}] = \tfrac{1}{2}\, m \int_0^t \left(\dot{\boldsymbol{q}}_c + \dot{\boldsymbol{\delta q}}\right)^2 dt' = S_0[\boldsymbol{q}_c] + S_0[\boldsymbol{\delta q}]. \quad (16.47)$$

The path integral therefore factorizes

$$\begin{aligned} \langle \boldsymbol{q}|e^{-i\,t\,H}|\boldsymbol{0}\rangle &= \int e^{iS_0[\boldsymbol{q}]}\, D\boldsymbol{q} = \int e^{iS_0[\boldsymbol{q}_c + \boldsymbol{\delta q}]}\, D\boldsymbol{\delta q} \\ &= \int e^{iS_0[\boldsymbol{q}_c]}\, e^{iS_0[\boldsymbol{\delta q}]}\, D\boldsymbol{\delta q} \\ &= e^{iS_0[\boldsymbol{q}_c]} \int e^{iS_0[\boldsymbol{\delta q}]}\, D\boldsymbol{\delta q} \end{aligned} \quad (16.48)$$

into the phase of the classical action times a path integral over the loops. The loop integral L is independent of the spatial points $\boldsymbol{q}$ and $\boldsymbol{0}$ and so can only depend upon the time interval, $L = L(t)$. Thus the amplitude is the product

$$\langle \boldsymbol{q}|e^{-i\,t\,H}|\boldsymbol{0}\rangle = e^{iS_0[\boldsymbol{q}_c]}\, L(t). \quad (16.49)$$

Since the classical velocity is $\dot{\boldsymbol{q}}_c = \boldsymbol{q}/t$, the classical action is

$$S_0[\boldsymbol{q}_c] = \int_0^T \frac{m}{2}\, \dot{\boldsymbol{q}}_c^2(t)\, dt = \frac{m}{2}\, \frac{\boldsymbol{q}^2}{t}. \quad (16.50)$$

So the amplitude is

$$\langle \boldsymbol{q}|e^{-i(t_2-t_1)H}|\boldsymbol{0}\rangle = e^{im\boldsymbol{q}^2/2t}\, L(t). \quad (16.51)$$

Since the position eigenstates are orthogonal, this amplitude must reduce to a delta function as $t \to 0$

$$\lim_{t\to 0} \langle \boldsymbol{q}|e^{-i\,t\,H}|\boldsymbol{0}\rangle = \langle \boldsymbol{q}|\boldsymbol{0}\rangle = \delta^3(\boldsymbol{q}). \quad (16.52)$$

One of the many representations of Dirac's delta function is

$$\delta^3(\boldsymbol{q}) = \lim_{t\to 0} \left(\frac{m}{2\pi i\hbar t}\right)^{3/2} e^{im\boldsymbol{q}^2/2\hbar t}. \quad (16.53)$$

Thus $N\, L(t) = (m/2\pi i t)^{3/2}$ and

$$\langle \boldsymbol{q}|e^{-itH/\hbar}|\boldsymbol{0}\rangle = \left(\frac{m}{2\pi i\hbar t}\right)^{3/2} e^{im\boldsymbol{q}^2/2\hbar t} \quad (16.54)$$

in unnatural units. You can verify (exercise 16.6) this result by inserting a complete set of momentum dyadics $|p\rangle\langle p|$ and doing the resulting Fourier transform.

Example 16.1 (The Bohm-Aharonov Effect) From our formula (11.311) for the action of a relativistic particle of mass m and charge q, we infer (exercise 16.7) that the action a nonrelativistic particle in an electromagnetic field with no scalar potential is

$$S = \int_{x_1}^{x_2} [m\boldsymbol{v} + q\,\boldsymbol{A}] \cdot d\boldsymbol{x}\,. \tag{16.55}$$

Now imagine that we shoot a beam of such particles past but not through a narrow cylinder in which a magnetic field is confined. The particles can go either way around the cylinder of area S but cannot enter the region of the magnetic field. The difference in the phases of the amplitudes is the loop integral from the source to the detector and back to the source

$$\frac{\Delta S}{\hbar} = \oint [m\boldsymbol{v} + q\,\boldsymbol{A}] \cdot \frac{d\boldsymbol{x}}{\hbar} = \oint \frac{m\boldsymbol{v} \cdot d\boldsymbol{x}}{\hbar} + \frac{q}{\hbar}\int_S \boldsymbol{B} \cdot d\boldsymbol{S} = \oint \frac{m\boldsymbol{v} \cdot d\boldsymbol{x}}{\hbar} + \frac{q\Phi}{\hbar} \tag{16.56}$$

in which Φ is the magnetic flux through the cylinder. □

16.6 Free Particle in Imaginary Time

If we mimic the steps of the preceding section (16.5) in which the hamiltonian is $H = \boldsymbol{p}^2/2m$, set $\beta = it/\hbar = 1/kT$, and use Dirac's delta function

$$\delta^3(\boldsymbol{q}) = \lim_{t\to 0} \left(\frac{m}{2\pi\hbar t}\right)^{3/2} e^{-m\boldsymbol{q}^2/2\hbar t} \tag{16.57}$$

then we get

$$\langle \boldsymbol{q}|e^{-\beta H}|\boldsymbol{0}\rangle = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} \exp\left[-\frac{m\boldsymbol{q}^2}{2\hbar^2\beta}\right] = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} e^{-mkT\boldsymbol{q}^2/2\hbar^2}. \tag{16.58}$$

To study the ground state of the system, we set $\beta = t/\hbar$ and let $t \to \infty$ in

$$\langle \boldsymbol{q}|e^{-tH/\hbar}|\boldsymbol{0}\rangle = \left(\frac{m}{2\pi\hbar t}\right)^{3/2} \exp\left[-\frac{m}{2}\frac{\boldsymbol{q}^2}{\hbar t}\right] \tag{16.59}$$

which for $D = \hbar/(2m)$ is the solution (3.206 & 13.107) of the diffusion equation.

16.7 Harmonic Oscillator in Real Time

Biologists have mice; physicists have harmonic oscillators with hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{16.60}$$

For this hamiltonian, our formula (16.29) for the coordinate matrix elements of the time-evolution operator $\exp(-itH)$ is

$$\langle q''|e^{-i\,t\,H}|q'\rangle = \int e^{iS[q]} Dq \tag{16.61}$$

with action

$$S[q] = \int_0^t \tfrac{1}{2}m\dot{q}^2(t') - \tfrac{1}{2}m\omega^2 q^2(t')\,dt'. \tag{16.62}$$

The classical solution $q_c(t) = q' \cos\omega t + \dot{q}_0 \sin(\omega t)/\omega$ in which $q' = q_c(0)$ and $\dot{q}_0 = \dot{q}_c(0)$ are the initial position and velocity makes the action $S[q]$ stationary

$$\frac{d}{d\epsilon} S[q + \epsilon h]\bigg|_{\epsilon=0} = 0 \tag{16.63}$$

and satisfies the classical equation of motion $m\ddot{q}_c(t) = -\,\omega^2 q_c(t)$.

We now apply the trick (16.46–16.48) we used for the free particle. We write an arbitrary process $q(t)$ is the sum of the classical process $q_c(t)$ and a loop $\delta q(t)$ with $\delta q(0) = \delta q(t) = 0$. Since the action $S[q]$ is quadratic in the variables q and $\dot{q}$, the functional Taylor series (15.31) for $S[q_c + \delta q]$ has only two terms

$$S[q_c + \delta q] = S[q_c] + S[\delta q]. \tag{16.64}$$

Thus we can write the path integral (16.61) as

$$\begin{aligned} \langle q''|e^{-itH}|q'\rangle &= \int e^{iS[q]}\, Dq = \int e^{iS[q_c+\delta q]}\, D\delta q \\ &= \int e^{iS[q_c]+iS[\delta q]}\, D\delta q = e^{iS[q_c]} \int e^{iS[\delta q]}\, D\delta q. \end{aligned} \tag{16.65}$$

The remaining path integral over the loops δq does not involve the end points q' and q'' and so must be a function $L(t)$ of the time t but not of q' or q''

$$\langle q''|e^{-itH}|q'\rangle = e^{iS[q_c]}\, L(t). \tag{16.66}$$

The action $S[q_c]$ is (exercise 16.8)

$$S[q_c] = \frac{m\omega}{2\sin(\omega t)} \left[\left(q'^2 + q''^2\right)\cos(\omega t) - 2q'q'' \right]. \tag{16.67}$$

The action $S[\delta q]$ of a loop

$$\delta q(t') = \sum_{n=1}^{n-1} a_j \, \sin \frac{j\pi t'}{t} \tag{16.68}$$

is (exercise 16.9)

$$S[\delta q] = \sum_{j=1}^{n-1} \frac{mt}{4} \, a_j^2 \left[\frac{(j\pi)^2}{t^2} - \omega^2 \right]. \tag{16.69}$$

The path integral over the loops is then, apart from a constant jacobian J,

$$\begin{aligned} \int e^{iS[\delta q]} \, D\delta q &= J \left(\frac{nm}{2\pi i t} \right)^{n/2} \int \exp \left\{ \sum_{j=1}^{n-1} \frac{imt}{4} \, a_j^2 \left[\frac{(j\pi)^2}{t^2} - \omega^2 \right] \right\} \prod_{j=1}^{n-1} da_j \\ &= J \left(\frac{nm}{2\pi i t} \right)^{n/2} \prod_{j=1}^{n-1} \int_{-\infty}^{\infty} \exp \left\{ \frac{imt}{4} \, a_j^2 \left[\frac{(j\pi)^2}{t^2} - \omega^2 \right] \right\} da_j . \end{aligned} \tag{16.70}$$

Using the gaussian integral (16.2) and the infinite product (4.140), we get

$$\begin{aligned} \int e^{iS[\delta q]} \, D\delta q &= J n^{n/2} \sqrt{\frac{m}{2\pi i t}} \prod_{j=1}^{n-1} \frac{\sqrt{2}}{j\pi} \left(1 - \frac{\omega^2 t^2}{\pi^2 j^2} \right)^{-1/2} \\ &= \sqrt{\frac{m\omega}{2\pi i \sin \omega t}} \left(\lim_{n\to\infty} J n^{n/2} \prod_{j=1}^{n-1} \frac{\sqrt{2}}{j\pi} \right) . \end{aligned} \tag{16.71}$$

Using (16.66) and (16.67), we see that the number within the parenthesis is unity because in that case we have (Feynman and Hibbs, 1965, ch. 3)

$$\langle q''|e^{-itH/\hbar}|q'\rangle = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t)}} \, \exp \left[i \frac{m\omega \left[\left(q'^2 + q''^2 \right) \cos(\omega t) - 2q'q'' \right]}{2\hbar \sin(\omega t)} \right] \tag{16.72}$$

which agrees with the amplitude (16.54) in the limit $t \to 0$ (exercise 16.10).

16.8 Harmonic Oscillator in Imaginary Time

For the harmonic oscillator with hamiltonian (16.60), our formula (16.25) for euclidean path integrals becomes

$$\langle q''|e^{-\beta H}|q'\rangle = \int \exp \left\{ - \int_0^{\beta} \tfrac{1}{2} m \dot{q}^2(t) + \tfrac{1}{2} m\omega^2 q^2(t) \, dt \right\} Dq. \tag{16.73}$$

The euclidean action, which is a time integral of the energy of the oscillator,

$$S_e[q] = \int_0^\beta \left[\tfrac{1}{2}m\dot{q}^2(t) + \tfrac{1}{2}m\omega^2 q^2(t)\right] dt \tag{16.74}$$

is purely quadratic, and so we may play the trick (15.32) if we can find a path $q_e(t)$ that makes it stationary

$$\begin{aligned}\delta S_e[q_e][h] &= \frac{d}{d\epsilon}\int_0^\beta \tfrac{1}{2}m(\dot{q}_e(t)+\epsilon\dot{h}(t))^2 + \tfrac{1}{2}m\omega^2(q_e(t)+\epsilon h(t))^2\, dt\Big|_{\epsilon=0}\\ &= \int_0^\beta m\dot{q}_e(t)\dot{h}(t) + m\omega^2 q_e(t)h(t)\, dt\\ &= \int_0^\beta \left[-m\ddot{q}_e(t) + m\omega^2 q_e(t)\right] h(t)dt = 0.\end{aligned} \tag{16.75}$$

The path $q_e(t)$ must satisfy the euclidean equation of motion

$$\ddot{q}_e(t) = \omega^2 q_e(t) \tag{16.76}$$

whose general solution is

$$q_e(t) = Ae^{\omega t} + Be^{-\omega t}. \tag{16.77}$$

The path from $q_e(0) = q'$ to $q_e(\beta) = q''$ must have

$$A = \frac{q'' e^{-\omega\beta} - q' e^{-2\omega\beta}}{1 - e^{-2\omega\beta}} \quad \text{and} \quad B = q' - A. \tag{16.78}$$

Its action $S_e[q_e]$ is (exercise 16.12)

$$S_e[q_e] = \tfrac{1}{2}m\omega\left[A^2\left(e^{2\omega\beta}-1\right) - B^2\left(e^{-2\omega\beta}-1\right)\right]. \tag{16.79}$$

Since the action is purely quadratic, the trick (15.32) tells us that the action $S_e[q]$ of the arbitrary path $q(t) = q_e(t) + \delta q(t)$ is the sum

$$S_e[q] = S_e[q_e] + S_e[\delta q] \tag{16.80}$$

in which the action $S_e[\delta q]$ of the loop $\delta q(t)$ depends only upon t and not upon q_β or q_0. It follows then that for some loop function $L(\beta)$ of β alone

$$\begin{aligned}\langle q''|e^{-\beta H}|q'\rangle &= \exp\left(-S_e[q_e]\right)\, L(\beta)\\ &= L(\beta)\, \exp\Big\{-\tfrac{1}{2}m\omega\left[A^2\left(e^{2\omega\beta}-1\right) - B^2\left(e^{-2\omega\beta}-1\right)\right]\Big\}.\end{aligned} \tag{16.81}$$

To study the ground state of the harmonic oscillator, we let $\beta \to \infty$ in this equation. Inserting a complete set of eigenstates $H|n\rangle = E_n|n\rangle$, we see that the limit of the left-hand side is

$$\lim_{\beta\to\infty} \langle q''|e^{-\beta H}|q'\rangle = \lim_{\beta\to\infty} \langle q''|n\rangle e^{-\beta E_n}\langle n|q'\rangle = e^{-\beta E_0}\langle q''|0\rangle\langle 0|q'\rangle. \quad (16.82)$$

Our formulas (16.78) for A and B say that $A \to q'' e^{-\omega\beta}$ and $B \to q'$ as $\beta \to \infty$, and so in this limit by (16.81 & 16.82) we have

$$e^{-\beta E_0}\langle q''|0\rangle\langle 0|q'\rangle = L(\beta)\, \exp\left[-\tfrac{1}{2}m\omega\left(q''^2 + q'^2\right)\right] \quad (16.83)$$

from which may infer our earlier formula (15.44) for the wave function of the ground state

$$\langle q|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{1}{2}\frac{m\omega q^2}{\hbar}\right] \quad (16.84)$$

in which the prefactor ensures the normalization

$$1 = \int_{-\infty}^{\infty} |\langle q|0\rangle|^2 dq. \quad (16.85)$$

Euclidean path integrals help one study ground states.

16.9 Euclidean Correlation Functions

In the Heisenberg picture, the position operator $q(t)$ is

$$q(t) = e^{itH} q\, e^{-itH} \quad (16.86)$$

in which $q = q(0)$ is the position operator at time $t = 0$ or equivalently the position operator in the Schrödinger picture. The analogous operator in imaginary time is the euclidean position operator $q_e(t)$ defined as

$$q_e(t) = e^{tH} q\, e^{-tH} \quad (16.87)$$

obtained from $q(t)$ by replacing t by $-it$.

The time-ordered product of two euclidean position operators is

$$\mathcal{T}\left[q_e(t_1)\, q_e(t_2)\right] = \theta(t_1 - t_2) q_e(t_1)\, q_e(t_2) + \theta(t_2 - t_1) q_e(t_2)\, q_e(t_1) \quad (16.88)$$

in which $\theta(x) = (x + |x|)/2|x|$ is Heaviside's function. We can use the method of section 16.3 to compute the matrix element of the **euclidean-time-ordered** product $\mathcal{T}\left[q(t_1)q(t_2)\right]$ sandwiched between two factors of $\exp(-tH)$. For $t_1 \ge t_2$, this matrix element is

$$\langle q_t|e^{-tH} q_e(t_1) q_e(t_2) e^{-tH}|q_{-t}\rangle = \langle q_t|e^{-(t-t_1)H} q e^{-(t_1-t_2)H} q e^{-(t+t_2)H}|q_{-t}\rangle. \quad (16.89)$$

Then instead of the path-integral formula (16.25), we get

$$\langle q_t | e^{-tH} \mathcal{T} \left[q_e(t_1) q_e(t_2) \right] e^{-tH} | q_{-t} \rangle = \int q(t_1) q(t_2) e^{-S_e[q,t,-t]} \, Dq \tag{16.90}$$

where as in (16.25) $S_e[q, t, -t]$ is the **euclidean action**

$$S_e[q, t, -t] = \int_{-t}^{t} \tfrac{1}{2} m \dot{q}^2(t') + V(q(t')) \, dt' \tag{16.91}$$

or the time-integral of the energy. As in the path integral (16.25), the integration is over all paths that go from $q(-t) = q_{-t}$ to $q(t) = q_t$. The analog of (16.25) is

$$\langle q_t | e^{-2tH} | q_{-t} \rangle = \int e^{-S_e[q,t,-t]} \, Dq \tag{16.92}$$

and the factors $(nm/2\pi\beta)^{n/2}$ cancel in the ratio of (16.90) to (16.92)

$$\frac{\langle q_t | e^{-tH} \mathcal{T} \left[q_e(t_1) q_e(t_2) \right] e^{-tH} | q_{-t} \rangle}{\langle q_t | e^{-2tH} | q_{-t} \rangle} = \frac{\displaystyle\int q(t_1) q(t_2) e^{-S_e[q,t,-t]} \, Dq}{\displaystyle\int e^{-S_e[q,t,-t]} \, Dq}. \tag{16.93}$$

In the limit $t \to \infty$, the operator $\exp(-tH)$ projects out the ground state $|0\rangle$ of the system

$$\lim_{t \to \infty} e^{-tH} | q_{-t} \rangle = \lim_{t \to \infty} \sum_{n=0}^{\infty} e^{-tH} |n\rangle \langle n | q_{-t} \rangle = \lim_{t \to \infty} e^{-tE_0} |0\rangle \langle 0 | q_{-t} \rangle \tag{16.94}$$

which we assume to be unique and normalized to unity. In the ratio (16.93), most of these factors cancel, leaving us with

$$\langle 0 | \mathcal{T} \left[q_e(t_1) q_e(t_2) \right] |0\rangle = \frac{\displaystyle\int q(t_1) q(t_2) e^{-S_e[q,\infty,-\infty]} \, Dq}{\displaystyle\int e^{-S_e[q,\infty,-\infty]} \, Dq}. \tag{16.95}$$

More generally, the mean value in the ground state $|0\rangle$ of *any* euclidean-time-ordered product of position operators $q(t_i)$ is a ratio of path integrals

$$\langle 0 | \, \mathcal{T} \left[q(t_1) \dots q(t_n) \right] |0\rangle = \frac{\displaystyle\int q(t_1) \dots q(t_n) \, e^{-S_e[q]} \, Dq}{\displaystyle\int e^{-S_e[q]} \, Dq} \tag{16.96}$$

in which $S_e[q]$ stands for $S_e[q, \infty, -\infty]$. Why do we need the time-ordered product $\mathcal{T}$ on the LHS? Because successive factors of $\exp(-(t_k - t_\ell)H)$ lead to the path integral of $\exp(-S_e[q])$. Why don't we need $\mathcal{T}$ on the RHS? Because the $q(t_i)$'s are real numbers which commute with each other.

The result (16.96) is important because it can be generalized to all quantum theories, including field theories.

16.10 Finite-Temperature Field Theory

Matrix elements of the operator $\exp(-\beta H)$ where $\beta = 1/(kT)$ tell us what a system is like at temperature T. In the low-temperature limit, they describe the ground state of the system.

Quantum mechanics imposes upon n coordinates q_i and conjugate momenta π_k the commutation relations

$$[q_i, p_k] = i\,\delta_{i,k} \quad \text{and} \quad [q_i, q_k] = [p_i, p_k] = 0. \tag{16.97}$$

In quantum field theory, we associate a coordinate $q_{\boldsymbol{x}} \equiv \phi(\boldsymbol{x})$ and a conjugate momentum $p_{\boldsymbol{x}} \equiv \pi(\boldsymbol{x})$ with each point $\boldsymbol{x}$ of space and impose upon them the very similar commutation relations

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = i\,\delta(\boldsymbol{x} - \boldsymbol{y}) \quad \text{and} \quad [\phi(\boldsymbol{x}), \phi(\boldsymbol{y})] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{y})] = 0. \tag{16.98}$$

Just as in quantum mechanics the time derivative of a coordinate is its commutator with a hamiltonian $\dot{q}_i = i[H, q_i]$, so too in quantum field theory the time derivative of a field $\dot{\phi}(\boldsymbol{x}, t) \equiv \dot{\phi}(x)$ is $\dot{\phi}(x) = i[H, \phi(x)]$. A typical hamiltonian for a single scalar field ϕ is

$$H = \int \left[\frac{1}{2}\pi^2(\boldsymbol{x}) + \frac{1}{2}\left(\nabla\phi(\boldsymbol{x})\right)^2 + \frac{1}{2}m^2\phi^2(\boldsymbol{x}) + P(\phi(\boldsymbol{x})) \right] d^3x \tag{16.99}$$

in which P is a quartic polynomial.

Since quantum field theory is just the quantum mechanics of many variables, we can use the methods of sections 16.3 & 16.4 to write matrix elements of $\exp(-\beta H)$ as path integrals. We define a potential

$$V(\phi(\boldsymbol{x})) = \frac{1}{2}\left(\nabla\phi(\boldsymbol{x})\right)^2 + \frac{1}{2}m^2\phi^2(\boldsymbol{x}) + P(\phi(\boldsymbol{x})) \tag{16.100}$$

and write the hamiltonian H as

$$H = \int \left[\frac{1}{2}\pi^2(\boldsymbol{x}) + V(\phi(\boldsymbol{x})) \right] d^3x. \tag{16.101}$$

Like $|q'\rangle$ and $|p'\rangle$, the states $|\phi'\rangle$ and $|\pi'\rangle$ are eigenstates of the hermitian operators $\phi(\boldsymbol{x}, 0)$ and $\pi(\boldsymbol{x}, 0)$

$$\phi(\boldsymbol{x}, 0)|\phi'\rangle = \phi'(\boldsymbol{x})|\phi'\rangle \quad \text{and} \quad \pi(\boldsymbol{x}, 0)|\pi'\rangle = \pi'(\boldsymbol{x})|\pi'\rangle. \tag{16.102}$$

The analog of $\langle q'|p'\rangle$ is

$$\langle \phi'|\pi'\rangle = f\ \exp\left[i\int \phi'(\boldsymbol{x})\pi'(\boldsymbol{x})d^3x\right] \tag{16.103}$$

in which f is a factor which eventually will cancel.

Repeating our derivation of Eq.(16.21) with

$$D\pi' \equiv \prod_{\boldsymbol{x}} d\pi'(\boldsymbol{x}) \tag{16.104}$$

we find in the limit $\epsilon \to 0$

$$\begin{aligned}\langle \phi''|\exp(-\epsilon H)|\phi'\rangle &= \int \langle \phi''|\exp\left(-\frac{\epsilon}{2}\int \pi^2(\boldsymbol{x})d^3x\right)|\pi'\rangle \\ &\qquad \times \langle \pi'|\exp\left(-\epsilon\int V(\phi(\boldsymbol{x}))\,d^3x\right)|\phi'\rangle\, D\pi' \\ &= |f|^2\ \exp\left(-\epsilon\int V(\phi'(\boldsymbol{x}))\,d^3x\right) \qquad (16.105)\\ &\quad \times\int \exp\left[\int -\tfrac{1}{2}\epsilon\pi'^2(\boldsymbol{x}) + i\pi'(\boldsymbol{x})[\phi''(\boldsymbol{x})-\phi'(\boldsymbol{x})]d^3x\right]D\pi'.\end{aligned}$$

Using the abbreviation

$$\dot\phi'(\boldsymbol{x}) \equiv \frac{\phi''(\boldsymbol{x})-\phi'(\boldsymbol{x})}{\epsilon} \tag{16.106}$$

and the integral formula (16.6), we get

$$\langle \phi''|\exp(-\epsilon H)|\phi'\rangle = f'\ \exp\left\{-\epsilon\int\left[\tfrac{1}{2}\,\dot\phi'^2(\boldsymbol{x}) + V(\phi'(\boldsymbol{x}))\right]d^3x\right\}.$$

Putting together $n = \beta/\epsilon$ such terms, integrating over the intermediate states $|\phi'''\rangle\langle\phi'''|$, and absorbing the normalizing factors into $D\phi$, we have

$$\langle \phi_\beta|e^{-\beta H}|\phi_0\rangle = \int_{\phi_0}^{\phi_\beta}\exp\left[-\int_0^\beta\int \tfrac{1}{2}\dot\phi^2(x) + V(\phi(x))\,d^3x dt\right]D\phi. \tag{16.107}$$

Replacing the potential $V(\phi)$ with its definition (16.100), we find

$$\langle \phi_\beta|e^{-\beta H}|\phi_0\rangle = \int_{\phi_0}^{\phi_\beta}\exp\left[-\int_0^\beta\int \tfrac{1}{2}\left[\dot\phi^2 + (\nabla\phi)^2 + m^2\phi^2\right] + P(\phi)\,d^3x dt\right]D\phi \tag{16.108}$$

in which the limits ϕ_0 and ϕ_β remind us that we are to integrate over all fields $\phi(\boldsymbol{x},t)$ that run from $\phi(\boldsymbol{x},0) = \phi_0(\boldsymbol{x})$ to $\phi(\boldsymbol{x},\beta) = \phi_\beta(\boldsymbol{x})$.

In terms of the energy density

$$\mathcal{H}(\phi) \equiv \tfrac{1}{2}\left[(\partial_a\phi)^2 + m^2\phi^2\right] + P(\phi) \tag{16.109}$$

in which a is summed from 0 to 3, the path integral (16.108) is

$$\langle \phi_\beta | e^{-\beta H} | \phi_0 \rangle = \int_{\phi_0}^{\phi_\beta} \exp \left[- \int_0^\beta \!\! \int \mathcal{H}(\phi) \, d^3x dt \right] D\phi. \tag{16.110}$$

The partition function $Z(\beta)$—defined as the trace $Z(\beta) \equiv \mathrm{Tr}\, e^{-\beta H}$ over all states of the system—is then an integral over all loop fields (ones for which ϕ_β and ϕ_0 coincide)

$$Z(\beta) \equiv \mathrm{Tr}\, e^{-\beta H} = \int_{\phi_0}^{\phi_0} \langle \phi | e^{-\beta H} | \phi \rangle D\phi = \int_{\phi_0}^{\phi_0} \exp \left[- \int_0^\beta \!\! \int \mathcal{H}(\phi) \, d^3x dt \right] D\phi. \tag{16.111}$$

Because the four space-time derivatives in $\mathcal{H}(\phi)$ occur with the same sign, finite-temperature field theory is called **euclidean** quantum field theory. The density operator ρ for the system described by the hamiltonian H in equilibrium at temperature T is $\rho = \exp(-\beta H)/Z(\beta)$.

Like the definition (16.87) of the euclidean position operator $q_e(t)$, the euclidean field operator $\phi_e(x)$ is defined as

$$\phi_e(\boldsymbol{x}, t) = e^{tH} \phi(\boldsymbol{x}, 0) e^{-tH}. \tag{16.112}$$

The **euclidean-time-ordered product** (16.88) of two fields is

$$\begin{aligned} \mathcal{T} \left[\phi_e(\boldsymbol{x}_1, t_1) \phi_e(\boldsymbol{x}_2, t_2) \right] = \; & \theta(t_1 - t_2) e^{t_1 H} \phi_e(\boldsymbol{x}_1, 0) e^{-(t_1 - t_2) H} \phi_e(\boldsymbol{x}_2, 0) e^{-t_2 H} \\ & + \theta(t_2 - t_1) e^{t_2 H} \phi_e(\boldsymbol{x}_2, 0) e^{-(t_2 - t_1) H} \phi_e(\boldsymbol{x}_1, 0) e^{-t_1 H}. \end{aligned}$$

The logic of equations (16.87–16.96) leads us to write its mean value in a system described by a stationary density operator— one that commutes with the hamiltonian—as the ratio

$$\begin{aligned} \langle \mathcal{T} \left[\phi_e(x_1) \phi_e(x_2) \right] \rangle & = \mathrm{Tr} \left\{ \rho \, \mathcal{T} \left[\phi_e(x_1) \phi_e(x_2) \right] \right\} \\ & = \frac{\mathrm{Tr} \left\{ e^{-\beta H} \, \mathcal{T} \left[\phi_e(x_1) \phi_e(x_2) \right] \right\}}{\mathrm{Tr} \left[\, e^{-\beta H} \right]} \\ & = \frac{\displaystyle \int_{\phi_0}^{\phi_0} \phi(x_1) \phi(x_2) \exp \left[- \int_0^\beta \!\! \int \mathcal{H}(\phi) \, d^3x \, dt \right] D\phi}{\displaystyle \int_{\phi_0}^{\phi_0} \exp \left[- \int_0^\beta \!\! \int \mathcal{H}(\phi) \, d^3x \, dt \right] D\phi} \end{aligned} \tag{16.113}$$

in which several factors have canceled.

In the zero-temperature ($\beta \to \infty$) limit, the density operator ρ becomes the projection operator $|0\rangle\langle 0|$ on the ground state, and mean-value formulas like (16.113) become

$$\langle 0|\mathcal{T}\left[\phi_e(x_1)\dots\phi_e(x_n)\right]|0\rangle = \frac{\displaystyle\int \phi_e(x_1)\dots\phi_e(x_n)\,\exp\left[-\int \mathcal{H}(\phi)\,d^4x\right] D\phi}{\displaystyle\int \exp\left[-\int \mathcal{H}(\phi)\,d^4x\right] D\phi} \tag{16.114}$$

in which Hamilton's density $\mathcal{H}(\phi)$ is integrated over all of euclidean space-time and over all fields that are periodic on the infinite time interval. Statistical field theory and lattice gauge theory are based upon such formulas.

16.11 Real-Time Field Theory

We now follow the derivation of section 16.10 using the same notation but for real time. In (16.105), we replace $-\epsilon H$ by $-i\epsilon H$ and follow the logic of sections (16.4 & 16.10). We find in the limit $\epsilon \to 0$ with $\dot{\phi}' \equiv (\phi'' - \phi')/\epsilon$

$$\begin{aligned}\langle \phi''|e^{-i\epsilon H}|\phi'\rangle &= \int \langle \phi''|e^{-i\epsilon\int \pi^2/2\,d^3x}|\pi'\rangle\langle\pi'|e^{-i\epsilon\int V(\phi)\,d^3x}|\phi'\rangle D\pi' \\ &= |f|^2 e^{-i\epsilon\int V(\phi')d^3x}\int e^{-i\epsilon\int \pi'^2/2 + i\pi'(\phi''-\phi')\,d^3x} D\pi' \\ &= f'\,\exp\left[i\epsilon\int \tfrac{1}{2}\dot{\phi}'^2 - V(\phi')\,d^3x\right].\end{aligned} \tag{16.115}$$

Putting together $2t/\epsilon$ similar factors and integrating over all the intermediate states $|\phi\rangle\langle\phi|$, we arrive at the path integral

$$\langle \phi''|e^{-i2tH}|\phi'\rangle = \int_{\phi'}^{\phi''} \exp\left[i\int \tfrac{1}{2}\dot{\phi}^2(x) - V(\phi(x))\,d^4x\right] D\phi \tag{16.116}$$

in which we integrate over all fields $\phi(x)$ that run from $\phi'(\boldsymbol{x}, -t)$ to $\phi''(\boldsymbol{x}, t)$. After expanding the definition (16.100) of the potential $V(\phi)$, we have

$$\langle \phi''|e^{-i2tH}|\phi'\rangle = \int_{\phi'}^{\phi''} \exp\left[i\int \tfrac{1}{2}\left[\dot{\phi}^2 - (\boldsymbol{\nabla}\phi)^2 - m^2\phi^2\right] - P(\phi)\,d^4x\right] D\phi. \tag{16.117}$$

This amplitude is a path integral

$$\langle \phi''|e^{-i2tH}|\phi'\rangle = \int_{\phi'}^{\phi''} e^{iS[\phi]}\,D\phi \tag{16.118}$$

of phases $\exp(iS[\phi])$ that are exponentials of the classical action

$$S[\phi] = \int \tfrac{1}{2}\left[\dot{\phi}^2 - (\boldsymbol{\nabla}\phi)^2 - m^2\phi^2\right] - P(\phi)\, d^4x. \tag{16.119}$$

The time dependence of the Heisenberg field operator is

$$\phi(\boldsymbol{x}, t) = e^{itH}\phi(\boldsymbol{x}, 0)e^{-itH}. \tag{16.120}$$

The **time-ordered product** of two fields, as in (16.88), is the sum

$$\mathcal{T}\left[\phi(x_1)\phi(x_2)\right] = \theta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) + \theta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1). \tag{16.121}$$

Between two factors of $\exp(-itH)$, it is for $t_1 > t_2$

$$e^{-itH}\mathcal{T}\left[\phi(x_1)\phi(x_2)\right]e^{-itH} = e^{-i(t-t_1)H}\phi(\boldsymbol{x}_1, 0)e^{-i(t_1-t_2)H}\phi(\boldsymbol{x}_2, 0)e^{-i(t-t_2)H}.$$

So by the logic that led to the path-integral formulas (16.113) and (16.118), we can write a matrix element of the time-ordered product (16.121) as

$$\langle\phi''|e^{-itH}\mathcal{T}\left[\phi(x_1)\phi(x_2)\right]e^{-itH}|\phi'\rangle = \int_{\phi'}^{\phi''} \phi(x_1)\,\phi(x_2)\,e^{iS[\phi]}\,D\phi \tag{16.122}$$

in which we integrate over fields that go from ϕ' at time $-t$ to ϕ'' at time t. The time-ordered product of any combination of fields is then

$$\langle\phi''|e^{-itH}\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]e^{-itH}|\phi'\rangle = \int \phi(x_1)\ldots\phi(x_n)\,e^{iS[\phi]}\,D\phi. \tag{16.123}$$

Like the position eigenstates $|q'\rangle$ of quantum mechanics, the eigenstates $|\phi'\rangle$ are states of infinite energy that overlap most states. Yet we often are interested in the ground state $|0\rangle$ or in states of a few particles. To form such matrix elements, we multiply both sides of equations (16.118 & 16.123) by $\langle 0|\phi''\rangle\langle\phi'|0\rangle$ and integrate over ϕ' and ϕ''. Since the ground state is a normalized eigenstate of the hamiltonian $H|0\rangle = E_0|0\rangle$ with eigenvalue E_0, we find from (16.118)

$$\begin{aligned}\int\langle 0|\phi''\rangle\langle\phi''|e^{-i2tH}|\phi'\rangle\langle\phi'|0\rangle D\phi'' D\phi' &= \langle 0|e^{-i2tH}|0\rangle \\ &= e^{-i2tE_0} = \int\langle 0|\phi''\rangle e^{iS[\phi]}\langle\phi'|0\rangle\, D\phi D\phi'' D\phi'\end{aligned} \tag{16.124}$$

and from (16.123)

$$e^{-2itE_0}\langle 0|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \int\langle 0|\phi''\rangle\phi(x_1)\ldots\phi(x_n)\,e^{iS[\phi]}\langle\phi'|0\rangle\,D\phi \tag{16.125}$$

in which we suppressed the differentials $D\phi'' D\phi'$. The mean value in the ground state of a time-ordered product of field operators is then a ratio of these path integrals

$$\langle 0|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \frac{\displaystyle\int \langle 0|\phi''\rangle\,\phi(x_1)\ldots\phi(x_n)\,e^{iS[\phi]}\langle\phi'|0\rangle\,D\phi}{\displaystyle\int \langle 0|\phi''\rangle\,e^{iS[\phi]}\langle\phi'|0\rangle\,D\phi} \tag{16.126}$$

in which factors involving E_0 have canceled and the integration is over all fields that go from $\phi(\boldsymbol{x},-t) = \phi'(\boldsymbol{x})$ to $\phi(\boldsymbol{x},t) = \phi''(\boldsymbol{x})$ and over $\phi'(\boldsymbol{x})$ and $\phi''(\boldsymbol{x})$.

16.12 Perturbation Theory

Field theories with hamiltonians that are quadratic in their fields like

$$H_0 = \int \tfrac{1}{2}\left[\pi^2(x) + (\nabla\phi(x))^2 + m^2\phi^2(x)\right] d^3x \tag{16.127}$$

are soluble. Their fields evolve in time as

$$\phi(\boldsymbol{x},t) = e^{itH_0}\phi(\boldsymbol{x},0)e^{-itH_0}. \tag{16.128}$$

The mean value in the ground state of H_0 of a time-ordered product of these fields is by (16.126) a ratio of path integrals

$$\langle 0|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \frac{\displaystyle\int \langle 0|\phi''\rangle\,\phi(x_1)\ldots\phi(x_n)\,e^{iS_0[\phi]}\langle\phi'|0\rangle\,D\phi}{\displaystyle\int \langle 0|\phi''\rangle\,e^{iS_0[\phi]}\langle\phi'|0\rangle\,D\phi} \tag{16.129}$$

in which the action $S_0[\phi]$ is quadratic in the fields

$$\begin{aligned} S_0[\phi] &= \int \tfrac{1}{2}\left[\dot\phi^2(x) - (\nabla\phi(x))^2 - m^2\phi^2(x)\right] d^4x \\ &= \int \tfrac{1}{2}\left[-\partial_a\phi(x)\partial^a\phi(x) - m^2\phi^2(x)\right] d^4x. \end{aligned} \tag{16.130}$$

So the path integrals in the ratio (16.129) are gaussian and doable.

The Fourier transforms

$$\tilde\phi(p) = \int e^{-ipx}\phi(x)\,d^4x \quad \text{and} \quad \phi(x) = \int e^{ipx}\tilde\phi(p)\,\frac{d^4p}{(2\pi)^4} \tag{16.131}$$

turn the space-time derivatives in the action into a quadratic form

$$S_0[\phi] = -\tfrac{1}{2}\int |\tilde\phi(p)|^2\,(p^2+m^2)\,\frac{d^4p}{(2\pi)^4} \tag{16.132}$$

in which $p^2 = \boldsymbol{p}^2 - p^{02}$, and $\tilde{\phi}(-p) = \tilde{\phi}^*(p)$ by (3.25) since the field ϕ is real.

The initial $\langle \phi' | 0 \rangle$ and final $\langle 0 | \phi'' \rangle$ wave functions produce the $i\epsilon$ in the Feynman propagator (5.233). Although its exact form doesn't matter here, the wave function $\langle \phi' | 0 \rangle$ of the ground state of H_0 is the exponential (15.52)

$$\langle \phi' | 0 \rangle = c \exp \left[-\frac{1}{2} \int |\tilde{\phi}'(\boldsymbol{p})|^2 \sqrt{\boldsymbol{p}^2 + m^2} \, \frac{d^3 p}{(2\pi)^3} \right] \tag{16.133}$$

in which $\tilde{\phi}'(\boldsymbol{p})$ is the spatial Fourier transform

$$\tilde{\phi}'(\boldsymbol{p}) = \int e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \, \phi'(\boldsymbol{x}) \, d^3 x \tag{16.134}$$

and c is a normalization factor that will cancel in ratios of path integrals.

Apart from $-2i \ln c$, which we will not keep track of, the wave functions $\langle \phi' | 0 \rangle$ and $\langle 0 | \phi'' \rangle$ add to the action $S_0[\phi]$ the term

$$\Delta S_0[\phi] = \frac{i}{2} \int \sqrt{\boldsymbol{p}^2 + m^2} \left(|\tilde{\phi}(\boldsymbol{p}, t)|^2 + |\tilde{\phi}(\boldsymbol{p}, -t)|^2 \right) \frac{d^3 p}{(2\pi)^3} \tag{16.135}$$

in which we envision taking the limit $t \to \infty$ with $\phi(\boldsymbol{x}, t) = \phi''(\boldsymbol{x})$ and $\phi(\boldsymbol{x}, -t) = \phi'(\boldsymbol{x})$. The identity (Weinberg, 1995, pp. 386–388)

$$f(+\infty) + f(-\infty) = \lim_{\epsilon \to 0+} \epsilon \int_{-\infty}^{\infty} f(t) \, e^{-\epsilon |t|} \, dt \tag{16.136}$$

allows us to write $\Delta S_0[\phi]$ as

$$\Delta S_0[\phi] = \lim_{\epsilon \to 0+} \frac{i\epsilon}{2} \int \sqrt{\boldsymbol{p}^2 + m^2} \int_{-\infty}^{\infty} |\tilde{\phi}(\boldsymbol{p}, t)|^2 \, e^{-\epsilon |t|} \, dt \, \frac{d^3 p}{(2\pi)^3}. \tag{16.137}$$

To first order in ϵ, the change in the action is (exercise 16.15)

$$\begin{aligned} \Delta S_0[\phi] &= \lim_{\epsilon \to 0+} \frac{i\epsilon}{2} \int \sqrt{\boldsymbol{p}^2 + m^2} \int_{-\infty}^{\infty} |\tilde{\phi}(\boldsymbol{p}, t)|^2 \, dt \, \frac{d^3 p}{(2\pi)^3} \\ &= \lim_{\epsilon \to 0+} \frac{i\epsilon}{2} \int \sqrt{\boldsymbol{p}^2 + m^2} \, |\tilde{\phi}(p)|^2 \, \frac{d^4 p}{(2\pi)^4}. \end{aligned} \tag{16.138}$$

The modified action is therefore

$$\begin{aligned} S_0[\phi, \epsilon] = S_0[\phi] + \Delta S_0[\phi] &= -\frac{1}{2} \int |\tilde{\phi}(p)|^2 \left(p^2 + m^2 - i\epsilon \sqrt{\boldsymbol{p}^2 + m^2} \right) \frac{d^4 p}{(2\pi)^4} \\ &= -\frac{1}{2} \int |\tilde{\phi}(p)|^2 \left(p^2 + m^2 - i\epsilon \right) \frac{d^4 p}{(2\pi)^4} \end{aligned} \tag{16.139}$$

since the square-root is positive. In terms of the modified action, our formula

(16.129) for the time-ordered product is the ratio

$$\langle 0|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \frac{\displaystyle\int \phi(x_1)\ldots\phi(x_n)\, e^{iS_0[\phi,\epsilon]}\, D\phi}{\displaystyle\int e^{iS_0[\phi,\epsilon]}\, D\phi}. \tag{16.140}$$

We can use this formula (16.140) to express the mean value in the vacuum $|0\rangle$ of the time-ordered exponential of a space-time integral of a classical (c-number, external) current $j(x)$ as the ratio

$$\begin{aligned} Z_0[j] &\equiv \langle 0|\,\mathcal{T}\left\{\exp\left[i\int j(x)\,\phi(x)\,d^4x\right]\right\}|0\rangle \\ &= \frac{\displaystyle\int \exp\left[i\int j(x)\,\phi(x)\,d^4x\right] e^{iS_0[\phi,\epsilon]}\, D\phi}{\displaystyle\int e^{iS_0[\phi,\epsilon]}\, D\phi}. \end{aligned} \tag{16.141}$$

Since the state $|0\rangle$ is normalized, the mean value $Z_0[0]$ is unity,

$$Z_0[0] = 1. \tag{16.142}$$

If we absorb the current into the action

$$S_0[\phi,\epsilon,j] = S_0[\phi,\epsilon] + \int j(x)\,\phi(x)\,d^4x \tag{16.143}$$

then in terms of the current's Fourier transform

$$\tilde{j}(p) = \int e^{-ipx}\, j(x)\, d^4x \tag{16.144}$$

the modified action $S_0[\phi,\epsilon,j]$ is (exercise 16.16)

$$S_0[\phi,\epsilon,j] = -\frac{1}{2}\int \left[|\tilde{\phi}(p)|^2\,\left(p^2+m^2-i\epsilon\right) - \tilde{j}^*(p)\tilde{\phi}(p) - \tilde{\phi}^*(p)\tilde{j}(p)\right]\frac{d^4p}{(2\pi)^4}. \tag{16.145}$$

Changing variables to

$$\tilde{\psi}(p) = \tilde{\phi}(p) - \tilde{j}(p)/(p^2+m^2-i\epsilon) \tag{16.146}$$

we write the action $S_0[\phi,\epsilon,j]$ as (exercise 16.17)

$$\begin{aligned} S_0[\phi,\epsilon,j] &= -\frac{1}{2}\int \left[|\tilde{\psi}(p)|^2\,\left(p^2+m^2-i\epsilon\right) - \frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2+m^2-i\epsilon)}\right]\frac{d^4p}{(2\pi)^4} \\ &= S_0[\psi,\epsilon] + \frac{1}{2}\int \left[\frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2+m^2-i\epsilon)}\right]\frac{d^4p}{(2\pi)^4}. \end{aligned} \tag{16.147}$$

And since $D\phi = D\psi$, our formula (16.141) gives simply (exercise 16.18)

$$Z_0[j] = \exp\left[\frac{i}{2}\int \frac{|\tilde{j}(p)|^2}{p^2 + m^2 - i\epsilon}\frac{d^4p}{(2\pi)^4}\right]. \tag{16.148}$$

Going back to position space, one finds (exercise 16.19)

$$Z_0[j] = \exp\left[\frac{i}{2}\int j(x)\,\Delta(x - x')\,j(x')\,d^4x\,d^4x'\right] \tag{16.149}$$

in which $\Delta(x - x')$ is Feynman's **propagator** (5.233)

$$\Delta(x - x') = \int \frac{e^{ip(x-x')}}{p^2 + m^2 - i\epsilon}\frac{d^4p}{(2\pi)^4}. \tag{16.150}$$

The functional derivative (chapter 15) of $Z_0[j]$, defined by (16.141), is

$$\frac{1}{i}\frac{\delta Z_0[j]}{\delta j(x)} = \langle 0|\,\mathcal{T}\left[\phi(x)\exp\left(i\int j(x)\phi(x)d^4x\right)\right]|0\rangle \tag{16.151}$$

while that of equation (16.149) is

$$\frac{1}{i}\frac{\delta Z_0[j]}{\delta j(x)} = Z_0[j]\int \Delta(x - x')\,j(x')\,d^4x'. \tag{16.152}$$

Thus the second functional derivative of $Z_0[j]$ evaluated at $j = 0$ gives

$$\langle 0|\,\mathcal{T}\left[\phi(x)\phi(x')\right]|0\rangle = \left.\frac{1}{i^2}\frac{\delta^2 Z_0[j]}{\delta j(x)\delta j(x')}\right|_{j=0} = -i\,\Delta(x - x'). \tag{16.153}$$

Similarly, one may show (exercise 16.20) that

$$\begin{aligned}\langle 0|\,\mathcal{T}\left[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right]|0\rangle &= \left.\frac{1}{i^4}\frac{\delta^4 Z_0[j]}{\delta j(x_1)\delta j(x_2)\delta j(x_3)\delta j(x_4)}\right|_{j=0}\\ &= -\Delta(x_1 - x_2)\Delta(x_3 - x_4) - \Delta(x_1 - x_3)\Delta(x_2 - x_4)\\ &\quad - \Delta(x_1 - x_4)\Delta(x_2 - x_3). \end{aligned} \tag{16.154}$$

Suppose now that we add a potential $V = P(\phi)$ to the free hamiltonian (16.127). Scattering amplitudes are matrix elements of the time-ordered exponential $\mathcal{T}\exp\left[-i\int P(\phi)\,d^4x\right]$. Our formula (16.140) for the mean value in the ground state $|0\rangle$ of the free hamiltonian H_0 of any time-ordered product of fields leads us to

$$\langle 0|\mathcal{T}\left\{\exp\left[-i\int P(\phi)\,d^4x\right]\right\}|0\rangle = \frac{\int \exp\left[-i\int P(\phi)\,d^4x\right]\,e^{iS_0[\phi,\epsilon]}\,D\phi}{\int e^{iS_0[\phi,\epsilon]}\,D\phi}. \tag{16.155}$$

Using (16.153 & 16.154), we can cast this expression into the magical form

$$\langle 0|\mathcal{T}\left\{\exp\left[-i\int P(\phi)\,d^4x\right]\right\}|0\rangle = \exp\left[-i\int P\left(\frac{\delta}{i\delta j(x)}\right)\,d^4x\right]\,Z_0[j]\Big|_{j=0}. \tag{16.156}$$

The generalization of the path-integral formula (16.140) to the ground state $|\Omega\rangle$ of an interacting theory with action S is

$$\langle\Omega|\mathcal{T}\left[\phi(x_1)\dots\phi(x_n)\right]|\Omega\rangle = \frac{\displaystyle\int \phi(x_1)\dots\phi(x_n)\,e^{iS[\phi,\epsilon]}\,D\phi}{\displaystyle\int e^{iS[\phi,\epsilon]}\,D\phi} \tag{16.157}$$

in which a term like $i\epsilon\phi^2$ is added to make the modified action $S[\phi,\epsilon]$.

These are some of the techniques one uses to make states of incoming and outgoing particles and to compute scattering amplitudes (Weinberg, 1995, 1996; Srednicki, 2007; Zee, 2010).

16.13 Application to Quantum Electrodynamics

In the Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$, the QED hamiltonian is

$$H = H_m + \int \left[\tfrac{1}{2}\boldsymbol{\pi}^2 + \tfrac{1}{2}(\boldsymbol{\nabla}\times\boldsymbol{A})^2 - \boldsymbol{A}\cdot\boldsymbol{j}\right]\,d^3x + V_C \tag{16.158}$$

in which H_m is the matter hamiltonian, and V_C is the Coulomb term

$$V_C = \frac{1}{2}\int \frac{j^0(\boldsymbol{x},t)j^0(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,d^3x d^3y. \tag{16.159}$$

The operators $\boldsymbol{A}$ and $\boldsymbol{\pi}$ are canonically conjugate, but they satisfy the Coulomb-gauge conditions

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \quad\text{and}\quad \boldsymbol{\nabla}\cdot\boldsymbol{\pi} = 0. \tag{16.160}$$

One may show (Weinberg, 1995, pp. 413–418) that in this theory, the analog of equation (16.157) is

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\,e^{iS_C}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]\,D\boldsymbol{A}\,D\psi}{\displaystyle\int e^{iS_C}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]\,D\boldsymbol{A}\,D\psi} \tag{16.161}$$

in which the Coulomb-gauge action is

$$S_C = \int \tfrac{1}{2}\dot{\boldsymbol{A}}^2 - \tfrac{1}{2}(\boldsymbol{\nabla}\times\boldsymbol{A})^2 + \boldsymbol{A}\cdot\boldsymbol{j} + \mathcal{L}_m\,d^4x - \int V_C\,dt \tag{16.162}$$

and the functional delta function is

$$\delta[\nabla \cdot \boldsymbol{A}] = \prod_x \delta(\nabla \cdot \boldsymbol{A}(x)). \tag{16.163}$$

It enforces the Coulomb-gauge condition. The term $\mathcal{L}_m$ is the action density of the matter field ψ.

Tricks are available. We introduce a new field $A^0(x)$ and consider the factor

$$F = \int \exp\left[i \int \frac{1}{2}\left(\nabla A^0 + \nabla \triangle^{-1} j^0\right)^2 d^4x\right] DA^0 \tag{16.164}$$

which is just a *number* independent of the charge density j^0 since we can cancel the j^0 term by shifting A^0. By integrating by parts, we can write the number F as (exercise 16.21)

$$\begin{aligned} F &= \int \exp\left[i \int \frac{1}{2}\left(\nabla A^0\right)^2 - A^0 j^0 - \frac{1}{2} j^0 \triangle^{-1} j^0 \; d^4x\right] DA^0 \\ &= \int \exp\left[i \int \frac{1}{2}\left(\nabla A^0\right)^2 - A^0 j^0 \; d^4x + i \int V_C \, dt\right] DA^0. \end{aligned} \tag{16.165}$$

So when we multiply the numerator and denominator of the amplitude (16.161) by F, the awkward Coulomb term cancels, and we get

$$\langle \Omega | \mathcal{T}\left[\mathcal{O}_1 \dots \mathcal{O}_n\right] | \Omega \rangle = \frac{\displaystyle\int \mathcal{O}_1 \dots \mathcal{O}_n \, e^{iS'} \, \delta[\nabla \cdot \boldsymbol{A}] \, DA \, D\psi}{\displaystyle\int e^{iS'} \, \delta[\nabla \cdot \boldsymbol{A}] \, DA \, D\psi} \tag{16.166}$$

where now DA includes all four components A^μ and

$$S' = \int \frac{1}{2} \dot{\boldsymbol{A}}^2 - \frac{1}{2}(\nabla \times \boldsymbol{A})^2 + \frac{1}{2}\left(\nabla A^0\right)^2 + \boldsymbol{A} \cdot \boldsymbol{j} - A^0 j^0 + \mathcal{L}_m \, d^4x. \tag{16.167}$$

Since the delta-function $\delta[\nabla \cdot \boldsymbol{A}]$ enforces the Coulomb-gauge condition, we can add to the action S' the term $(\nabla \cdot \dot{\boldsymbol{A}})\, A^0$ which is $-\dot{\boldsymbol{A}} \cdot \nabla A^0$ after we integrate by parts and drop the surface term. This extra term makes the action gauge invariant

$$\begin{aligned} S &= \int \frac{1}{2}(\dot{\boldsymbol{A}} - \nabla \boldsymbol{A}^{\mathbf{0}})^2 - \frac{1}{2}(\nabla \times \boldsymbol{A})^2 + \boldsymbol{A} \cdot \boldsymbol{j} - A^0 j^0 + \mathcal{L}_m \, d^4x \\ &= \int -\frac{1}{4} F_{ab} \, F^{ab} - A^b j_b + \mathcal{L}_m \; d^4x. \end{aligned} \tag{16.168}$$

Thus at this point we have

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]\, DA\, D\psi}{\displaystyle\int e^{iS}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]\, DA\, D\psi} \tag{16.169}$$

in which S is the gauge-invariant action (16.168), and the integral is over all fields. The only relic of the Coulomb gauge is the gauge-fixing delta functional $\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]$.

We now make the gauge transformation

$$A'_b(x) = A_b(x) + \partial_b\Lambda(x) \quad \text{and} \quad \psi'(x) = e^{iq\Lambda(x)}\psi(x) \tag{16.170}$$

and replace the fields $A_b(x)$ and $\psi(x)$ everywhere in the numerator and (separately) in the denominator in the ratio (16.166) of path integrals by their gauge transforms (16.170) $A'_\mu(x)$ and $\psi'(x)$. This change of variables changes nothing; it's like replacing $\int_{-\infty}^{\infty} f(x)\,dx$ by $\int_{-\infty}^{\infty} f(y)\,dy$, and so

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle' \tag{16.171}$$

in which the prime refers to the gauge transformation (16.170).

We've seen that the action S is gauge invariant. So is the measure $DA\,D\psi$, and we now restrict ourselves to operators $\mathcal{O}_1\ldots\mathcal{O}_n$ that are *gauge invariant*. So in the right-hand side of equation (16.171), the replacement of the fields by their gauge transforms affects only the term $\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]$ that enforces the Coulomb-gauge condition

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A} + \triangle\Lambda)\, DA\, D\psi}{\displaystyle\int e^{iS}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A} + \triangle\Lambda)\, DA\, D\psi}. \tag{16.172}$$

We now have two choices. If we integrate over all gauge functions $\Lambda(x)$ in both the numerator and the denominator of this ratio (16.172), then apart from over-all constants that cancel, the mean value in the vacuum of the time-ordered product is the ratio

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\, DA\, D\psi}{\displaystyle\int e^{iS}\, DA\, D\psi} \tag{16.173}$$

in which we integrate over all matter fields, gauge fields, and gauges. That is, **we do not fix the gauge**.

The analogous formula for the euclidean time-ordered product is

$$\langle\Omega|\mathcal{T}_e\left[\mathcal{O}_1 \ldots \mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1 \ldots \mathcal{O}_n \, e^{-S_e} \, DA\, D\psi}{\displaystyle\int e^{-S_e} \, DA\, D\psi} \tag{16.174}$$

in which the euclidean action S_e is the space-time integral of the energy density. This formula is quite general; it holds in nonabelian gauge theories and is important in lattice gauge theory.

Our second choice is to multiply the numerator and the denominator of the ratio (16.172) by the exponential $\exp[-i\frac{1}{2}\alpha \int (- \triangle\Lambda)^2 \, d^4x]$ and then integrate over $\Lambda(x)$ separately in the numerator and denominator. This operation just multiplies the numerator and denominator by the same constant factor, which cancels. But if before integrating over all gauge transformations $\Lambda(x)$, we shift Λ so that $\triangle\Lambda$ decreases by $\dot{A}^0$, then the exponential factor is $\exp[-i\frac{1}{2}\alpha \int (\dot{A}^0 - \triangle\Lambda)^2 \, d^4x]$. Now when we integrate over $\Lambda(x)$, the delta function $\delta(\boldsymbol{\nabla} \cdot \boldsymbol{A} + \triangle\Lambda)$ replaces $\triangle\Lambda$ by $-\boldsymbol{\nabla} \cdot \boldsymbol{A}$ in the inserted exponential, converting it to $\exp[-i\frac{1}{2}\alpha \int (\dot{A}^0 + \boldsymbol{\nabla} \cdot \boldsymbol{A})^2 \, d^4x]$. The result is to replace the gauge-invariant action (16.168) with the gauge-fixed action

$$S_\alpha = \int -\frac{1}{4} F_{ab} \, F^{ab} - \frac{\alpha}{2} (\partial_b A^b)^2 - A^b j_b + \mathcal{L}_m \; d^4x. \tag{16.175}$$

This action is Lorentz invariant and so is much easier to work with than the one (16.162) with the Coulomb term. We can use it to compute scattering amplitudes perturbatively. The mean value of a time-ordered product of operators in the ground state $|0\rangle$ of the free theory is

$$\langle 0|\mathcal{T}\left[\mathcal{O}_1 \ldots \mathcal{O}_n\right]|0\rangle = \frac{\displaystyle\int \mathcal{O}_1 \ldots \mathcal{O}_n \, e^{iS_\alpha} \, DA\, D\psi}{\displaystyle\int e^{iS_\alpha} \, DA\, D\psi}. \tag{16.176}$$

By following steps analogous to those the led to (16.150), one may show (exercise 16.22) that in Feynman's gauge, $\alpha = 1$, the photon propagator is

$$\langle 0|\mathcal{T}\left[A_\mu(x) A_\nu(y)\right]|0\rangle = -i\triangle_{\mu\nu}(x-y) = -i \int \frac{\eta_{\mu\nu}}{q^2 - i\epsilon} \, e^{iq\cdot(x-y)} \, \frac{d^4q}{(2\pi)^4}. \tag{16.177}$$

16.14 Fermionic Path Integrals

In our brief introduction (1.11–1.12) and (1.43–1.45), to Grassmann variables, we learned that because

$$\theta^2 = 0 \tag{16.178}$$

the most general function $f(\theta)$ of a single Grassmann variable θ is

$$f(\theta) = a + b\,\theta. \tag{16.179}$$

So a complete integral table consists of the integral of this linear function

$$\int f(\theta)\,d\theta = \int a + b\,\theta\,d\theta = a\int d\theta + b\int \theta\,d\theta. \tag{16.180}$$

This equation has two unknowns, the integral $\int d\theta$ of unity and the integral $\int \theta\,d\theta$ of θ. We choose them so that the integral of $f(\theta+\zeta)$

$$\int f(\theta+\zeta)\,d\theta = \int a + b\,(\theta+\zeta)\,d\theta = (a+b\,\zeta)\int d\theta + b\int \theta\,d\theta \tag{16.181}$$

is the same as the integral (16.180) of $f(\theta)$. Thus the integral $\int d\theta$ of unity must vanish, while the integral $\int \theta\,d\theta$ of θ can be any constant, which we choose to be unity. Our complete table of integrals is then

$$\int d\theta = 0 \quad \text{and} \quad \int \theta\,d\theta = 1. \tag{16.182}$$

The anticommutation relations for a fermionic degree of freedom ψ are

$$\{\psi,\psi^\dagger\} \equiv \psi\,\psi^\dagger + \psi^\dagger\psi = 1 \quad \text{and} \quad \{\psi,\psi\} = \{\psi^\dagger,\psi^\dagger\} = 0. \tag{16.183}$$

Because ψ has $\psi^\dagger$, it is conventional to introduce a variable $\theta^* = \theta^\dagger$ that anti-commutes with itself and with θ

$$\{\theta^*,\theta^*\} = \{\theta^*,\theta\} = \{\theta,\theta\} = 0. \tag{16.184}$$

The logic that led to (16.182) now gives

$$\int d\theta^* = 0 \quad \text{and} \quad \int \theta^*\,d\theta^* = 1. \tag{16.185}$$

We define the reference state $|0\rangle$ as $|0\rangle \equiv \psi|s\rangle$ for a state $|s\rangle$ that is not annihilated by ψ. Since $\psi^2 = 0$, the operator ψ annihilates the state $|0\rangle$

$$\psi|0\rangle = \psi^2|s\rangle = 0. \tag{16.186}$$

The effect of the operator ψ on the state

$$|\theta\rangle = \exp\left(\psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta\right)|0\rangle = \left(1 + \psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta\right)|0\rangle \tag{16.187}$$

is

$$\psi|\theta\rangle = \psi(1+\psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta)|0\rangle = \psi\psi^\dagger\theta|0\rangle = (1-\psi^\dagger\psi)\theta|0\rangle = \theta|0\rangle \quad (16.188)$$

while that of θ on $|\theta\rangle$ is

$$\theta|\theta\rangle = \theta(1+\psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta)|0\rangle = \theta|0\rangle. \quad (16.189)$$

The state $|\theta\rangle$ therefore is an eigenstate of ψ with eigenvalue θ

$$\psi|\theta\rangle = \theta|\theta\rangle. \quad (16.190)$$

The bra corresponding to the ket $|\zeta\rangle$ is

$$\langle\zeta| = \langle 0|\left(1+\zeta^*\psi - \frac{1}{2}\zeta^*\zeta\right) \quad (16.191)$$

and the inner product $\langle\zeta|\theta\rangle$ is (exercise 16.23)

$$\begin{aligned}\langle\zeta|\theta\rangle &= \langle 0|\left(1+\zeta^*\psi - \frac{1}{2}\zeta^*\zeta\right)\left(1+\psi^\dagger\theta - \frac{1}{2}\theta^*\theta\right)|0\rangle \\ &= \langle 0|1+\zeta^*\psi\psi^\dagger\theta - \frac{1}{2}\zeta^*\zeta - \frac{1}{2}\theta^*\theta + \frac{1}{4}\zeta^*\zeta\theta^*\theta|0\rangle \\ &= \langle 0|1+\zeta^*\theta - \frac{1}{2}\zeta^*\zeta - \frac{1}{2}\theta^*\theta + \frac{1}{4}\zeta^*\zeta\theta^*\theta|0\rangle \\ &= \exp\left[\zeta^*\theta - \frac{1}{2}\left(\zeta^*\zeta + \theta^*\theta\right)\right]. \end{aligned} \quad (16.192)$$

Example 16.2 (A Gaussian Integral) For any number c, we can compute the integral of $\exp(c\,\theta^*\theta)$ by expanding the exponential

$$\int e^{c\,\theta^*\theta}\,d\theta^* d\theta = \int (1+c\,\theta^*\theta)\,d\theta^* d\theta = \int (1-c\,\theta\,\theta^*)\,d\theta^* d\theta = -c \quad (16.193)$$

a formula that we'll use over and over. □

The identity operator for the space of states

$$c|0\rangle + d|1\rangle \equiv c|0\rangle + d\psi^\dagger|0\rangle \quad (16.194)$$

is (exercise 16.24) the integral

$$I = \int |\theta\rangle\langle\theta|\,d\theta^* d\theta = |0\rangle\langle 0| + |1\rangle\langle 1| \quad (16.195)$$

in which the differentials anti-commute with each other and with other fermionic variables: $\{d\theta, d\theta^*\} = 0$, $\{d\theta, \theta\} = 0$, $\{d\theta, \psi\} = 0$, and so forth.

The case of several Grassmann variables $\theta_1, \theta_2, \ldots, \theta_n$ and several Fermi operators $\psi_1, \psi_2, \ldots, \psi_n$ is similar. The θ_k anticommute among themselves

$$\{\theta_i, \theta_j\} = \{\theta_i, \theta_j^*\} = \{\theta_i^*, \theta_j^*\} = 0 \quad (16.196)$$

while the ψ_k satisfy

$$\{\psi_k, \psi_\ell^\dagger\} = \delta_{k\ell} \quad \text{and} \quad \{\psi_k, \psi_l\} = \{\psi_k^\dagger, \psi_\ell^\dagger\} = 0. \tag{16.197}$$

The reference state $|0\rangle$ is

$$|0\rangle = \left(\prod_{k=1}^{n} \psi_k \right) |s\rangle \tag{16.198}$$

in which $|s\rangle$ is any state not annihilated by any ψ_k (so the resulting $|0\rangle$ isn't zero). The direct-product state

$$|\theta\rangle \equiv \exp\left(\sum_{k=1}^{n} \psi_k^\dagger \theta_k - \frac{1}{2}\theta_k^* \theta_k \right) |0\rangle = \left[\prod_{k=1}^{n} \left(1 + \psi_k^\dagger \theta_k - \frac{1}{2}\theta_k^* \theta_k \right) \right] |0\rangle \tag{16.199}$$

is (exercise 16.25) a simultaneous eigenstate of each ψ_k

$$\psi_k |\theta\rangle = \theta_k |\theta\rangle. \tag{16.200}$$

It follows that

$$\psi_\ell \psi_k |\theta\rangle = \psi_\ell \theta_k |\theta\rangle = -\theta_k \psi_\ell |\theta\rangle = -\theta_k \theta_\ell |\theta\rangle = \theta_\ell \theta_k |\theta\rangle \tag{16.201}$$

and so too $\psi_k \psi_\ell |\theta\rangle = \theta_k \theta_\ell |\theta\rangle$. Since the ψ's anticommute, their eigenvalues must also

$$\theta_\ell \theta_k |\theta\rangle = \psi_\ell \psi_k |\theta\rangle = -\psi_k \psi_\ell |\theta = -\theta_k \theta_\ell |\theta\rangle \tag{16.202}$$

which is why they must be Grassmann variables.

The inner product $\langle \zeta | \theta \rangle$ is

$$\begin{aligned} \langle \zeta | \theta \rangle &= \langle 0 | \left[\prod_{k=1}^{n} (1 + \zeta_k^* \psi_k - \frac{1}{2} \zeta_k^* \zeta_k) \right] \left[\prod_{\ell=1}^{n} (1 + \psi_\ell^\dagger \theta_\ell - \frac{1}{2} \theta_\ell^* \theta_\ell) \right] |0\rangle \\ &= \exp\left[\sum_{k=1}^{n} \zeta_k^* \theta_k - \frac{1}{2} \left(\zeta_k^* \zeta_k + \theta_k^* \theta_k \right) \right] = e^{\zeta^\dagger \theta - (\zeta^\dagger \zeta + \theta^\dagger \theta)/2}. \end{aligned} \tag{16.203}$$

The identity operator is

$$I = \int |\theta\rangle \langle \theta| \prod_{k=1}^{n} d\theta_k^* d\theta_k. \tag{16.204}$$

Example 16.3 (Gaussian Grassmann Integral) For any 2×2 matrix A, we may compute the gaussian integral

$$g(A) = \int e^{-\theta^\dagger A \theta} \, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \tag{16.205}$$

by expanding the exponential. The only terms that survive are the ones that have exactly one of each of the four variables θ_1, θ_2, θ_1^*, and θ_2^*. Thus, the integral is the determinant of the matrix A

$$\begin{aligned} g(A) &= \int \frac{1}{2}\left(\theta_k^\dagger A_{k\ell}\theta_\ell\right)^2 \, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \\ &= \int \left(\theta_1^* A_{11}\theta_1 \, \theta_2^* A_{22}\theta_2 + \theta_1^* A_{12}\theta_2 \, \theta_2^* A_{21}\theta_1\right) \, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \\ &= A_{11}A_{22} - A_{12}A_{21} = \det A. \end{aligned} \tag{16.206}$$

The natural generalization to n dimensions

$$\int e^{-\theta^\dagger A\theta} \prod_{k=1}^n d\theta_k^* d\theta_k = \det A \tag{16.207}$$

is true for any $n \times n$ matrix A. If A is invertible, then the invariance of Grassmann integrals under translations implies that

$$\begin{aligned} \int e^{-\theta^\dagger A\theta + \theta^\dagger\zeta + \zeta^\dagger\theta} \prod_{k=1}^n d\theta_k^* d\theta_k &= \int e^{-\theta^\dagger A(\theta + A^{-1}\zeta) + \theta^\dagger\zeta + \zeta^\dagger(\theta + A^{-1}\zeta)} \prod_{k=1}^n d\theta_k^* d\theta_k \\ &= \int e^{-\theta^\dagger A\theta + \zeta^\dagger\theta + \zeta^\dagger A^{-1}\zeta} \prod_{k=1}^n d\theta_k^* d\theta_k \\ &= \int e^{-(\theta^\dagger + \zeta^\dagger A^{-1})A\theta + \zeta^\dagger\theta + \zeta^\dagger A^{-1}\zeta} \prod_{k=1}^n d\theta_k^* d\theta_k \\ &= \int e^{-\theta^\dagger A\theta + \zeta^\dagger A^{-1}\zeta} \prod_{k=1}^n d\theta_k^* d\theta_k \\ &= \det A \; e^{\zeta^\dagger A^{-1}\zeta}. \end{aligned} \tag{16.208}$$

The values of θ and $\theta^\dagger$ that make the argument $-\theta^\dagger A\theta + \theta^\dagger\zeta + \zeta^\dagger\theta$ of the exponential stationary are $\overline{\theta} = A^{-1}\zeta$ and $\overline{\theta^\dagger} = \zeta^\dagger A^{-1}$. So a gaussian Grassmann integral is equal to its exponential evaluated at its stationary point, apart from a prefactor involving the determinant $\det A$. This result is a fermionic echo of the bosonic results (16.13–16.15). □

One may further extend these definitions to a Grassmann field $\chi_m(x)$ and an associated Dirac field $\psi_m(x)$. The $\chi_m(x)$'s anticommute among themselves and with all fermionic variables at all points of space-time

$$\{\chi_m(x), \chi_n(x')\} = \{\chi_m^*(x), \chi_n(x')\} = \{\chi_m^*(x), \chi_n^*(x')\} = 0 \tag{16.209}$$

and the Dirac field $\psi_m(x)$ obeys the equal-time anticommutation relations

$$\begin{aligned} \{\psi_m(\boldsymbol{x},t), \psi_n^\dagger(\boldsymbol{x'},t)\} &= \delta_{mn}\, \delta(\boldsymbol{x}-\boldsymbol{x'}) \\ \{\psi_m(\boldsymbol{x},t), \psi_n(\boldsymbol{x'},t)\} &= \{\psi_m^\dagger(\boldsymbol{x},t), \psi_n^\dagger(\boldsymbol{x'},t)\} = 0. \end{aligned} \tag{16.210}$$

As in (16.102 & 16.198), we use eigenstates of the field ψ at $t = 0$. If $|0\rangle$ is defined in terms of a state $|s\rangle$ that is not annihilated by any $\psi_m(\boldsymbol{x},0)$ as

$$|0\rangle = \left[\prod_{m,\boldsymbol{x}} \psi_m(\boldsymbol{x},0)\right] |s\rangle \tag{16.211}$$

then (exercise 16.26) the state

$$\begin{aligned} |\chi\rangle &= \exp\left(\int \sum_m \psi_m^\dagger(\boldsymbol{x},0)\, \chi_m(\boldsymbol{x}) - \frac{1}{2}\chi_m^*(\boldsymbol{x})\chi_m(\boldsymbol{x})\, d^3x\right)|0\rangle \\ &= \exp\left(\int \psi^\dagger\chi - \tfrac{1}{2}\chi^\dagger\chi\, d^3x\right)|0\rangle \end{aligned} \tag{16.212}$$

is an eigenstate of the operator $\psi_m(\boldsymbol{x},0)$ with eigenvalue $\chi_m(\boldsymbol{x})$

$$\psi_m(\boldsymbol{x},0)|\chi\rangle = \chi_m(\boldsymbol{x})|\chi\rangle. \tag{16.213}$$

The inner product of two such states is (exercise 16.27)

$$\langle\chi'|\chi\rangle = \exp\left[\int \chi'^\dagger\chi - \frac{1}{2}\chi'^\dagger\chi' - \frac{1}{2}\chi^\dagger\chi\, d^3x\right]. \tag{16.214}$$

The identity operator is the integral

$$I = \int |\chi\rangle\langle\chi|\, D\chi^* D\chi \tag{16.215}$$

in which

$$D\chi^* D\chi \equiv \prod_{m,\boldsymbol{x}} d\chi_m^*(\boldsymbol{x}) d\chi_m(\boldsymbol{x}). \tag{16.216}$$

The hamiltonian for a free Dirac field ψ of mass m is the spatial integral

$$H_0 = \int \overline{\psi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\psi\ d^3x \tag{16.217}$$

in which $\overline{\psi} \equiv i\psi^\dagger\gamma^0$ and the gamma matrices (10.286) satisfy

$$\{\gamma^a, \gamma^b\} = 2\,\eta^{ab} \tag{16.218}$$

where η is the 4×4 diagonal matrix with entries $(-1, 1, 1, 1)$. Since $\psi|\chi\rangle =$

$\chi|\chi\rangle$ and $\langle\chi'|\psi^\dagger = \langle\chi'|\chi'^\dagger$, the quantity $\langle\chi'|\exp(-i\epsilon H_0)|\chi\rangle$ is by (16.214)

$$\begin{aligned}\langle\chi'|e^{-i\epsilon H_0}|\chi\rangle &= \int\langle\chi'|\chi\rangle \exp\left[-i\epsilon\int\overline{\chi}'\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\;d^3x\right] \qquad (16.219)\\ &= \int\exp\left[\int\tfrac{1}{2}(\chi'^\dagger-\chi^\dagger)\chi-\tfrac{1}{2}\chi'^\dagger(\chi'-\chi)-i\epsilon\overline{\chi}'(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\chi d^3x\right]\\ &= \int\exp\left\{\epsilon\int\left[\tfrac{1}{2}\dot{\chi}^\dagger\chi-\tfrac{1}{2}\chi'^\dagger\dot{\chi}-i\overline{\chi}'\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\right]\;d^3x\right\}\end{aligned}$$

in which $\chi'^\dagger - \chi^\dagger = \epsilon\dot{\chi}^\dagger$ and $\chi' - \chi = \epsilon\dot{\chi}$. Everything within the square brackets is multiplied by ϵ, so we replace $\chi'^\dagger$ by $\chi'^\dagger$ and $\overline{\chi}'$ by $\overline{\chi}'$ so as to write to first order in ϵ

$$\langle\chi'|e^{-i\epsilon H_0}|\chi\rangle = \int\exp\left[\epsilon\int\tfrac{1}{2}\dot{\chi}^\dagger\chi-\tfrac{1}{2}\chi^\dagger\dot{\chi}-i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\;d^3x\right] \qquad (16.220)$$

in which the dependence upon χ' is through the time derivatives.

Putting together $n = 2t/\epsilon$ such matrix elements, integrating over all intermediate-state dyadics $|\chi\rangle\langle\chi|$, and using our formula (16.215), we find

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int\exp\left[\int\tfrac{1}{2}\dot{\chi}^\dagger\chi-\tfrac{1}{2}\chi^\dagger\dot{\chi}-i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi d^4x\right]D\chi^*D\chi. \qquad (16.221)$$

Integrating $\dot{\chi}^\dagger\chi$ by parts and dropping the surface term, we get

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int\exp\left[\int-\chi^\dagger\dot{\chi}-i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\,d^4x\right]D\chi^*D\chi. \qquad (16.222)$$

Since $-\chi^\dagger\dot{\chi} = -i\overline{\chi}\gamma^0\dot{\chi}$, the argument of the exponential is

$$i\int-\overline{\chi}\gamma^0\dot{\chi}-\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\;d^4x = i\int-\overline{\chi}\,(\gamma^\mu\partial_\mu+m)\,\chi\;d^4x. \qquad (16.223)$$

We then have

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int\exp\left(i\int\mathcal{L}_0(\chi)\,d^4x\right)D\chi^*D\chi \qquad (16.224)$$

in which $\mathcal{L}_0(\chi) = -\overline{\chi}\,(\gamma^\mu\partial_\mu+m)\,\chi$ is the action density (10.289) for a free Dirac field. Thus the amplitude is a path integral with phases given by the classical action $S_0[\chi]$

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int e^{i\int\mathcal{L}_0(\chi)\,d^4x}D\chi^*D\chi = \int e^{iS_0[\chi]}D\chi^*D\chi \qquad (16.225)$$

and the integral is over all fields that go from $\chi(\boldsymbol{x},-t) = \chi_{-t}(\boldsymbol{x})$ to $\chi(\boldsymbol{x},t) = \chi_t(\boldsymbol{x})$. Any normalization factor will cancel in ratios of such integrals.

Since Fermi fields anticommute, their time-ordered product has an extra minus sign

$$\mathcal{T}\left[\overline{\psi}(x_1)\psi(x_2)\right] = \theta(x_1^0 - x_2^0)\,\overline{\psi}(x_1)\,\psi(x_2) - \theta(x_2^0 - x_1^0)\,\psi(x_2)\,\overline{\psi}(x_1). \quad (16.226)$$

The logic behind our formulas (16.123) and (16.129) for the time-ordered product of bosonic fields now leads to an expression for the time-ordered product of $2n$ Dirac fields

$$\langle 0|\mathcal{T}\left[\overline{\psi}(x_1)\dots\psi(x_{2n})\right]|0\rangle = \frac{\int \langle 0|\chi''\rangle\,\overline{\chi}(x_1)\dots\chi(x_{2n})\,e^{iS_0[\chi]}\langle\chi'|0\rangle\,D\chi^* D\chi}{\int \langle 0|\chi''\rangle\,e^{iS_0[\chi]}\langle\chi'|0\rangle\,D\chi^* D\chi}. \quad (16.227)$$

As in (16.140), the effect of the inner products $\langle 0|\chi''\rangle$ and $\langle\chi'|0\rangle$ is to insert ϵ-terms which modify the Dirac propagators

$$\langle 0|\mathcal{T}\left[\overline{\psi}(x_1)\dots\psi(x_{2n})\right]|0\rangle = \frac{\int \overline{\chi}(x_1)\dots\chi(x_{2n})\,e^{iS_0[\chi,\epsilon]}\,D\chi^* D\chi}{\int e^{iS_0[\chi,\epsilon]}\,D\chi^* D\chi}. \quad (16.228)$$

Imitating (16.141), we introduce a Grassmann external current $\zeta(x)$ and define a fermionic analog of $Z_0[j]$

$$Z_0[\zeta] \equiv \langle 0|\,\mathcal{T}\left[e^{i\int \overline{\zeta}\psi + \overline{\psi}\zeta\,d^4x}\right]|0\rangle = \frac{\int e^{i\int \overline{\zeta}\chi + \overline{\chi}\zeta\,d^4x} e^{iS_0[\chi,\epsilon]} D\chi^* D\chi}{\int e^{iS_0[\chi,\epsilon]} D\chi^* D\chi}. \quad (16.229)$$

16.15 Application to nonabelian gauge theories

The action of a (fairly) generic non-abelian gauge theory is

$$S = \int -\tfrac{1}{4} F_{a\mu\nu} F_a^{\mu\nu} - \overline{\psi}\,(\gamma^\mu D_\mu + m)\,\psi\;d^4x \quad (16.230)$$

in which the Maxwell field is

$$F_{a\mu\nu} \equiv \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g\,f_{abc}\,A_{b\mu}\,A_{c\nu} \quad (16.231)$$

and the covariant derivative is

$$D_\mu\psi \equiv \partial_\mu\psi - ig\,t_a\,A_{a\mu}\,\psi. \quad (16.232)$$

Here g is a coupling constant, f_{abc} is a structure constant (10.63), and t_a is a generator (10.55) of the Lie algebra (section 10.15) of the gauge group.

One may show (Weinberg, 1996, pp. 14–18) that the analog of equation (16.169) for quantum electrodynamics is

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1 \dots \mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1 \dots \mathcal{O}_n \, e^{iS} \, \delta[A_{a3}] \, DA \, D\psi}{\displaystyle\int e^{iS} \, \delta[A_{a3}] \, DA \, D\psi} \tag{16.233}$$

in which the functional delta function

$$\delta[A_{a3}] \equiv \prod_{x,b} \delta(A_{a3}(x)) \tag{16.234}$$

enforces the axial-gauge condition, and $D\psi$ stands for $D\psi^* D\psi$.

Initially, physicists had trouble computing nonabelian amplitudes beyond the lowest order of perturbation theory. Then DeWitt showed how to compute to second order (DeWitt, 1967), and Faddeev and Popov, using path integrals, showed how to compute to all orders (Faddeev and Popov, 1967).

16.16 The Faddeev-Popov trick

The path-integral tricks of Faddeev and Popov are described in (Weinberg, 1996, pp. 19–27). We will use gauge-fixing functions $G_a(x)$ to impose a gauge condition on our non-abelian gauge fields $A^a_\mu(x)$. For instance, we can use $G_a(x) = A^3_a(x)$ to impose an axial gauge or $G_a(x) = i\partial_\mu A^\mu_a(x)$ to impose a Lorentz-invariant gauge.

Under an infinitesimal gauge transformation (11.511)

$$A^\lambda_{a\mu} = A_{a\mu} - \partial_\mu \lambda_a - g \, f_{abc} \, A_{b\mu} \, \lambda_c \tag{16.235}$$

the gauge fields change, and so the gauge-fixing functions $G_b(x)$, which depend upon them, also change. The jacobian J of that change is

$$J = \det\left(\frac{\delta G^\lambda_a(x)}{\delta \lambda_b(y)}\right)\bigg|_{\lambda=0} \equiv \frac{DG^\lambda}{D\lambda}\bigg|_{\lambda=0} \tag{16.236}$$

and it typically involves the delta function $\delta^4(x-y)$.

Let $B[G]$ be any functional of the gauge-fixing functions $G_b(x)$ such as

$$B[G] = \prod_{x,a} \delta(G_a(x)) = \prod_{x,a} \delta(A^3_a(x)) \tag{16.237}$$

in an axial gauge or

$$B[G] = \exp\left[\frac{i}{2}\int (G_a(x))^2 \, d^4x\right] = \exp\left[-\frac{i}{2}\int \left(\partial_\mu A^\mu_a(x)\right)^2 \, d^4x\right] \tag{16.238}$$

in a Lorentz-invariant gauge.

We want to understand functional integrals like (16.233)

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\, B[G]\, J\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[G]\, J\, DA\, D\psi} \tag{16.239}$$

in which the operators $\mathcal{O}_k$, the action functional $S[A]$, and the differentials $DAD\psi$ (but not the gauge-fixing functional $B[G]$ or the Jacobian J) are gauge invariant. The axial-gauge formula (16.233) is a simple example in which $B[G] = \delta[A_{a3}]$ enforces the axial-gauge condition $A_{a3}(x) = 0$ and the determinant $J = \det\left(\delta_{ab}\partial_3\delta(x-y)\right)$ is a constant that cancels.

If we translate the gauge fields by a gauge transformation Λ, then the ratio (16.239) does not change

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1^\Lambda\dots\mathcal{O}_n^\Lambda\, e^{iS^\Lambda}\, B[G^\Lambda]\, J^\Lambda\, DA^\Lambda\, D\psi^\Lambda}{\displaystyle\int e^{iS^\Lambda}\, B[G^\Lambda]\, J^\Lambda\, DA^\Lambda\, D\psi^\Lambda} \tag{16.240}$$

any more than $\int f(y)\,dy$ is different from $\int f(x)\,dx$. Since the operators $\mathcal{O}_k$, the action functional $S[A]$, and the differentials $DAD\psi$ are gauge invariant, most of the Λ-dependence goes away

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\, B[G^\Lambda]\, J^\Lambda\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[G^\Lambda]\, J^\Lambda\, DA\, D\psi}. \tag{16.241}$$

Let $\Lambda\lambda$ be a gauge transformation Λ followed by an infinitesimal gauge transformation λ. The jacobian J^Λ is a determinant of a product of matrices which is a product of their determinants

$$\begin{aligned} J^\Lambda &= \det\left(\frac{\delta G_a^{\Lambda\lambda}(x)}{\delta\lambda_b(y)}\right)\bigg|_{\lambda=0} = \det\left(\int \frac{\delta G_a^{\Lambda\lambda}(x)}{\delta\Lambda\lambda_c(z)}\frac{\delta\Lambda\lambda_c(z)}{\delta\lambda_b(y)}\, d^4z\right)\bigg|_{\lambda=0} \\ &= \det\left(\frac{\delta G_a^{\Lambda\lambda}(x)}{\delta\Lambda\lambda_c(z)}\right)\bigg|_{\lambda=0} \det\left(\frac{\delta\Lambda\lambda_c(z)}{\delta\lambda_b(y)}\right)\bigg|_{\lambda=0} \\ &= \det\left(\frac{\delta G_a^{\Lambda}(x)}{\delta\Lambda_c(z)}\right) \det\left(\frac{\delta\Lambda\lambda_c(z)}{\delta\lambda_b(y)}\right)\bigg|_{\lambda=0} \equiv \frac{DG^\Lambda}{D\Lambda}\frac{D\Lambda\lambda}{D\lambda}\bigg|_{\lambda=0}. \end{aligned} \tag{16.242}$$

Now we integrate over the gauge transformation Λ with weight function

$\rho(\Lambda) = (D\Lambda\lambda/D\lambda|_{\lambda=0})^{-1}$ and find, since the ratio (16.241) is Λ-independent

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1 \dots \mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1 \dots \mathcal{O}_n \, e^{iS} \, B[G^\Lambda] \, \frac{DG^\Lambda}{D\Lambda} \, D\Lambda \, DA \, D\psi}{\displaystyle\int e^{iS} \, B[G^\Lambda] \, \frac{DG^\Lambda}{D\Lambda} \, D\Lambda \, DA \, D\psi}$$
$$= \frac{\displaystyle\int \mathcal{O}_1 \dots \mathcal{O}_n \, e^{iS} \, B[G^\Lambda] \, DG^\Lambda \, DA \, D\psi}{\displaystyle\int e^{iS} \, B[G^\Lambda] \, DG^\Lambda \, DA \, D\psi}$$
$$= \frac{\displaystyle\int \mathcal{O}_1 \dots \mathcal{O}_n \, e^{iS} \, DA \, D\psi}{\displaystyle\int e^{iS} \, DA \, D\psi}. \tag{16.243}$$

Thus the mean-value in the vacuum of a time-ordered product of gauge-invariant operators is a ratio of path integrals over all gauge fields without any gauge fixing. No matter what gauge condition G or gauge-fixing functional $B[G]$ we use, the resulting gauge-fixed ratio (16.239) is equal to the ratio (16.243) of path integrals over all gauge fields without any gauge fixing. All gauge-fixed ratios (16.239) give the same time-ordered products, and so we can use whatever gauge condition G or gauge-fixing functional $B[G]$ is most convenient.

The analogous formula for the euclidean time-ordered product is

$$\langle\Omega|\mathcal{T}_e\left[\mathcal{O}_1 \dots \mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1 \dots \mathcal{O}_n \, e^{-S_e} \, DA \, D\psi}{\displaystyle\int e^{-S_e} \, DA \, D\psi} \tag{16.244}$$

where the euclidean action S_e is the space-time integral of the energy density. This formula is the basis for lattice gauge theory.

The path-integral formulas (16.173 & 16.244) derived for quantum electrodynamics therefore also apply to nonabelian gauge theories.

16.17 Ghosts

Faddeev and Popov showed how to do perturbative calculations in which one does fix the gauge. To continue our description of their tricks, we return to gauge-fixed expression (16.239) for the time-ordered product

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\, B[G]\, J\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[G]\, J\, DA\, D\psi} \tag{16.245}$$

set $G_b(x) = -i\partial_\mu A^\mu_b(x)$ and use (16.238) as the gauge-fixing functional $B[G]$

$$B[G] = \exp\left[\frac{i}{2}\int (G_a(x))^2\; d^4x\right] = \exp\left[-\frac{i}{2}\int (\partial_\mu A^\mu_a(x))^2\; d^4x\right]. \tag{16.246}$$

This functional adds to the action density the term $-(\partial_\mu A^\mu_a)^2/2$ which leads to a gauge-field propagator like the photon's (16.177)

$$\langle 0|\mathcal{T}\left[A^a_\mu(x)A^b_\nu(y)\right]|0\rangle = -i\delta_{ab}\triangle_{\mu\nu}(x-y) = -i\int \frac{\eta_{\mu\nu}\delta_{ab}}{q^2 - i\epsilon}\; e^{iq\cdot(x-y)}\; \frac{d^4q}{(2\pi)^4}. \tag{16.247}$$

What about the determinant J? Under an infinitesimal gauge transformation (16.235), the gauge field becomes

$$A^\lambda_{a\mu} = A_{a\mu} - \partial_\mu\lambda_a - g\, f_{abc}\, A_{b\mu}\,\lambda_c \tag{16.248}$$

and so $G^\lambda_a(x) = i\partial^\mu A^\lambda_{a\mu}(x)$ is

$$G^\lambda_a(x) = i\partial^\mu A_{a\mu}(x) + i\partial^\mu \int [-\delta_{ac}\partial_\mu - g\, f_{abc}A_{b\mu}(x)]\,\delta^4(x-y)\lambda_c(y)\, d^4y. \tag{16.249}$$

The jacobian J then is the determinant (16.236) of the matrix

$$\left(\frac{\delta G^\lambda_a(x)}{\delta\lambda_c(y)}\right)\Bigg|_{\lambda=0} = -i\delta_{ac}\,\Box\,\delta^4(x-y) - ig\, f_{abc}\,\frac{\partial}{\partial x^\mu}\left[A^\mu_b(x)\delta^4(x-y)\right] \tag{16.250}$$

that is

$$J = \det\left(-i\delta_{ac}\,\Box\,\delta^4(x-y) - ig\, f_{abc}\,\frac{\partial}{\partial x^\mu}\left[A^\mu_b(x)\delta^4(x-y)\right]\right). \tag{16.251}$$

But we've seen (16.207) that a determinant can be written as a fermionic path integral

$$\det A = \int e^{-\theta^\dagger A\,\theta}\;\prod_{k=1}^{n} d\theta^*_k d\theta_k. \tag{16.252}$$

So we can write the jacobian J as

$$J = \int \exp\left[\int i\omega^*_a\Box\omega_a + igf_{abc}\omega^*_a\partial_\mu(A^\mu_b\omega_c)\, d^4x\right] D\omega^* D\omega \tag{16.253}$$

which contributes the terms $-\partial_\mu \omega_a^* \partial^\mu \omega_a$ and

$$-\partial_\mu \omega_a^* \, g \, f_{abc} \, A_b^\mu \omega_c = \partial_\mu \omega_a^* \, g \, f_{abc} \, A_c^\mu \omega_b \tag{16.254}$$

to the action density.

Thus we can do perturbation theory by using the modified action density

$$\mathcal{L}' = -\tfrac{1}{4} F_{a\mu\nu} F_a^{\mu\nu} - \tfrac{1}{2} \left(\partial_\mu A_a^\mu\right)^2 - \partial_\mu \omega_a^* \partial^\mu \omega_a + \partial_\mu \omega_a^* \, g \, f_{abc} \, A_c^\mu \omega_b - \overline{\psi} \, (\not{D} + m) \, \psi \tag{16.255}$$

in which $\not{D} \equiv \gamma^\mu D_\mu = \gamma^\mu (\partial_\mu - i g t^a A_{a\mu})$. The **ghost** field ω is a mathematical device, not a physical field describing real particles, which would be spinless fermions violating the spin-statistics theorem (example 10.19).

Further Reading

Quantum Field Theory (Srednicki, 2007), *The Quantum Theory of Fields I, II, & III* (Weinberg, 1995, 1996, 2005), and *Quantum Field Theory in a Nutshell* (Zee, 2010) all provide excellent treatments of path integrals.

Exercises

16.1 Derive the multiple gaussian integral (16.8) from (5.167).

16.2 Derive the multiple gaussian integral (16.12) from (5.166).

16.3 Show that the vector $\overline{Y}$ that makes the argument of the multiple gaussian integral (16.12) stationary is given by (16.13), and that the multiple gaussian integral (16.12) is equal to its exponential evaluated at its stationary point $\overline{Y}$ apart from a prefactor involving $\det iS$.

16.4 Repeat the previous exercise for the multiple gaussian integral (16.11).

16.5 Compute the double integral (16.23) for the case $V(q_j) = 0$.

16.6 Insert a complete set of momentum dyadics $|p\rangle\langle p|$, use the inner product $\langle q|p\rangle = \exp(iqp)/\sqrt{2\pi}$, do the resulting Fourier transform, and so verify the free-particle path integral (16.54).

16.7 By taking the nonrelativistic limit of the formula (11.311) for the action of a relativistic particle of mass m and charge q, derive the expression (16.55) for the action a nonrelativistic particle in an electromagnetic field with no scalar potential.

16.8 Show that for the hamiltonian (16.60) of the simple harmonic oscillator the action $S[q_c]$ of the classical path is (16.67).

16.9 Show that the harmonic-oscillator action of the loop (16.68) is (16.69).

16.10 Show that the harmonic-oscillator amplitude (16.72) for $q' = 0$ and $q'' = q$ reduces as $t \to 0$ to the one-dimensional version of the free-particle amplitude (16.54).

16.11 Work out the path-integral formula for the amplitude for a mass m to fall to the ground from height h in a gravitational field of local acceleration g to lowest order and then including loops. Hint: use the technique of section 16.7.

16.12 Show that the action (16.74) of the stationary solution (16.77) is (16.79).

16.13 Derive formula (16.132) for the action $S_0[\phi]$ from (16.130 & 16.131).

16.14 Derive identity (16.136). Split the time integral at $t = 0$ into two halves, use

$$\epsilon\, e^{\pm \epsilon t} = \pm \frac{d}{dt}\, e^{\pm \epsilon t} \tag{16.256}$$

and then integrate each half by parts.

16.15 Derive the third term in equation (16.138) from its second term.

16.16 Use (16.143) and the Fourier transform (16.144) of the external current j to derive the formula (16.145) for the modified action $S_0[\phi, \epsilon, j]$.

16.17 Derive equation (16.147) from equations (16.145) and (16.146).

16.18 Derive the formula (16.148) for $Z_0[j]$ from the expression (16.147) for $S_0[\phi, \epsilon, j]$.

16.19 Derive equations (16.149 & 16.150) from formula (16.148).

16.20 Derive equation (16.154) from the formula (16.149) for $Z_0[j]$.

16.21 Show that the time integral of the Coulomb term (16.159) is the negative of the term that is quadratic in j^0 in the number F defined by (16.164).

16.22 By following steps analogous to those the led to (16.150), derive the formula (16.177) for the photon propagator in Feynman's gauge.

16.23 Derive expression (16.192) for the inner product $\langle \zeta | \theta \rangle$.

16.24 Derive the representation (16.195) of the identity operator I for a single fermionic degree of freedom from the rules (16.182 & 16.185) for Grassmann integration and the anticommutation relations (16.178 & 16.184).

16.25 Derive the eigenvalue equation (16.200) from the definition (16.198 & 16.199) of the eigenstate $|\theta\rangle$ and the anticommutation relations (16.196 & 16.197).

16.26 Derive the eigenvalue relation (16.213) for the Fermi field $\psi_m(\boldsymbol{x}, t)$ from the anticommutation relations (16.209 & 16.210) and the definitions (16.211 & 16.212).

16.27 Derive the formula (16.214) for the inner product from the definition (16.212) of the ket $|\chi\rangle$.