1 Scalar Fields

Let us consider the theory described by the Lagrange density

\[ \mathcal{L} = -\frac{1}{2} \partial_\nu \phi \partial^\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{24} \phi^4 \]  

which describes a scalar field of mass \( m \). The amplitude for the elastic scattering of two bosons of initial four-momenta \( p_1 \) and \( p_2 \) into two of final momenta \( p_1' \) and \( p_2' \) is

\[ \langle p_1', p_2' | S | p_1, p_2 \rangle = \langle p_1', p_2' | T \left[ \exp \left( -i \int V_i(\phi(x)) \, d^4x \right) \right] | p_1, p_2 \rangle \]

\[ = \langle p_1', p_2' | T \left[ \exp \left( -i (g/24) \int \phi^4(x) \, d^4x \right) \right] | p_1, p_2 \rangle. \]  

If the momenta are all different, then to lowest order the amplitude is

\[ \langle p_1', p_2' | S | p_1, p_2 \rangle_1 = -i \frac{g}{24} \langle p_1', p_2' | \int \phi^4(x) \, d^4x \ | p_1, p_2 \rangle. \]
The field is
\[
\phi(x) = \int \left[ a(k)e^{ikx} + a^\dagger(k)e^{-ikx} \right] \frac{d^3k}{\sqrt{(2\pi)^32k^0}} \tag{4}
\]
in which \(k^0 \equiv \omega = \sqrt{m^2 + k^2}\). Any one of the 4 fields in \(\phi^4(x)\) can absorb the boson with momentum \(p_1\); any one of the remaining 3 fields can absorb the other incoming boson; and either of the 2 other fields can emit the boson of momentum \(p'_1\). These factors are \(4! = 24\), they cancel \(1/24\), and we have to lowest order
\[
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_1 = -i\frac{g}{24} \int \mathcal{T} [\phi^4(x) \phi^4(x')] d^4xd^4x' | p_1, p_2 \rangle. \tag{5}
\]

To second order in the coupling constant \(g\), the amplitude is
\[
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_2 = \frac{1}{2} \left( -i\frac{g}{24} \right)^2 \langle p'_1, p'_2 \rangle \int \mathcal{T} [\phi^4(x) \phi^4(x')] d^4xd^4x' | p_1, p_2 \rangle. \tag{6}
\]

Either \(\phi^4(x)\) or \(\phi^4(x')\) can absorb the boson of momentum \(p_1\). We take it to be absorbed at \(x\) and cancel the leading factor of \(1/2\). Since any one of the 4 fields at \(x\) can absorb this particle, we also get a factor of 4. The other incoming boson can be absorbed at either \(x\) or \(x'\). If it is absorbed at \(x\), then 3 fields could absorb it, and we get a factor of 3. The two outgoing bosons must then be emitted at \(x'\) or the diagram would be disconnected. (We are ignoring disconnected diagrams.) Since any one of the 4 fields \(\phi^4(x')\) can emit the particle with momentum \(p'_1\), and any one of the 3 remaining fields can emit the other outgoing particle, we get a factor of \(4 \cdot 3 = 12\). So the second-order amplitude that both momenta \(p_1\) and \(p_2\) are absorbed at \(x\) is
\[
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_{2xx'} = -\frac{g^2}{4(2\pi)^64\sqrt{p'_1 p'_2 p_1 p_2}} \int e^{i\mathbf{x}(p_1+p_2)}e^{-i\mathbf{x'}(p'_1+p'_2)} \langle 0 | \mathcal{T} [\phi^2(x) \phi^2(x')] | 0 \rangle d^4xd^4x'. \tag{7}
\]

If we continue to ignore disconnected diagrams, then the time-ordered product is
\[
\langle 0 | \mathcal{T} [\phi^2(x) \phi^2(x')] | 0 \rangle = \theta(x^0 - x'^0) \langle 0 | \phi^2(x) \phi^2(x') | 0 \rangle + \theta(x'^0 - x^0) \langle 0 | \phi^2(x') \phi^2(x) | 0 \rangle
\]
\[
= \theta(x^0 - x'^0) \langle 0 | \phi^{(+)}(x) \phi^{(-)}(x') | 0 \rangle + \theta(x'^0 - x^0) \langle 0 | \phi^{(+)}(x') \phi^{(-)}(x) | 0 \rangle
\]
\[
= 2\theta(x^0 - x'^0) \langle 0 | \phi^{(+)}(x) \phi^{(-)}(x') | 0 \rangle^2 + 2\theta(x'^0 - x^0) \langle 0 | \phi^{(+)}(x') \phi^{(-)}(x) | 0 \rangle^2. \tag{8}
\]
Since the cross-terms vanish, we can write this as

$$
\langle 0 | T \left[ \phi^2(\tau(x)) \phi^2(\tau(x')) \right] | 0 \rangle = 2 \left[ \theta(x_0 - x_0^0) \langle 0 | \phi(\tau(x)) \phi(\tau(x')) | 0 \rangle + \theta(x_0^0 - x_0) \langle 0 | \phi(\tau(x')) \phi(\tau(x)) | 0 \rangle \right] 
= 2 \langle 0 | T [\phi(\tau(x)) \phi(\tau(x'))] | 0 \rangle^2.
$$

We then have

$$
\langle p_1', p_2' | S | p_1, p_2 \rangle_{2xx'} = -\frac{g^2}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1^0 p_2^0}} \int e^{ix(p_1 + p_2)} e^{-ix'(p_1' + p_2')} \langle 0 | T [\phi(\tau(x')) | 0 \rangle^2 d^4x d^4x'.
$$

Each time-ordered product is $-i$ times the Feynman propagator

$$
\langle 0 | T [\phi(\tau(x)) \phi(\tau(x'))] | 0 \rangle = -i \Delta_F(x - x') = -i \int \frac{e^{ik(x-x')}}{k^2 + m^2 - i\epsilon(2\pi)^4}.
$$

So the partial amplitude is

$$
\langle p_1', p_2' | S | p_1, p_2 \rangle_{2xx'} = \frac{g^2}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1^0 p_2^0}} \int e^{ix(p_1 + p_2)} e^{-ix'(p_1' + p_2')} d^4x d^4x' 
\times \int \frac{e^{ik(x-x')}}{k^2 + m^2 - i\epsilon} d^4k \int \frac{e^{ik'(x-x')}}{k'^2 + m^2 - i\epsilon} d^4k'
= \frac{g^2}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1^0 p_2^0}} \int \frac{\delta^4(p_1 + p_2 + k + k')}{k^2 + m^2 - i\epsilon} \frac{\delta^4(p_1' + p_2' + k + k')}{k'^2 + m^2 - i\epsilon} d^4k d^4k'
= \frac{g^2}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1^0 p_2^0}} \int \frac{1}{k^2 + m^2 - i\epsilon} \frac{1}{(p_1 + p_2 + k)^2 + m^2 - i\epsilon} d^4k.
$$

Shifting $k$ by $p_2$, we get the more symmetrical expression

$$
\langle p_1', p_2' | S | p_1, p_2 \rangle_{2xx'} = \frac{g^2 \delta^4(p_1' + p_2' - p_1 - p_2)}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1^0 p_2^0}} \int \frac{1}{(k + p_1)^2 + m^2 - i\epsilon} \frac{1}{(p_2 - k)^2 + m^2 - i\epsilon} d^4k.
$$
which diverges logarithmically. (The disconnected terms are worse.)

The next step is to use one of Feynman’s many tricks

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[(1-x)A + xB]^2} \tag{14}$$

to combine the two denominators. Setting

$$A = (k + p_1)^2 + m^2 - i\epsilon = k^2 + 2kp_1 - i\epsilon$$
$$B = (p_2 - k)^2 + m^2 - i\epsilon = k^2 - 2kp_2 - i\epsilon, \tag{15}$$

we find

$$\frac{1}{(k + p_1)^2 + m^2 - i\epsilon} \frac{1}{(p_2 - k)^2 + m^2 - i\epsilon} = \int_0^1 \frac{dx}{[(1-x)A + xB]^2}$$

$$= \int_0^1 \frac{dx}{[(1-x)[k^2 + 2kp_1 - i\epsilon] + x[k^2 - 2kp_2 - i\epsilon]]^2}$$

$$= \int_0^1 \frac{dx}{[k^2 + 2k((1-x)p_1 - xp_2) - i\epsilon]^2}. \tag{16}$$

To get rid of the term linear in $k$, we again shift $k$, replacing it by $k - (1-x)p_1 + xp_2$. We then have

$$\frac{1}{(k + p_1)^2 + m^2 - i\epsilon} \frac{1}{(p_2 - k)^2 + m^2 - i\epsilon} = \int_0^1 \frac{dx}{[k^2 - ((1-x)p_1 - xp_2)^2 - i\epsilon]^2}$$

$$= \int_0^1 \frac{dx}{[k^2 + m^2(1 - 2x + 2x^2) + 2x(1-x)p_1p_2 - i\epsilon]^2}$$

$$= \int_0^1 \frac{dx}{[k^2 + m^2 - x(1-x)2(m^2 - p_1p_2) - i\epsilon]^2}. \tag{17}$$
In terms of Mandelstam’s variables
\begin{align}
    s &= -(p_1 + p_2)^2 \\
    t &= -(p_1 - p'_1)^2 \\
    u &= -(p_1 - p'_2)^2,
\end{align}
which satisfy
\begin{equation}
    s + t + u = 4m^2,
\end{equation}
\begin{equation}
    1/AB \text{ is}
\end{equation}
\begin{equation}
    \frac{1}{(k + p_1)^2 + m^2 - i\epsilon} \frac{1}{(p_2 - k)^2 + m^2 - i\epsilon} = \int_0^1 \frac{dx}{[k^2 + m^2 - s x(1 - x) - i\epsilon]^2}.
\end{equation}

The partial amplitude is
\begin{equation}
    \langle p'_1, p'_2 | S | p_1, p_2 \rangle_{2x'x} = \frac{g^2 \delta^4(p'_1 + p'_2 - p_1 - p_2)}{8(2\pi)^6 \sqrt{p_1^{00} p_2^{00} p'^1_2 p'^0_2}} \int d^4k \int_0^1 \frac{dx}{[k^2 + m^2 - s x(1 - x) - i\epsilon]^2}.
\end{equation}

The partial amplitude in which \(p_1\) is absorbed and \(p'_1\) emitted at \(x\), while \(p_2\) is absorbed and \(p'_2\) emitted at \(x'\) is
\begin{equation}
    \langle p'_1, p'_2 | S | p_1, p_2 \rangle_{2x'x'x} = \frac{g^2 \delta^4(p'_1 + p'_2 - p_1 - p_2)}{8(2\pi)^6 \sqrt{p_1^{00} p_2^{00} p'^1_2 p'^0_2}} \int d^4k \int_0^1 \frac{dx}{[k^2 + m^2 - t x(1 - x) - i\epsilon]^2}.
\end{equation}

Finally, the partial amplitude in which \(p_1\) is absorbed and \(p'_2\) emitted at \(x\), while \(p_2\) is absorbed and \(p'_1\) emitted at \(x'\) is
\begin{equation}
    \langle p'_1, p'_2 | S | p_1, p_2 \rangle_{2x'x'x} = \frac{g^2 \delta^4(p'_1 + p'_2 - p_1 - p_2)}{8(2\pi)^6 \sqrt{p_1^{00} p_2^{00} p'^1_2 p'^0_2}} \int d^4k \int_0^1 \frac{dx}{[k^2 + m^2 - u x(1 - x) - i\epsilon]^2}.
\end{equation}

Homework set 5: Use either the Feynman rules or brute force to derive (22).
The full amplitude at order $g^2$ is then
\begin{align*}
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_2 &= \frac{g^2 \delta^4(p'_1 + p'_2 - p_1 - p_2)}{8(2\pi)^6 \sqrt{p'_1 p'_2 p_1 p_2}} \int d^4k \int_0^1 \frac{1}{[k^2 + m^2 - s x(1 - x) - i\epsilon]^2 + \frac{1}{[k^2 + m^2 - t x(1 - x) - i\epsilon]^2} + \frac{1}{[k^2 + m^2 - u x(1 - x) - i\epsilon]^2}} dx.
\end{align*}

(24)

It is still logarithmically divergent. The shifts we made in $k$ are not justified unless something regularizes these integrals. But we're going to assume some sort of regularization and change the contour of the integration over $k^0$.

The $k^0$ integral runs from $-\infty$ to $+\infty$ along the real axis of the complex $k^0$-plane. The poles in the denominator of the first integral in (24) are at
\begin{equation}
\begin{aligned}
k^0 &= \pm \sqrt{k^2 + m^2 - s x(1 - x) - i\epsilon}
\end{aligned}
\end{equation}

(25)

and as long as $s \leq 4(m^2 + k^2)$ they lie just under the positive $k^0$ axis and just above the negative $k^0$ axis. Since $t \leq 0$ and $u \leq 0$, the poles in the second and third terms always lie just under the positive $k^0$ axis and just above the negative $k^0$ axis. Since the integrand is otherwise analytic, we can rotate the contour of the $k^0$ integral from along the real axis to along the imaginary axis. The new $k^0$ integral runs from $-i\infty$ to $+i\infty$ up the imaginary axis of the complex $k^0$-plane. Letting $f(k^0)$ stand for the function within the square brackets, we write
\begin{align*}
\int_{-\infty}^{\infty} f(k^0) \, dk^0 = \int_{-i\infty}^{i\infty} f(k^0) \, dk^0 = \int_{-\infty}^{\infty} f(ik^4) \, i \, dk^4.
\end{align*}

(26)

This change is known as the **Wick rotation** (Gian Carlo Wick, 1909–1992). After the Wick rotation, the amplitude (24) is
\begin{align*}
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_2 &= \frac{ig^2 \delta^4(p'_1 + p'_2 - p_1 - p_2)}{8(2\pi)^6 \sqrt{p'_1 p'_2 p_1 p_2}} \int d^4k_e \int_0^1 \frac{1}{[k_e^2 + m^2 - s x(1 - x)]^2 + \frac{1}{[k_e^2 + m^2 - t x(1 - x)]^2} + \frac{1}{[k_e^2 + m^2 - u x(1 - x)]^2}} dx
\end{align*}

(27)
in which \( k^2_e = k_1^2 + k_2^2 + k_3^2 + k_4^2 \) and \( d^4 k_e = dk_1 dk_2 dk_3 dk_4 \) and \( k_\ell = k^\ell \) for \( \ell = 1, 2, 3, \) and \( 4. \)

The next step is to regularize the integrals by means of an ultraviolet cutoff at \( k^2_e = \Lambda^2 \). This cutoff breaks Lorentz invariance. The area of a sphere of radius \( k \) in \( d \) dimensions is

\[
A_d = \frac{2 \pi^{d/2} k^{d-1}}{\Gamma(d/2)}
\]

which for \( d = 4 \) is

\[
A_4 = \frac{2 \pi^2 k^3}{\Gamma(2)} = 2 \pi^2 k^3.
\]

Thus, the cutoff amplitude is

\[
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_2 = \frac{ig^2 \delta^4(p'_1 + p'_2 - p_1 - p_2)}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1^0 p_2^0}} \int_0^\Lambda 2 \pi^2 k^3 dk \int_0^1 \left[ \frac{1}{[k^2 + m^2 - s x(1-x)]^2} \right] \frac{1}{[k^2 + m^2 - t x(1-x)]^2} \frac{1}{[k^2 + m^2 - u x(1-x)]^2} dx
\]

\[
= \frac{i2\pi^2 g^4 \delta^4(p'_1 + p'_2 - p_1 - p_2)}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1^0 p_2^0}} \frac{1}{2} \int_0^1 \left\{ \ln \left[ \frac{\Lambda^2 + m^2 - s x(1-x)}{m^2 - s x(1-x)} \right] - \frac{\Lambda^2}{\Lambda^2 + m^2 - s x(1-x)} \right\} \frac{\Lambda^2}{\Lambda^2 + m^2 - t x(1-x)} dx
\]

\[
+ \ln \left[ \frac{\Lambda^2 + m^2 - t x(1-x)}{m^2 - t x(1-x)} \right] - \frac{\Lambda^2}{\Lambda^2 + m^2 - t x(1-x)}\right\} \frac{\Lambda^2}{\Lambda^2 + m^2 - u x(1-x)} dx
\]

\[
+ \ln \left[ \frac{\Lambda^2 + m^2 - u x(1-x)}{m^2 - u x(1-x)} \right] - \frac{\Lambda^2}{\Lambda^2 + m^2 - u x(1-x)}\right\} \frac{\Lambda^2}{\Lambda^2 + m^2 - u x(1-x)} dx.
\]
For $\Lambda \gg m$, we find that
\[
\ln \left[ \frac{\Lambda^2 + m^2 - s x(1-x)}{m^2 - s x(1-x)} \right] = \ln \left[ \frac{\Lambda^2}{m^2 - s x(1-x)} \right] + \ln \left[ \frac{\Lambda^2 + m^2 - s x(1-x)}{\Lambda^2} \right]
\]
\[
= \ln \left[ \frac{\Lambda^2}{m^2 - s x(1-x)} \right] + \ln \left[ \frac{\Lambda^2 + m^2 - s x(1-x)}{m^2 - s x(1-x)} \right] \rightarrow \ln \left[ \frac{\Lambda^2}{m^2 - s x(1-x)} \right]
\]
(31)
as $\Lambda \to \infty$. Similarly,
\[
\lim_{\Lambda \to \infty} \frac{\Lambda^2}{\Lambda^2 + m^2 - u x(1-x)} = \lim_{\Lambda \to \infty} \frac{1}{1 + [m^2 - u x(1-x)]/\Lambda^2} = 1.
\]
(32)
Thus, the cutoff amplitude in this limit is
\[
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_2 = \frac{ig^2 \delta^4(p'_1 + p'_2 - p_1 - p_2)}{2^9 \pi^4 \sqrt{p'_1 p'_2 p_1 p_2}} \int_0^1 \left\{ \ln \left[ \frac{\Lambda^2}{m^2 - s x(1-x)} \right] \right. \\
+ \ln \left[ \frac{\Lambda^2}{m^2 - t x(1-x)} \right] + \ln \left[ \frac{\Lambda^2}{m^2 - u x(1-x)} \right] - 3 \left\} dx. \tag{33}
\]
So to second order the amplitude in this limit is
\[
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_{1+2} = \frac{-i g^4 (p'_1 + p'_2 - p_1 - p_2)}{2^4 \pi^4 \sqrt{p'_1 p'_2 p_1 p_2}} \left[ g - \frac{g^2}{2^5 \pi^2} \int_0^1 \left\{ \ln \left[ \frac{\Lambda^2}{m^2 - s x(1-x)} \right] \right. \\
+ \ln \left[ \frac{\Lambda^2}{m^2 - t x(1-x)} \right] + \ln \left[ \frac{\Lambda^2}{m^2 - u x(1-x)} \right] - 3 \left\} dx \right]. \tag{34}
\]
We define the renormalized coupling $g_r$ as
\[
g_r \equiv g - \frac{g^2}{2^5 \pi^2} \int_0^1 \left\{ \ln \left[ \frac{\Lambda^2}{m^2 - s x(1-x)} \right] + \ln \left[ \frac{\Lambda^2}{m^2 - t x(1-x)} \right] + \ln \left[ \frac{\Lambda^2}{m^2 - u x(1-x)} \right] - 3 \right\} dx \tag{35}
\]
at a specific but arbitrary combination of momenta. The standard choice is the off-shell point

\[ p_1^2 = p_2^2 = p_1'^2 = p_2'^2 = \mu^2 \]

\[ s = t = u = -\frac{4\mu^2}{3}. \]

(36)

That is

\[
gr \equiv g - \frac{g^2}{25\pi^2} \int_0^1 \left\{ \ln \left[ \frac{\Lambda^2}{m^2 + \frac{4\mu^2}{3} x(1-x)} \right] + \ln \left[ \frac{\Lambda^2}{m^2 + \frac{4\mu^2}{3} x(1-x)} \right] + \ln \left[ \frac{\Lambda^2}{m^2 + \frac{4\mu^2}{3} x(1-x)} \right] - 3 \right\} dx
\]

\[
= g - \frac{3g^2}{32\pi^2} \int_0^1 \left\{ \ln \left[ \frac{\Lambda^2}{m^2 + (4\mu^2/3) x(1-x)} \right] - 1 \right\} dx
\]

(37)

So \( g_r \) is really a function of this mass \( \mu \):

\[ g_r = g_r(\mu). \]

(38)

The physical amplitude at \( s = t = u = -4\mu^2/3 \) is

\[
\langle p_1', p_2'|S|p_1, p_2\rangle_{1+2} = \frac{-i\delta^4(p_1' + p_2' - p_1 - p_2)}{24\pi^2 \sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}} g_r.
\]

(39)

So \( g_r \) is a finite physically meaningful quantity. But then the original coupling \( g \) must have been

\[
g = g_r + \frac{3g^2}{32\pi^2} \int_0^1 \left\{ \ln \left[ \frac{\Lambda^2}{m^2 + (4\mu^2/3) x(1-x)} \right] - 1 \right\} dx \equiv g_r + B g^2
\]

(40)

in which \( B \) diverges logarithmically.

(We can solve this quadratic equation

\[ B g^2 - g + g_r = 0 \]

(41)
and get
\[ g = \frac{1 \pm \sqrt{1 - 4Bg_r}}{2B} \quad (42) \]
which vanishes in the limit \( B \to \infty \). But then \( g \) would be infinitesimal and complex.

At other \( s, t, u \) points, the amplitude is
\[
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_{1+2} = -i\delta^4(p'_1 + p'_2 - p_1 - p_2) \left[ g - \frac{g^2}{2^5\pi^2} \int_0^1 \left\{ \ln \left[ \frac{\Lambda^2}{m^2 - s x(1-x)} \right] \\
+ \ln \left[ \frac{\Lambda^2}{m^2 - t x(1-x)} \right] + \ln \left[ \frac{\Lambda^2}{m^2 - u x(1-x)} \right] - 3 \right\} dx \right]
\]
\[
= -i\delta^4(p'_1 + p'_2 - p_1 - p_2) \left[ g_r + \frac{3g^2}{32\pi^2} \int_0^1 \left\{ \ln \left[ \frac{\Lambda^2}{m^2 + (4\mu^2/3) x(1-x)} \right] - 1 \right\} dx \right. \]
\[
- \frac{g^2}{32\pi^2} \int_0^1 \left\{ \ln \left[ \frac{\Lambda^2}{m^2 - s x(1-x)} \right] \\
+ \ln \left[ \frac{\Lambda^2}{m^2 - t x(1-x)} \right] + \ln \left[ \frac{\Lambda^2}{m^2 - u x(1-x)} \right] - 3 \right\} dx \right] \quad (43)
\]
The big bad \( \ln \Lambda^2 \) terms cancel, as do the \(-1\) and the \(-3\). We then have
\[
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_{1+2} = -i\delta^4(p'_1 + p'_2 - p_1 - p_2) \left[ g_r - \frac{g^2}{32\pi^2} \int_0^1 \left\{ \ln \left[ \frac{m^2 + 4\mu^2 x(1-x)/3}{m^2 - s x(1-x)} \right] \\
+ \ln \left[ \frac{m^2 + 4\mu^2 x(1-x)}{m^2 - t x(1-x)} \right] + \ln \left[ \frac{m^2 + 4\mu^2 x(1-x)}{m^2 - u x(1-x)} \right] \right\} dx \right] \quad (44)
\]
which we rewrite as

\[
\langle p'_1, p'_2 | S | p_1, p_2 \rangle_{1+2} = \frac{-i\delta^4(p'_1 + p'_2 - p_1 - p_2)}{2^{1/2} \sqrt{p'_1 p'_2 p_1 p_2}} \left[ g_r - \frac{g_r^2}{32\pi^2} \int_0^1 \left\{ \ln \left[ \frac{m^2 + 4\mu^2 x(1-x)/3}{m^2 - s x(1-x)} \right] + \ln \left[ \frac{m^2 + 4\mu^2 x(1-x)/3}{m^2 - t x(1-x)} \right] + \ln \left[ \frac{m^2 + 4\mu^2 x(1-x)/3}{m^2 - u x(1-x)} \right] \right\} dx \right]
\]

(45)

to order \(g_r^2\). Because \(g_r = g_r(\mu)\) depends upon the renormalization point \(\mu\), \(g_r(\mu)\) is called a running coupling constant or a running coupling. The quantity \(\mu^2\) can be any real number greater than \(-3m^2\), which makes the integral real.

The physical amplitude must not change when we change \(\mu\), so changes in \(\mu^2\) are compensated by changes in \(g_r(\mu)\). The \(\mu\)-dependent part of \(\langle p'_1, p'_2 | S | p_1, p_2 \rangle_{1+2}\) is inside the square bracket of (45)

\[
s(\mu) = g_r - \frac{3g_r^2}{32\pi^2} \int_0^1 \ln \left[ m^2 + (4\mu^2/3) x(1-x) \right] dx
\]

(46)

and in the limit \(\mu^2 \gg m^2\)

\[
s(\mu) \approx g_r - \frac{3g_r^2}{32\pi^2} \ln \mu^2.
\]

(47)

The condition that \(s(\mu)\) be independent of \(\mu\) then is

\[
0 = \frac{ds(\mu)}{d\mu} = \frac{dg_r(\mu)}{d\mu} - \frac{3g_r(\mu) ln \mu}{8\pi^2} \frac{dg_r(\mu)}{d\mu} - \frac{3g_r^2}{32\pi^2} \frac{2}{\mu}
\]

(48)

or

\[
\left(1 - \frac{3g_r(\mu) \ln \mu}{8\pi^2} \right) \frac{dg_r(\mu)}{d\mu} = \frac{3g_r^2}{16\pi^2} \frac{1}{\mu}.
\]

(49)

To lowest order in \(g_r\), we have

\[
\frac{dg_r(\mu)}{d\mu} = \frac{3g_r^2}{16\pi^2} \frac{1}{\mu}
\]

(50)
or

\[
\mu \frac{dg_r(\mu)}{d\mu} \equiv \beta(g_r(\mu)) = \frac{3g_r^2}{16\pi^2}
\]

which is an example of the **Callan-Symanzik equation**.

It is usual to write \( g_\mu \equiv g_r(\mu) \). We can integrate the Callan-Symanzik equation

\[
\ln \frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_E} \frac{dg_\mu}{\beta(g_\mu)} = \frac{16\pi^2}{3} \int_{g_M}^{g_E} \frac{dg_\mu}{g_\mu^2} = \frac{16\pi^2}{3} \left( \frac{1}{g_M} - \frac{1}{g_E} \right)
\]

(52)

to find the running physical coupling constant \( g_\mu \) at energy \( \mu = E \)

\[
g_E = \frac{g_M}{1 - 3g_M \ln(E/M)/16\pi^2}.
\]

(53)

As the energy \( E = \sqrt{s} \) rises above \( M \), while staying below the singular value \( E = M \exp(16\pi^2/3g_M) \), the running coupling \( g_E \) slowly increases. And so does the scattering amplitude, \( s(\mu) \approx g_E \).