Renormalization

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1 Scalar Fields

Let us consider the theory described by the Lagrange density

$$\mathcal{L} = -\frac{1}{2}\partial_\nu \phi \, \partial^\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{24}\phi^4 \tag{1}$$

which describes a scalar field of mass m. The amplitude for the elastic scattering of two bosons of initial four-momenta p_1 and p_2 into two of final momenta p_1' and p_2' is

$$\begin{aligned}\langle p_1', p_2' | S | p_1, p_2 \rangle &= \langle p_1', p_2' | \mathcal{T} \left[\exp \left(-\, i \int V_i(\phi(x)) \, d^4x \right) \right] | p_1, p_2 \rangle \\ &= \langle p_1', p_2' | \mathcal{T} \left[\exp \left(-\, i (g/24) \int \phi^4(x) \, d^4x \right) \right] | p_1, p_2 \rangle .\end{aligned} \tag{2}$$

If the momenta are all different, then to lowest order the amplitude is

$$\langle p_1', p_2' | S | p_1, p_2 \rangle_1 = \, - \, i \frac{g}{24} \langle p_1', p_2' | \int \phi^4(x) \, d^4x \; | p_1, p_2 \rangle . \tag{3}$$

The field is

$$\phi(x) = \int \left[a(k) e^{ikx} + a^\dagger(k) e^{-ikx} \right] \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} \tag{4}$$

in which $k^0 \equiv \omega = \sqrt{m^2 + \boldsymbol{k}^2}$. Any one of the 4 fields in $\phi^4(x)$ can absorb the boson with momentum p_1; any one of the remaining 3 fields can absorb the other incoming boson; and either of the 2 other fields can emit the boson of momentum p_1'. These factors are $4! = 24$, they cancel $1/24$, and we have to lowest order

$$\langle p_1', p_2' | S | p_1, p_2 \rangle_1 = -ig \frac{(2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')}{\sqrt{(2\pi)^{12} 2^4 p_1^0 p_2^0 p_1'^0 p_2'^0}}. \tag{5}$$

To second order in the coupling constant g, the amplitude is

$$\langle p_1', p_2' | S | p_1, p_2 \rangle_2 = \frac{1}{2} \left(-i \frac{g}{24} \right)^2 \langle p_1', p_2' | \int \mathcal{T} \left[\phi^4(x) \, \phi^4(x') \right] \, d^4x \, d^4x' \, | p_1, p_2 \rangle. \tag{6}$$

Either $\phi^4(x)$ or $\phi^4(x')$ can absorb the boson of momentum p_1. We take it to be absorbed at x and cancel the leading factor of $1/2$. Since any one of the 4 fields at x can absorb this particle, we also get a factor of 4. The other incoming boson can be absorbed at either x or x'. If it is absorbed at x, then 3 fields could absorb it, and we get a factor of 3. The two outgoing bosons must then be emitted at x' or the diagram would be disconnected. (We are ignoring disconnected diagrams.) Since any one of the 4 fields $\phi^4(x')$ can emit the particle with momentum p_1', and any one of the 3 remaining fields can emit the other outgoing particle, we get a factor of $4 \cdot 3 = 12$. So the second-order amplitude that both momenta p_1 and p_2 are absorbed at x is

$$\langle p_1', p_2' | S | p_1, p_2 \rangle_{2xx'x'} = -\frac{g^2}{4(2\pi)^6 4 \sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}} \int e^{ix(p_1+p_2)} e^{-ix'(p_1'+p_2')} \langle 0 | \mathcal{T} \left[\phi^2(x) \, \phi^2(x') \right] | 0 \rangle \, d^4x \, d^4x'. \tag{7}$$

If we continue to ignore disconnected diagrams, then the time-ordered product is

$$\begin{aligned} \langle 0 | \mathcal{T} \left[\phi^2(x) \, \phi^2(x') \right] | 0 \rangle &= \theta(x^0 - x'^0) \langle 0 | \phi^2(x) \, \phi^2(x') | 0 \rangle + \theta(x'^0 - x^0) \langle 0 | \phi^2(x') \, \phi^2(x) | 0 \rangle \\ &= \theta(x^0 - x'^0) \langle 0 | \phi^{(+)2}(x) \, \phi^{(-)2}(x') | 0 \rangle + \theta(x'^0 - x^0) \langle 0 | \phi^{(+)2}(x') \, \phi^{(-)2}(x) | 0 \rangle \\ &= 2\theta(x^0 - x'^0) \langle 0 | \phi^{(+)}(x) \, \phi^{(-)}(x') | 0 \rangle^2 + 2\theta(x'^0 - x^0) \langle 0 | \phi^{(+)}(x') \, \phi^{(-)}(x) | 0 \rangle^2. \end{aligned} \tag{8}$$

Since the cross-terms vanish, we can write this as

$$\begin{aligned}\langle 0|\mathcal{T}\left[\phi^2(x)\,\phi^2(x')\right]|0\rangle &= 2\left[\theta(x^0-x'^0)\langle 0|\phi^{(+)}(x)\,\phi^{(-)}(x')|0\rangle + \theta(x'^0-x^0)\langle 0|\phi^{(+)}(x')\,\phi^{(-)}(x)|0\rangle\right]^2 \\ &= 2\langle 0|\mathcal{T}\left[\phi(x)\,\phi(x')\right]|0\rangle^2.\end{aligned} \tag{9}$$

We then have

$$\langle p_1', p_2'|S|p_1, p_2\rangle_{2xx'x'} = -\frac{g^2}{8(2\pi)^6\sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}}\int e^{ix(p_1+p_2)}e^{-ix'(p_1'+p_2')}\langle 0|\mathcal{T}\left[\phi(x)\,\phi(x')\right]|0\rangle^2\, d^4x\, d^4x'. \tag{10}$$

Each time-ordered product is $-i$ times the Feynman propagator

$$\langle 0|\mathcal{T}\left[\phi(x)\,\phi(x')\right]|0\rangle = -i\Delta_F(x-x') = -i\int \frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon}\frac{d^4k}{(2\pi)^4}. \tag{11}$$

So the partial amplitude is

$$\begin{aligned}\langle p_1', p_2'|S|p_1, p_2\rangle_{2xx'x'} &= \frac{g^2}{8(2\pi)^6\sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}}\int e^{ix(p_1+p_2)}e^{-ix'(p_1'+p_2')}\, d^4x\, d^4x' \\ &\quad\times\int \frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon}\frac{d^4k}{(2\pi)^4}\int \frac{e^{ik'(x-x')}}{k'^2+m^2-i\epsilon}\frac{d^4k'}{(2\pi)^4} \\ &= \frac{g^2}{8(2\pi)^6\sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}}\int \frac{\delta^4(p_1+p_2+k+k')}{k^2+m^2-i\epsilon}\frac{\delta^4(p_1'+p_2'+k+k')}{k'^2+m^2-i\epsilon}\, d^4k\, d^4k' \\ &= \frac{g^2}{8(2\pi)^6\sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}}\int \frac{1}{k^2+m^2-i\epsilon}\,\frac{\delta^4(p_1'+p_2'-p_1-p_2)}{(p_1+p_2+k)^2+m^2-i\epsilon}\, d^4k.\end{aligned} \tag{12}$$

Shifting k by p_2, we get the more symmetrical expression

$$\langle p_1', p_2'|S|p_1, p_2\rangle_{2xx'x'} = \frac{g^2\delta^4(p_1'+p_2'-p_1-p_2)}{8(2\pi)^6\sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}}\int \frac{1}{(k+p_1)^2+m^2-i\epsilon}\,\frac{1}{(p_2-k)^2+m^2-i\epsilon}\, d^4k \tag{13}$$

which diverges logarithmically. (The disconnected terms are worse.)

The next step is to use one of Feynman's many tricks

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[(1-x)A + xB]^2} \tag{14}$$

to combine the two denominators. Setting

$$\begin{aligned} A &= (k+p_1)^2 + m^2 - i\epsilon = k^2 + 2kp_1 - i\epsilon \\ B &= (p_2-k)^2 + m^2 - i\epsilon = k^2 - 2kp_2 - i\epsilon, \end{aligned} \tag{15}$$

we find

$$\begin{aligned} \frac{1}{(k+p_1)^2 + m^2 - i\epsilon} \frac{1}{(p_2-k)^2 + m^2 - i\epsilon} &= \int_0^1 \frac{dx}{[(1-x)A + xB]^2} \\ &= \int_0^1 \frac{dx}{[(1-x)[k^2 + 2kp_1 - i\epsilon] + x[k^2 - 2kp_2 - i\epsilon]]^2} \\ &= \int_0^1 \frac{dx}{[k^2 + 2k((1-x)p_1 - xp_2) - i\epsilon]^2}. \end{aligned} \tag{16}$$

To get rid of the term linear in k, we again shift k, replacing it by $k - (1-x)p_1 + xp_2$. We then have

$$\begin{aligned} \frac{1}{(k+p_1)^2 + m^2 - i\epsilon} \frac{1}{(p_2-k)^2 + m^2 - i\epsilon} &= \int_0^1 \frac{dx}{[k^2 - ((1-x)p_1 - xp_2)^2 - i\epsilon]^2} \\ &= \int_0^1 \frac{dx}{[k^2 + m^2(1 - 2x + 2x^2) + 2x(1-x)p_1p_2 - i\epsilon]^2} \\ &= \int_0^1 \frac{dx}{[k^2 + m^2 - x(1-x)2(m^2 - p_1p_2) - i\epsilon]^2}. \end{aligned} \tag{17}$$

In terms of Mandelstam's variables

$$\begin{aligned} s &= -(p_1+p_2)^2 \\ t &= -(p_1-p_1')^2 \\ u &= -(p_1-p_2')^2, \end{aligned} \tag{18}$$

which satisfy

$$s+t+u = 4m^2, \tag{19}$$

$1/AB$ is

$$\frac{1}{(k+p_1)^2+m^2-i\epsilon}\,\frac{1}{(p_2-k)^2+m^2-i\epsilon} = \int_0^1 \frac{dx}{\left[k^2+m^2-s\,x(1-x)-i\epsilon\right]^2}. \tag{20}$$

The partial amplitude is

$$\langle p_1', p_2'|S|p_1,p_2\rangle_{2xx'x'} = \frac{g^2\delta^4(p_1'+p_2'-p_1-p_2)}{8(2\pi)^6\sqrt{p_1^0p_2^0p_1'^0p_2'^0}}\int d^4k\int_0^1 \frac{dx}{\left[k^2+m^2-s\,x(1-x)-i\epsilon\right]^2}. \tag{21}$$

The partial amplitude in which p_1 is absorbed and p_1' emitted at x, while p_2 is absorbed and p_2' emitted at x' is

$$\langle p_1', p_2'|S|p_1,p_2\rangle_{2x'xx'} = \frac{g^2\delta^4(p_1'+p_2'-p_1-p_2)}{8(2\pi)^6\sqrt{p_1^0p_2^0p_1'^0p_2'^0}}\int d^4k\int_0^1 \frac{dx}{\left[k^2+m^2-t\,x(1-x)-i\epsilon\right]^2}. \tag{22}$$

Finally, the partial amplitude in which p_1 is absorbed and p_2' emitted at x, while p_2 is absorbed and p_1' emitted at x' is

$$\langle p_1', p_2'|S|p_1,p_2\rangle_{2x'x'x} = \frac{g^2\delta^4(p_1'+p_2'-p_1-p_2)}{8(2\pi)^6\sqrt{p_1^0p_2^0p_1'^0p_2'^0}}\int d^4k\int_0^1 \frac{dx}{\left[k^2+m^2-u\,x(1-x)-i\epsilon\right]^2}. \tag{23}$$

Homework set 5: Use either the Feynman rules or brute force to derive (22).

The full amplitude at order g^2 is then

$$\langle p_1', p_2'|S|p_1, p_2\rangle_2 = \frac{g^2 \delta^4(p_1' + p_2' - p_1 - p_2)}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}} \int d^4k \int_0^1 \left[\frac{1}{\left[k^2 + m^2 - s\,x(1-x) - i\epsilon\right]^2} + \frac{1}{\left[k^2 + m^2 - t\,x(1-x) - i\epsilon\right]^2} + \frac{1}{\left[k^2 + m^2 - u\,x(1-x) - i\epsilon\right]^2} \right] dx. \tag{24}$$

It is still logarithmically divergent. The shifts we made in k are not justified unless something regularizes these integrals. But we're going to assume some sort of regularization and change the contour of the integration over k^0.

The k^0 integral runs from $-\infty$ to $+\infty$ along the real axis of the complex k^0-plane. The poles in the denominator of the first integral in (24) are at

$$k^0 = \pm\sqrt{\boldsymbol{k}^2 + m^2 - s\,x(1-x) - i\epsilon} \tag{25}$$

and as long as $s \le 4(m^2 + \boldsymbol{k}^2)$ they lie just under the positive k^0 axis and just above the negative k^0 axis. Since $t \le 0$ and $u \le 0$, the poles in the second and third terms always lie just under the positive k^0 axis and just above the negative k^0 axis. Since the integrand is otherwise analytic, we can rotate the contour of the k^0 integral from along the real axis to along the imaginary axis. The new k^0 integral runs from $-i\infty$ to $+i\infty$ up the imaginary axis of the complex k^0-plane. Letting $f(k^0)$ stand for the function within the square brackets, we write

$$\int_{-\infty}^{\infty} f(k^0)\, dk^0 = \int_{-i\infty}^{i\infty} f(k^0)\, dk^0 = \int_{-\infty}^{\infty} f(ik^4)\, i\, dk^4. \tag{26}$$

This change is known as the **Wick rotation** (Gian Carlo Wick, 1909–1992). After the Wick rotation, the amplitude (24) is

$$\langle p_1', p_2'|S|p_1, p_2\rangle_2 = \frac{ig^2 \delta^4(p_1' + p_2' - p_1 - p_2)}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}} \int d^4k_e \int_0^1 \left[\frac{1}{\left[k_e^2 + m^2 - s\,x(1-x)\right]^2} + \frac{1}{\left[k_e^2 + m^2 - t\,x(1-x)\right]^2} + \frac{1}{\left[k_e^2 + m^2 - u\,x(1-x)\right]^2} \right] dx \tag{27}$$

in which $k_e^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2$ and $d^4k_e = dk_1 dk_2 dk_3 dk_4$ and $k_\ell = k^\ell$ for $\ell = 1$, 2, 3, and 4.

The next step is to regularize the integrals by means of an ultraviolet cutoff at $k_e^2 = \Lambda^2$. This cutoff breaks Lorentz invariance. The area of a sphere of radius k in d dimensions is

$$A_d = \frac{2\,\pi^{d/2} k^{d-1}}{\Gamma(d/2)} \tag{28}$$

which for $d = 4$ is

$$A_4 = \frac{2\pi^2 k^3}{\Gamma(2)} = 2\pi^2 k^3. \tag{29}$$

Thus, the cutoff amplitude is

$$\begin{aligned}
\langle p_1', p_2' | S | p_1, p_2 \rangle_2 = {} & \frac{i g^2 \delta^4(p_1' + p_2' - p_1 - p_2)}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}} \int_0^\Lambda 2\pi^2 k^3 dk \int_0^1 \left[\frac{1}{[k^2 + m^2 - s\,x(1-x)]^2} \right. \\
& \left. + \frac{1}{[k^2 + m^2 - t\,x(1-x)]^2} + \frac{1}{[k^2 + m^2 - u\,x(1-x)]^2} \right] dx \\
= {} & \frac{i 2\pi^2 g^2 \delta^4(p_1' + p_2' - p_1 - p_2)}{8(2\pi)^6 \sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}} \frac{1}{2} \int_0^1 \left\{ \ln\left[\frac{\Lambda^2 + m^2 - s\,x(1-x)}{m^2 - s\,x(1-x)} \right] - \frac{\Lambda^2}{\Lambda^2 + m^2 - s\,x(1-x)} \right. \\
& + \ln\left[\frac{\Lambda^2 + m^2 - t\,x(1-x)}{m^2 - t\,x(1-x)} \right] - \frac{\Lambda^2}{\Lambda^2 + m^2 - t\,x(1-x)} \\
& \left. + \ln\left[\frac{\Lambda^2 + m^2 - u\,x(1-x)}{m^2 - u\,x(1-x)} \right] - \frac{\Lambda^2}{\Lambda^2 + m^2 - u\,x(1-x)} \right\} dx.
\end{aligned} \tag{30}$$

For $\Lambda >> m$, we find that

$$\begin{aligned} \ln\left[\frac{\Lambda^2+m^2-s\,x(1-x)}{m^2-s\,x(1-x)}\right] &= \ln\left[\frac{\Lambda^2}{m^2-s\,x(1-x)}\frac{\Lambda^2+m^2-s\,x(1-x)}{\Lambda^2}\right] \\ &= \ln\left[\frac{\Lambda^2}{m^2-s\,x(1-x)}\right]+\ln\left[\frac{\Lambda^2+m^2-s\,x(1-x)}{\Lambda^2}\right] \\ &= \ln\left[\frac{\Lambda^2}{m^2-s\,x(1-x)}\right]+\ln\left[1+\frac{m^2-s\,x(1-x)}{\Lambda^2}\right] \to \ln\left[\frac{\Lambda^2}{m^2-s\,x(1-x)}\right] \end{aligned} \tag{31}$$

as $\Lambda \to \infty$. Similarly,

$$\lim_{\Lambda\to\infty}\frac{\Lambda^2}{\Lambda^2+m^2-u\,x(1-x)} = \lim_{\Lambda\to\infty}\frac{1}{1+[m^2-u\,x(1-x)]/\Lambda^2} = 1. \tag{32}$$

Thus, the cutoff amplitude in this limit is

$$\begin{aligned} \langle p_1', p_2'|S|p_1, p_2\rangle_2 = \frac{ig^2\delta^4(p_1'+p_2'-p_1-p_2)}{2^9\pi^4\sqrt{p_1^0p_2^0p_1'^0p_2'^0}} \int_0^1 \Bigg\{ \ln\left[\frac{\Lambda^2}{m^2-s\,x(1-x)}\right] \\ +\ln\left[\frac{\Lambda^2}{m^2-t\,x(1-x)}\right]+\ln\left[\frac{\Lambda^2}{m^2-u\,x(1-x)}\right]-3\Bigg\}dx. \end{aligned} \tag{33}$$

So to second order the amplitude in this limit is

$$\begin{aligned} \langle p_1', p_2'|S|p_1, p_2\rangle_{1+2} = \frac{-i\delta^4(p_1'+p_2'-p_1-p_2)}{2^4\pi^2\sqrt{p_1^0p_2^0p_1'^0p_2'^0}} \Bigg[g-\frac{g^2}{2^5\pi^2}\int_0^1 \Bigg\{ \ln\left[\frac{\Lambda^2}{m^2-s\,x(1-x)}\right] \\ +\ln\left[\frac{\Lambda^2}{m^2-t\,x(1-x)}\right]+\ln\left[\frac{\Lambda^2}{m^2-u\,x(1-x)}\right]-3\Bigg\}dx\Bigg]. \end{aligned} \tag{34}$$

We **define the renormalized coupling g_r as**

$$g_r \equiv g-\frac{g^2}{2^5\pi^2}\int_0^1 \Bigg\{ \ln\left[\frac{\Lambda^2}{m^2-s\,x(1-x)}\right]+\ln\left[\frac{\Lambda^2}{m^2-t\,x(1-x)}\right]+\ln\left[\frac{\Lambda^2}{m^2-u\,x(1-x)}\right]-3\Bigg\}dx \tag{35}$$

at a specific but arbitrary combination of momenta. The standard choice is the off-shell point

$$\begin{aligned} p_1^2 = p_2^2 = p_1'^2 = p_2'^2 = \mu^2 \\ s = t = u = -\frac{4\mu^2}{3}. \end{aligned} \tag{36}$$

That is

$$\begin{aligned} g_r \equiv g &- \frac{g^2}{2^5\pi^2} \int_0^1 \left\{ \ln \left[\frac{\Lambda^2}{m^2 + \frac{4\mu^2}{3}\, x(1-x)} \right] + \ln \left[\frac{\Lambda^2}{m^2 + \frac{4\mu^2}{3}\, x(1-x)} \right] + \ln \left[\frac{\Lambda^2}{m^2 + \frac{4\mu^2}{3}\, x(1-x)} \right] - 3 \right\} dx \\ &= g - \frac{3g^2}{32\pi^2} \int_0^1 \left\{ \ln \left[\frac{\Lambda^2}{m^2 + (4\mu^2/3)\, x(1-x)} \right] - 1 \right\} dx \end{aligned} \tag{37}$$

So g_r is really a function of this mass μ:

$$g_r = g_r(\mu). \tag{38}$$

The physical amplitude at $s = t = u = -4\mu^2/3$ is

$$\langle p_1', p_2' | S | p_1, p_2 \rangle_{1+2} = \frac{-\, i\delta^4(p_1' + p_2' - p_1 - p_2)}{2^4\pi^2 \sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}}\; g_r. \tag{39}$$

So g_r is a finite physically meaningful quantity. But then the original coupling g must have been

$$g = g_r + \frac{3g^2}{32\pi^2} \int_0^1 \left\{ \ln \left[\frac{\Lambda^2}{m^2 + (4\mu^2/3)\, x(1-x)} \right] - 1 \right\} dx \equiv g_r + B\, g^2 \tag{40}$$

in which B diverges logarithmically.

(We can solve this quadratic equation

$$B\, g^2 - g + g_r = 0 \tag{41}$$

and get

$$g = \frac{1 \pm \sqrt{1 - 4Bg_r}}{2B} \tag{42}$$

which vanishes in the limit $B \to \infty$. But then g would be infinitesimal and complex.)

At other s, t, u points, the amplitude is

$$\begin{aligned}
\langle p_1', p_2'|S|p_1, p_2\rangle_{1+2} &= \frac{-i\delta^4(p_1' + p_2' - p_1 - p_2)}{2^4\pi^2\sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}} \Bigg[g - \frac{g^2}{2^5\pi^2} \int_0^1 \Bigg\{ \ln\left[\frac{\Lambda^2}{m^2 - s\,x(1-x)}\right] \\
&\quad + \ln\left[\frac{\Lambda^2}{m^2 - t\,x(1-x)}\right] + \ln\left[\frac{\Lambda^2}{m^2 - u\,x(1-x)}\right] - 3 \Bigg\} dx \Bigg] \\
&= \frac{-i\delta^4(p_1' + p_2' - p_1 - p_2)}{2^4\pi^2\sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}} \Bigg[g_r + \frac{3g^2}{32\pi^2} \int_0^1 \Bigg\{ \ln\left[\frac{\Lambda^2}{m^2 + (4\mu^2/3)\,x(1-x)}\right] - 1 \Bigg\} dx \\
&\quad - \frac{g^2}{32\pi^2} \int_0^1 \Bigg\{ \ln\left[\frac{\Lambda^2}{m^2 - s\,x(1-x)}\right] \\
&\quad + \ln\left[\frac{\Lambda^2}{m^2 - t\,x(1-x)}\right] + \ln\left[\frac{\Lambda^2}{m^2 - u\,x(1-x)}\right] - 3 \Bigg\} dx \Bigg].
\end{aligned} \tag{43}$$

The big bad $\ln \Lambda^2$ terms cancel, as do the -1 and the -3. We then have

$$\begin{aligned}
\langle p_1', p_2'|S|p_1, p_2\rangle_{1+2} &= \frac{-i\delta^4(p_1' + p_2' - p_1 - p_2)}{2^4\pi^2\sqrt{p_1^0 p_2^0 p_1'^0 p_2'^0}} \Bigg[g_r - \frac{g^2}{32\pi^2} \int_0^1 \Bigg\{ \ln\left[\frac{m^2 + 4\mu^2\,x(1-x)/3}{m^2 - s\,x(1-x)}\right] \\
&\quad + \ln\left[\frac{m^2 + 4\mu^2\,x(1-x)/3}{m^2 - t\,x(1-x)}\right] + \ln\left[\frac{m^2 + 4\mu^2\,x(1-x)/3}{m^2 - u\,x(1-x)}\right] \Bigg\} dx \Bigg]
\end{aligned} \tag{44}$$

which we rewrite as

$$
\begin{aligned}
\langle p_1', p_2'|S|p_1, p_2\rangle_{1+2} = \frac{-\,i\delta^4(p_1'+p_2'-p_1-p_2)}{2^4\pi^2\sqrt{p_1^0p_2^0p_1'^0p_2'^0}} \Bigg[g_r - \frac{g_r^2}{32\pi^2}\int_0^1 \Bigg\{ \ln\left[\frac{m^2+4\mu^2\,x(1-x)/3}{m^2-s\,x(1-x)}\right] \\
+ \ln\left[\frac{m^2+4\mu^2\,x(1-x)/3}{m^2-t\,x(1-x)}\right] + \ln\left[\frac{m^2+4\mu^2\,x(1-x)/3}{m^2-u\,x(1-x)}\right] \Bigg\} dx \Bigg]
\end{aligned}
\tag{45}
$$

to order g_r^2. Because $g_r = g_r(\mu)$ depends upon the renormalization point μ, $g_r(\mu)$ is called a **running coupling constant** or a **running coupling**. The quantity μ^2 can be any real number greater than $-3m^2$, which makes the integral real.

The physical amplitude must not change when we change μ, so changes in μ^2 are compensated by changes in $g_r(\mu)$. The μ-dependent part of $\langle p_1', p_2'|S|p_1, p_2\rangle_{1+2}$ is inside the square bracket of (45)

$$
s(\mu) = g_r - \frac{3g_r^2}{32\pi^2}\int_0^1 \ln\left[m^2 + (4\mu^2/3)\,x(1-x)\right]\,dx \tag{46}
$$

and in the limit $\mu^2 >> m^2$

$$
s(\mu) \approx g_r - \frac{3g_r^2}{32\pi^2}\ln\mu^2. \tag{47}
$$

The condition that $s(\mu)$ be independent of μ then is

$$
0 = \frac{ds(\mu)}{d\mu} = \frac{dg_r(\mu)}{d\mu} - \frac{3g_r(\mu)\ln\mu}{8\pi^2}\frac{dg_r(\mu)}{d\mu} - \frac{3g_r^2}{32\pi^2}\frac{2}{\mu} \tag{48}
$$

or

$$
\left(1 - \frac{3g_r(\mu)\ln\mu}{8\pi^2}\right)\frac{dg_r(\mu)}{d\mu} = \frac{3g_r^2}{16\pi^2}\,\frac{1}{\mu}. \tag{49}
$$

To lowest order in g_r, we have

$$
\frac{dg_r(\mu)}{d\mu} = \frac{3g_r^2}{16\pi^2}\,\frac{1}{\mu} \tag{50}
$$

or

$$\mu \frac{dg_r(\mu)}{d\mu} \equiv \beta(g_r(\mu)) = \frac{3g_r^2}{16\pi^2} \tag{51}$$

which is an example of the **Callan-Symanzik equation**.

It is usual to write $g_\mu \equiv g_r(\mu)$. We can integrate the Callan-Symanzik equation

$$\ln \frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_E} \frac{dg_\mu}{\beta(g_\mu)} = \frac{16\pi^2}{3} \int_{g_M}^{g_E} \frac{dg_\mu}{g_\mu^2} = \frac{16\pi^2}{3} \left(\frac{1}{g_M} - \frac{1}{g_E} \right) \tag{52}$$

to find the running physical coupling constant g_μ at energy $\mu = E$

$$g_E = \frac{g_M}{1 - 3\, g_M \, \ln(E/M)/16\pi^2}. \tag{53}$$

As the energy $E = \sqrt{s}$ rises above M, while staying below the singular value $E = M \exp(16\pi^2/3g_M)$, the running coupling g_E slowly increases. And so does the scattering amplitude, $s(\mu) \approx g_E$.