Perturbation Theory

Kevin Cahill for 523

January 22, 2013

1 The Interaction Picture

As you learned in your courses on quantum mechanics, to do perturbation theory one splits the hamiltonian $H = H_0 + V$ into an easy part H_0 and a hard part V. A good way to do time-dependent perturbation theory is to have the operators vary with time as if they were the Heisenberg field operators of the theory in which the hamiltonian is the easy one H_0. So the electromagnetic field $A(x)$ and a Dirac field $\psi(x)$ will change with time as

$$A(x) = e^{iH_0 t} A(0, \vec{x}) e^{-iH_0 t} \quad \text{and} \quad \psi(x) = e^{iH_0 t} \psi(0, \vec{x}) e^{-iH_0 t}. \tag{1}$$

Now the hamiltonian $H = H_0 + V$ and its easy H_0 and hard V parts are spatial integrals of some combinations of the fields and their derivatives, usually polynomials of low degree. The easy part H_0 of the hamiltonian for quantum electrodynamics (QED) just represents the energy of sets of noninteracting electrons, positrons, and photons.

The easy hamiltonian H_0 for the field of the electron is

$$H_0 = \int \overline{\psi}(x) \left(\vec{\gamma} \cdot \vec{\nabla} + m \right) \psi(x)\, d^3x = i \int \psi^\dagger(x) \gamma^0 \left(\vec{\gamma} \cdot \vec{\nabla} + m \right) \psi(x)\, d^3x. \tag{2}$$

The electron field and its adjoint are

$$
\begin{aligned}
\psi_\ell(x) &= \sum_{s=-}^{+} \int \left[u_\ell(\vec{p},s) b(p,s) e^{ipx} + v_\ell(\vec{p},s) c^\dagger(p,s) e^{-ipx} \right] \frac{d^3p}{(2\pi)^{3/2}} \\
\psi_\ell^\dagger(x) &= \sum_{s=-}^{+} \int \left[u_\ell^*(\vec{p},s) b^\dagger(p,s) e^{-ipx} + v_\ell^*(\vec{p},s) c(p,s) e^{ipx} \right] \frac{d^3p}{(2\pi)^{3/2}}.
\end{aligned} \tag{3}
$$

The Dirac spinors satisfy

$$
\begin{aligned}
(i\vec{\gamma} \cdot \vec{p} + m) u(p,s) &= i\gamma^0 p^0 u(p,s) \\
(-i\vec{\gamma} \cdot \vec{p} + m) v(p,s) &= -i\gamma^0 p^0 v(p,s).
\end{aligned} \tag{4}
$$

So

$$
\begin{aligned}
\left(\vec{\gamma} \cdot \vec{\nabla} + m\right) \psi(x) &= \sum_{s=-}^{+} \int \left[\left(\vec{\gamma} \cdot \vec{\nabla} + m\right) u(\vec{p},s) b(p,s) e^{ipx} + \left(\vec{\gamma} \cdot \vec{\nabla} + m\right) v(\vec{p},s) c^\dagger(p,s) e^{-ipx} \right] \frac{d^3p}{(2\pi)^{3/2}} \\
&= \sum_{s=-}^{+} \int \left[(i\vec{\gamma} \cdot \vec{p} + m)\, u(\vec{p},s) b(p,s) e^{ipx} + (-i\vec{\gamma} \cdot \vec{p} + m)\, v(\vec{p},s) c^\dagger(p,s) e^{-ipx} \right] \frac{d^3p}{(2\pi)^{3/2}} \\
&= \sum_{s=-}^{+} \int \left[i\gamma^0 p^0 u(\vec{p},s) b(p,s) e^{ipx} - i\gamma^0 p^0 v(\vec{p},s) c^\dagger(p,s) e^{-ipx} \right] \frac{d^3p}{(2\pi)^{3/2}}
\end{aligned} \tag{5}
$$

and thus H_0 is

$$
\begin{aligned}
H_0 &= i \int \psi^\dagger(x)\gamma^0 \left(\vec{\gamma}\cdot\vec{\nabla} + m\right) \psi(x)\, d^3x \\
&= i \int \sum_{s'=-}^{+} \int \left[u^\dagger(\vec{p}', s')b^\dagger(p', s')e^{-ip'x} + v^\dagger(\vec{p}', s')c(p', s')e^{ip'x}\right] \frac{d^3p'}{(2\pi)^{3/2}}\gamma^0 \\
&\quad \times \sum_{s=-}^{+} \int \left[i\gamma^0 p^0 u(\vec{p}, s)b(p, s)e^{ipx} - i\gamma^0 p^0 v(\vec{p}, s)c^\dagger(p, s)e^{-ipx}\right] \frac{d^3p}{(2\pi)^{3/2}} d^3x \\
&= \int \sum_{s'=-}^{+} \int \left[u^\dagger(\vec{p}', s')b^\dagger(p', s')e^{-ip'x} + v^\dagger(\vec{p}', s')c(p', s')e^{ip'x}\right] \frac{d^3p'}{(2\pi)^3} \\
&\quad \times \sum_{s=-}^{+} \int \left[p^0 u(\vec{p}, s)b(p, s)e^{ipx} - p^0 v(\vec{p}, s)c^\dagger(p, s)e^{-ipx}\right] d^3p\, d^3x \\
&= \sum_{s'=-}^{+}\sum_{s=-}^{+} \int d^3p' d^3p \Big[u^\dagger(\vec{p}', s')b^\dagger(p', s')p^0 u(\vec{p}, s)b(p, s)e^{i(p'^0-p^0)x^0}\delta(\vec{p}-\vec{p}') \\
&\quad - v^\dagger(\vec{p}', s')c(p', s')p^0 v(\vec{p}, s)c^\dagger(p, s)e^{-i(p'^0-p^0)x^0}\delta(\vec{p}-\vec{p}') \\
&\quad - u^\dagger(\vec{p}', s')b^\dagger(p', s')p^0 v(\vec{p}, s)c^\dagger(p, s)e^{i(p'^0+p^0)x^0}\delta(\vec{p}+\vec{p}') \\
&\quad + v^\dagger(\vec{p}', s')c(p', s')p^0 u(\vec{p}, s)b(p, s)e^{-i(p'^0+p^0)x^0}\delta(\vec{p}+\vec{p}')\Big] \\
&= \sum_{s'=-}^{+}\sum_{s=-}^{+} \int d^3p \Big[u^\dagger(\vec{p}, s')b^\dagger(p, s')p^0 u(\vec{p}, s)b(p, s) \\
&\quad - v^\dagger(\vec{p}, s')c(p, s')p^0 v(\vec{p}, s)c^\dagger(p, s) \\
&\quad - u^\dagger(-\vec{p}, s')b^\dagger(-p, s')p^0 v(\vec{p}, s)c^\dagger(p, s)e^{2ip^0x^0} \\
&\quad + v^\dagger(-\vec{p}, s')c(-p, s')p^0 u(\vec{p}, s)b(p, s)e^{-2ip^0x^0}\Big] .
\end{aligned} \tag{6}
$$

We saw in equation (20) of the notes on Dirac spinors that the spinors are suitably orthonormal:

$$\begin{aligned} u^\dagger(\vec{p},s)u(\vec{p},s') &= \delta_{s,s'} \\ v^\dagger(\vec{p},s)v(\vec{p},s') &= \delta_{s,s'} \\ u^\dagger(\vec{p},s)v(-\vec{p},s') &= 0 \\ v^\dagger(\vec{p},s)u(-\vec{p},s') &= 0. \end{aligned} \tag{7}$$

Thus since $c(p,s)c^\dagger(p,s) + c^\dagger(p,s)c(p,s) = \delta^{(3)}(\vec{0}) = L^3/(2\pi)^3$, the easy hamiltonian is

$$\begin{aligned} H_0 &= \sum_{s'=-}^{+}\sum_{s=-}^{+}\int d^3p \left[\delta_{s,s'} b^\dagger(p,s')p^0 b(p,s) - \delta_{s,s'} c(p,s')p^0 v(\vec{p},s)c^\dagger(p,s)\right] \\ &= \sum_{s=-}^{+}\int d^3p \; p^0 \left[b^\dagger(p,s)b(p,s) - c(p,s)c^\dagger(p,s)\right] \\ &= \sum_{s=-}^{+}\int d^3p \; p^0 \left[b^\dagger(p,s)b(p,s) + c^\dagger(p,s)c(p,s) - \delta(\vec{0})\right] \\ &= \sum_{s=-}^{+}\int d^3p \; p^0 \left[b^\dagger(p,s)b(p,s) + c^\dagger(p,s)c(p,s) - L^3/(2\pi)^3\right] \end{aligned} \tag{8}$$

in which we used the formula

$$\delta(\vec{p}) = \int e^{\vec{p}\cdot\vec{x}} \, d^3x/(2\pi)^3. \tag{9}$$

So the easy hamiltonian for the system of noninteracting electrons, positrons, and photons is

$$H_0 = \int \sum_{s=-}^{+} \sqrt{\vec{p}^2 + m^2}\left[b^\dagger(p,s)b(p,s) + c^\dagger(p,s)c(p,s) - \frac{L^3}{(2\pi)^3}\right] + |\vec{p}|\left[a^\dagger(p,s)a(p,s) + \frac{1}{2}\frac{L^3}{(2\pi)^3}\right] d^3p. \tag{10}$$

The $\pm\frac{1}{2}$ terms may have something to do with dark energy, but since they are quartically divergent (quite apart from the L^3 volume factors), we'll remove them by normally ordering the hamiltonian. That is, we'll write our operators including H_0 and V in terms of the fields A and ψ and then we'll move every annihilation operator to the right past any creation operators without taking into account the nonzero value of the corresponding commutator or anticommutator but keeping any minus signs that arise from anticommutation. When we're done, we'll surround the operator with colons to indicate normal ordering

$$: H_0 := \int \sum_{s=-}^{+} \sqrt{\vec{p}^2 + m^2} \left[b^\dagger(p,s) b(p,s) + c^\dagger(p,s) c(p,s) \right] + |\vec{p}| \left[a^\dagger(p,s) a(p,s) \right] d^3p. \tag{11}$$

Now the normally ordered easy part $: H_0 :$ just counts up the particles of each kind and multiplies by their energy.

The hard part $: V :$ is cubic

$$: V := ie \int : \overline{\psi}(x) \gamma^b A_b(x) \psi(x) : \, d^3x = ie \int : \overline{\psi}(x) \not{A}(x) \psi(x) : \, d^3x \tag{12}$$

in which the electromagnetic field is

$$\boldsymbol{A}(x) = \sum_{\lambda=-}^{+} \int \left[\boldsymbol{\epsilon}(\boldsymbol{k},\lambda) a(\boldsymbol{k},\lambda) e^{ikx} + \boldsymbol{\epsilon}^*(\boldsymbol{k},\lambda) a^\dagger(\boldsymbol{k},\lambda) e^{-ikx} \right] \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \tag{13}$$

and $e > 0$ is the charge of the positron. We use this Fourier expansion of the electromagnetic field for incoming and outgoing photons. For internal photons, also called virtual photons, we use a 4-vector form $A_b(x)$ of the field with the Lorentz-covariant propagator we derived in the notes on path integrals

$$\langle 0 | \mathcal{T} \left[A_a(x) A_b(y) \right] | 0 \rangle^{eff} = -i \Delta_{ab}^{eff}(x-y) = -i \int \frac{\eta_{ab}}{k^2 - i\epsilon} \, e^{ik(x-y)} \, \frac{d^4k}{(2\pi)^4} \tag{14}$$

and mentioned in equation (17) of the notes on coherent states. Here $|0\rangle$ is the ground state or vacuum of the easy hamiltonian H_0. We'll often omit the colons that denote normal ordering.

When doing (time-dependent) perturbation theory, we compute transition rates

$$R(t, t_0) = \langle f | e^{-iH(t-t_0)} | i \rangle \tag{15}$$

between initial $|i\rangle$ and final $|f\rangle$ eigenstates of the easy hamiltonian H_0. An overall phase factor doesn't matter, so we can just as well compute

$$r(t,t_0) = \langle f|e^{iH_0t}e^{-iH(t-t_0)}e^{-iH_0t_0}|i\rangle \equiv \langle f|U(t,t_0)|i\rangle \tag{16}$$

which differs by the phase $e^{i(E_ft-E_it_0)}$. The unitary operator $U(t,t_0)$ obeys the differential equation

$$\begin{aligned}\partial_t U(t,t_0) = \dot{U}(t,t_0) &= \partial_t\left(e^{iH_0t}e^{-iH(t-t_0)}e^{-iH_0t_0}\right) = \partial_t\left(e^{iH_0t}\right)e^{-iHt}e^{-iH_0t_0} + e^{iH_0t}\partial_t\left(e^{-iHt}\right)e^{-iH_0t_0}\\ &= e^{iH_0t}\left(iH_0 - iH\right)e^{-iHt}e^{-iH_0t_0} = e^{iH_0t}(-iV)e^{-iHt}e^{-iH_0t_0} = e^{iH_0t}(-iV)e^{-iH_0t}e^{iH_0t}e^{-iHt}e^{-iH_0t_0}\\ &= -iV(t)U(t,t_0)\end{aligned} \tag{17}$$

in which the nonlinear part of the hamiltonian has the time dependence induced by the easy hamiltonian

$$V(t) = e^{iH_0t}Ve^{-iH_0t}. \tag{18}$$

That is, the fields $\psi(x)$ and $A(x)$ have the simple time dependence that turns $e^{\pm i\vec{p}\cdot\vec{x}}$ into $e^{\pm ipx} = e^{\pm i(\vec{p}\cdot\vec{x}-p^0x^0)}$ in their Fourier expansions. The solution to the differential equation (17) is the time-ordered exponential

$$U(t,t_0) = \mathcal{T}\left\{\exp\left[-i\int_{t_0}^{t} V(t')\,dt'\right]\right\}. \tag{19}$$

So our problem is to compute matrix elements of this time-ordered exponential

$$\begin{aligned}\langle f|\mathcal{T}\left\{\exp\left[-i\int_{-\infty}^{\infty} V(t')\,dt'\right]\right\}|i\rangle &= \langle f|\mathcal{T}\left\{\exp\left[-i\int_{-\infty}^{\infty} ie\int \overline{\psi}(x)\not{A}(x)\psi(x)\,d^3x\,dt'\right]\right\}|i\rangle\\ &= \langle f|\mathcal{T}\left\{\exp\left[e\int \overline{\psi}(x)\not{A}(x)\psi(x)\,d^4x\right]\right\}|i\rangle\end{aligned} \tag{20}$$

in which the integration is over all of space-time and the fields evolve under their non interacting, free-field hamiltonian H_0.

2 Electron-Electron Scattering

The initial state is $|p,s;q,t\rangle = a^\dagger(p,s)a^\dagger(q,t)|0\rangle$ in which $|0\rangle$ is the perturbative vacuum, which every annihilation operator maps to zero, $a(p,s)|0\rangle = 0$. The final state is $|p',s';q',t'\rangle = a^\dagger(p',s')a^\dagger(q',t')|0\rangle$. Let's assume that p' and q' are both different from p and q.

Homework problem 1: Show that the first-order term, the term with one factor of e, does not contribute to electron-electron scattering.

So the lowest-order contribution to electron-electron scattering is with $U \equiv U(\infty, -\infty)$

$$\langle p',s';q',t'|U|p,s;q,t\rangle = \langle p',s';q',t'|\,\frac{e^2}{2}\int \mathcal{T}\left[\overline{\psi}(x)A\!\!\!/(x)\psi(x)\,\overline{\psi}(y)A\!\!\!/(y)\psi(y)\right]\,d^4x d^4y|p,s;q,t\rangle. \tag{21}$$

The electron of momentum p can be absorbed either at x or at y. The two choices give the same number. So if we say that the electron of momentum p is absorbed at x, not at y, then we can cancel the factor of 2. Next, the outgoing electron of momentum p' can be emitted either at x or at y. These are two different processes and are represented by two different amplitudes which we add.

To turn the formula (21) into numbers, we use the Fourier expansions (3) of the electron field. The amplitude for p in and p' out at x is then

$$\langle p',s';q',t'|U|p,s;q,t\rangle = \langle p',s';q',t'|\,e^2\int \mathcal{T}\left[\overline{\psi}^-(x)\overline{\psi}(y)A\!\!\!/(y)\psi(y)\,A\!\!\!/(x)\psi^+(x)\right]\,d^4x d^4y|p,s;q,t\rangle \tag{22}$$

in which two minus signs have cancelled. The annihilating $\psi(x)^+$ part of the field could annihilate either initial electron, but we agreed it'd do the one of momentum p, so using the anticommutation relations

$$\begin{aligned}\{b(p,s),b(p',s')\} &= 0 = \{c(p,s)\,c(p',s')\} = \{b(p,s),c^\dagger(p',s')\}\\ \{b(p,s),b^\dagger(p',s')\} &= \delta_{s,s'}\,\delta^{(3)}(\vec{p}-\vec{p}') = \{c(p,s),c^\dagger(p',s')\},\end{aligned} \tag{23}$$

we find

$$\begin{aligned}\psi^+_\ell(x)|p,s;q,t\rangle &= \sum_{s''=-}^{+}\int u_\ell(\vec{p}'',s'')b(p'',s)e^{ip''x}\frac{d^3p''}{(2\pi)^{3/2}}b^\dagger(p,s)|q,t\rangle\\ &= u_\ell(\vec{p},s)e^{ipx}\frac{1}{(2\pi)^{3/2}}|q,t\rangle.\end{aligned} \tag{24}$$

As a second homework problem, show that the effect of $\overline{\psi}^{-}_{\ell'}(x)$ on the final state (assuming it acts on the p', s' electron) is

$$\langle p', s'; q', t'|\,\overline{\psi}^{-}_{\ell'}(x) = \langle q', t'|\,\overline{u}_{\ell'}(\vec{p}', s')\, e^{-ip'x}\,\frac{1}{(2\pi)^{3/2}}. \tag{25}$$

As a third homework problem, show that the electron fields at y convert the vacuum state $|0\rangle$ into the q, t and q', t' states of an electron

$$\begin{aligned} \psi^{+}_{\ell}(y)|q,t\rangle &= u_{\ell}(\vec{q},t)e^{iqy}\frac{1}{(2\pi)^{3/2}}|0\rangle \\ \langle q', t'|\,\overline{\psi}^{-}_{\ell''}(y) &= \langle 0|\,\overline{u}_{\ell''}(\vec{q}', t')\, e^{-iq'y}\,\frac{1}{(2\pi)^{3/2}}. \end{aligned} \tag{26}$$

So since the fields at y turn the q, t electron into a q', t' electron, the amplitude for p in and p' out at x is

$$\begin{aligned} \langle p', s'; q', t'|U|p, s; q, t\rangle &= \frac{e^2}{(2\pi)^3}\langle q', t'|\int \mathcal{T}\left[\overline{\psi}(y)\not{A}(y)\psi(y)\,\overline{u}(\vec{p}', s')\not{A}(x)u(\vec{p}, s)e^{i(p-p')x}\right] d^4x d^4y\,|q, t\rangle \\ &= \frac{e^2}{(2\pi)^6}\langle 0|\int \mathcal{T}\left[\overline{u}(\vec{q}', t')\not{A}(y)u(q, t)\,\overline{u}(\vec{p}', s')\not{A}(x)u(\vec{p}, s)e^{i(p-p')x}e^{i(q-q')y}\right] d^4x d^4y\,|0\rangle \\ &= \frac{e^2}{(2\pi)^6}\overline{u}(\vec{q}', t')\gamma^a u(q, t)\,\overline{u}(\vec{p}', s')\gamma^b u(p, s)\int \langle 0|\mathcal{T}\left[A_a(y)A_b(x)\right]|0\rangle e^{i(p-p')x}e^{i(q-q')y}d^4x d^4y \end{aligned} \tag{27}$$

We now use our formula (14) for the mean value of the time-ordered product of two electromagnetic fields

$$\langle 0|\mathcal{T}\left[A_a(x)A_b(y)\right]|0\rangle^{eff} = -i\Delta^{eff}_{ab}(x-y) = -i\int \frac{\eta_{ab}}{k^2 - i\epsilon}\, e^{ik(x-y)}\,\frac{d^4k}{(2\pi)^4} \tag{28}$$

in the vacuum $|0\rangle$. We call this the photon propagator. So the amplitude for p in and p' out at x is

$$\begin{aligned}
\langle p', s'; q', t'|U|p, s; q, t\rangle &= \frac{e^2}{(2\pi)^6}\overline{u}(\vec{q}', t')\gamma^a u(q,t)\,\overline{u}(\vec{p}', s')\gamma^b u(p,s)\int \langle 0|\mathcal{T}\left[A_a(y)A_b(x)\right]|0\rangle e^{i(p-p')x}e^{i(q-q')y}d^4x d^4y \\
&= \frac{-ie^2}{(2\pi)^{10}}\overline{u}(\vec{q}', t')\gamma^a u(q,t)\,\overline{u}(\vec{p}', s')\gamma^b u(p,s)\int \frac{\eta_{ab}}{k^2 - i\epsilon}\, e^{ik(x-y)}e^{i(p-p')x}e^{i(q-q')y}d^4x d^4y d^4k \\
&= \frac{-ie^2}{(2\pi)^2}\overline{u}(\vec{q}', t')\gamma^a u(q,t)\,\overline{u}(\vec{p}', s')\gamma_a u(p,s)\int \frac{\delta(k+p-p')\delta(-k+q-q')}{k^2 - i\epsilon}\, d^4k \\
&= \frac{-ie^2}{(2\pi)^2}\,\delta(p+q-p'-q')\,\frac{\overline{u}(\vec{q}', t')\gamma^a u(q,t)\,\overline{u}(\vec{p}', s')\gamma_a u(p,s)}{(p'-p)^2 - i\epsilon}.
\end{aligned} \tag{29}$$

The remaining delta function enforces conservation of energy and momentum.

What about the amplitude for the q', t' electron to be made at x? Since

$$\begin{aligned}
b(q'', t'')|p', s'; q', t'\rangle &= b(q'', t'')b^\dagger(p', s')b^\dagger(q', t')|0\rangle = -b(q'', t'')b^\dagger(q', t')b^\dagger(p', s')|0\rangle \\
&= -\,\delta_{t',t''}\delta(\vec{q}'' - \vec{q}')b^\dagger(p', s')|0\rangle = -\delta_{t',t''}\delta(\vec{q}'' - \vec{q}')|p', s'\rangle,
\end{aligned} \tag{30}$$

the amplitude for p in and q' out at x is

$$\begin{aligned}
\langle p', s'; q', t'|U|p, s; q, t\rangle &= \langle p', s'; q', t'|\, e^2 \int \mathcal{T}\left[\overline{\psi}^-(x)\overline{\psi}(y)\not{A}(y)\psi(y)\,\not{A}(x)\psi^+(x)\right]\, d^4x d^4y|p, s; q, t\rangle \\
&= -\frac{e^2}{(2\pi)^3}\langle p', s'|\int \mathcal{T}\left[\overline{\psi}(y)\not{A}(y)\psi(y)\,\overline{u}(\vec{q}', s')\not{A}(x)u(\vec{p}, s)e^{i(p-q')x}\right] d^4x d^4y\,|q, t\rangle \\
&= -\frac{e^2}{(2\pi)^6}\langle 0|\int \mathcal{T}\left[\overline{u}(\vec{p}', s')\not{A}(y)u(q, t)\,\overline{u}(\vec{q}', t')\not{A}(x)u(\vec{p}, s)e^{i(p-q')x}e^{i(q-p')y}\right] d^4x d^4y\,|0\rangle \\
&= -\frac{e^2}{(2\pi)^6}\overline{u}(\vec{p}', s')\gamma^a u(q, t)\,\overline{u}(\vec{q}', t')\gamma^b u(p, s)\int \langle 0|\mathcal{T}\left[A_a(y)A_b(x)\right]\,|0\rangle e^{i(p-q')x}e^{i(q-p')y}d^4x d^4y \\
&= \frac{ie^2}{(2\pi)^{10}}\overline{u}(\vec{p}', s')\gamma^a u(q, t)\,\overline{u}(\vec{q}', t')\gamma^b u(p, s)\int \frac{\eta_{ab}}{k^2 - i\epsilon}\, e^{ik(x-y)}e^{i(p-q')x}e^{i(q-p')y}d^4x d^4y d^4k \\
&= \frac{ie^2}{(2\pi)^2}\overline{u}(\vec{p}', s')\gamma^a u(q, t)\,\overline{u}(\vec{q}', t')\gamma_a u(p, s)\int \frac{\delta(k + p - q')\delta(-k + q - p')}{k^2 - i\epsilon}\, d^4k \\
&= \frac{ie^2}{(2\pi)^2}\,\delta(p + q - p' - q')\,\frac{\overline{u}(\vec{p}', s')\gamma^a u(q, t)\,\overline{u}(\vec{q}', t')\gamma_a u(p, s)}{(q' - p)^2 - i\epsilon}.
\end{aligned} \tag{31}$$

Fermi statistics has provided a relative minus sign between the two amplitudes (29 & 31). The total amplitude is the sum

$$\langle p', s'; q', t'|U|p, s; q, t\rangle = \frac{-ie^2\delta(p + q - p' - q')}{(2\pi)^2}\left[\frac{\overline{u}(\vec{q}', t')\gamma^a u(q, t)\,\overline{u}(\vec{p}', s')\gamma_a u(p, s)}{(p' - p)^2 - i\epsilon} - \frac{\overline{u}(\vec{p}', s')\gamma^a u(q, t)\,\overline{u}(\vec{q}', t')\gamma_a u(p, s)}{(q' - p)^2 - i\epsilon}\right]. \tag{32}$$

Before turning this amplitude into a cross-section, let us look at another scattering process.

3 Electron-Positron Scattering

After canceling the factor of 2, the amplitude for the process in which an electron of momentum p and spin index s is absorbed at x and emitted from x as an electron of momentum p' and spin index s' while a positron momentum q and spin index t is absorbed at y and emitted from y as a positron of momentum q' and spin index t' is

$$\begin{aligned}
\langle p', s'; q', t'|U|p, s; q, t\rangle_1 &= \langle p', s'; q', t'|\, e^2 \int \mathcal{T}\left[\overline{\psi}^+(y) \not{A}(y) \psi^-(y)\, \overline{\psi}^-(x) \not{A}(x) \psi^+(x)\right]\, d^4x d^4y\, |p, s; q, t\rangle \\
&= \frac{e^2}{(2\pi)^3} \langle q', t'| \int \mathcal{T}\left[\overline{\psi}^+(y) \not{A}(y) \psi^-(y)\, \overline{u}(\vec{p}', s') \not{A}(x) u(\vec{p}, s) e^{i(p-p')x}\right] d^4x d^4y\, |q, t\rangle \\
&= -\frac{e^2}{(2\pi)^6} \int \langle 0| \mathcal{T}\left[\overline{v}(q, t) \not{A}(y) v(q', t') e^{i(q-q')y}\, \overline{u}(\vec{p}', s') \not{A}(x) u(\vec{p}, s) e^{i(p-p')x}\right] |0\rangle d^4x d^4y \\
&= -\frac{e^2}{(2\pi)^6} \int \overline{v}(q, t) \gamma^b v(q', t') e^{i(q-q')y}\, \overline{u}(\vec{p}', s') \gamma^a u(\vec{p}, s) e^{i(p-p')x} \\
&\quad \times \langle 0| \mathcal{T}\left[A_b(y) A_a(x)\right] |0\rangle d^4x d^4y.
\end{aligned} \tag{33}$$

The time-ordered product $\langle 0|\mathcal{T}\left[A_a(x)A_b(y)\right]|0\rangle$ is the photon propagator (28), and so the amplitude is

$$\begin{aligned}
\langle p', s'; q', t'|U|p, s; q, t\rangle_1 &= \frac{ie^2}{(2\pi)^6}\int \overline{v}(q,t)\gamma^b v(q',t')e^{i(q-q')y}\,\overline{u}(\vec{p}',s')\gamma^a u(\vec{p},s)e^{i(p-p')x} \\
&\quad\times\int \frac{\eta_{ab}}{k^2-i\epsilon}\,e^{ik(y-x)}\,\frac{d^4k}{(2\pi)^4}d^4x d^4y \\
&= \frac{ie^2}{(2\pi)^6}\int \overline{v}(q,t)\gamma^b v(q',t')e^{i(q-q')y}\,\overline{u}(\vec{p}',s')\gamma^a u(\vec{p},s)e^{i(p-p')x} \\
&\quad\times\int \frac{\eta_{ab}}{k^2-i\epsilon}\,e^{ik(y-x)}\,\frac{d^4k}{(2\pi)^4}d^4x d^4y \qquad (34)\\
&= \frac{ie^2}{(2\pi)^2}\int \overline{v}(q,t)\gamma_a v(q',t')\,\overline{u}(\vec{p}',s')\gamma^a u(\vec{p},s) \\
&\quad\times\int \frac{1}{k^2-i\epsilon}\,\delta(p-p'-k)\delta(q-q'+k)\,d^4k \\
&= \frac{ie^2}{(2\pi)^2}\overline{v}(q,t)\gamma_a v(q',t')\,\overline{u}(\vec{p}',s')\gamma^a u(\vec{p},s)\frac{1}{(p-p')^2-i\epsilon}\,\delta(p+q-p'-q').
\end{aligned}$$

But there is another process: the incoming electron and positron can both be absorbed at x, and the final electron and positron can both be emitted at y. The amplitude for this process is

$$
\begin{aligned}
\langle p', s'; q', t'|U|p, s; q, t\rangle_2 &= \langle p', s'; q', t'|\, e^2 \int \mathcal{T}\left[\overline{\psi}^-(y)\not{A}(y)\psi^-(y)\,\overline{\psi}^+(x)\not{A}(x)\psi^+(x)\right]\, d^4x d^4y\,|p, s; q, t\rangle \\
&= \frac{e^2}{(2\pi)^3}\langle p', s'; q', t'|\int \mathcal{T}\left[\overline{\psi}^-(y)\not{A}(y)\psi^-(y)\,\overline{v}(q, t)\not{A}(x)u(p, s)e^{i(p+q)x}\right]\, d^4x d^4y\,|0\rangle \\
&= \frac{e^2}{(2\pi)^6}\langle 0|\int \mathcal{T}\left[\overline{u}(p', s')\not{A}(y)v(q', t')e^{-i(p'+q')y}\,\overline{v}(q, t)\not{A}(x)u(p, s)e^{i(p+q)x}\right] d^4x d^4y\,|0\rangle \\
&= \frac{e^2}{(2\pi)^6}\overline{u}(p', s')\gamma^a v(q', t')\overline{v}(q, t)\gamma^b u(p, s) \\
&\quad \times \int \langle 0|\mathcal{T}\left[A_a(y)A_b(x)\right]|0\rangle e^{-i(p'+q')y}\, e^{i(p+q)x} d^4x d^4y \\
&= \frac{-ie^2}{(2\pi)^6}\overline{u}(p', s')\gamma^a v(q', t')\overline{v}(q, t)\gamma^b u(p, s) \\
&\quad \times \int \frac{\eta_{ab}}{k^2 - i\epsilon}\, e^{ik(y-x)}\, \frac{d^4k}{(2\pi)^4} e^{-i(p'+q')y}\, e^{i(p+q)x} d^4x d^4y \\
&= \frac{-ie^2}{(2\pi)^6}\overline{u}(p', s')\gamma^a v(q', t')\overline{v}(q, t)\gamma_a u(p, s) \\
&\quad \times \int \frac{1}{k^2 - i\epsilon}\, e^{ik(y-x)}\, \frac{d^4k}{(2\pi)^4} e^{-i(p'+q')y}\, e^{i(p+q)x} d^4x d^4y \\
&= \frac{-ie^2}{(2\pi)^2}\overline{u}(p', s')\gamma^a v(q', t')\overline{v}(q, t)\gamma_a u(p, s)\int \frac{1}{k^2 - i\epsilon}\, d^4k\, \delta(-k + p + q)\delta(k - p' - q') \\
&= \frac{-ie^2}{(2\pi)^2}\overline{u}(p', s')\gamma^a v(q', t')\overline{v}(q, t)\gamma_a u(p, s)\delta(p' + q' - p - q)\frac{1}{(p + q)^2 - i\epsilon}.
\end{aligned}
\tag{35}
$$

So we must add these two amplitudes

$$
\begin{aligned}
\langle p', s'; q', t'|U|p, s; q, t\rangle = & \frac{ie^2}{(2\pi)^2}\overline{v}(q,t)\gamma_a v(q',t')\,\overline{u}(\vec{p}',s')\gamma^a u(\vec{p},s)\frac{1}{(p-p')^2-i\epsilon}\,\delta(p+q-p'-q') \\
& +\frac{-ie^2}{(2\pi)^2}\overline{u}(p',s')\gamma^a v(q',t')\overline{v}(q,t)\gamma_a u(p,s)\delta(p'+q'-p-q)\frac{1}{(p+q)^2-i\epsilon} \\
= & \frac{ie^2\delta(p'+q'-p-q)}{(2\pi)^2}\left[\frac{\overline{v}(q,t)\gamma_a v(q',t')\,\overline{u}(\vec{p}',s')\gamma^a u(\vec{p},s)}{(p-p')^2} - \frac{\overline{u}(p',s')\gamma^a v(q',t')\overline{v}(q,t)\gamma_a u(p,s)}{(p+q)^2}\right].
\end{aligned}
\tag{36}
$$

4 Pair Production of Muons

The production of muon pairs is a simpler process because there is only one lowest-order diagram. The interaction now is

$$
V = \int \overline{\psi}_e(x)\not{A}(x)\psi_e(x) + \int \overline{\psi}_\mu(x)\not{A}(x)\psi_\mu(x)\, d^3x \tag{37}
$$

in which the muon field ψ_μ is just like the electron field ψ_e but in which the mass is that of the muon, $m_\mu \approx 105.7$ MeV. The easy hamiltonian now has a term that counts the number of muons and antimuons with $p^0 = \sqrt{\vec{p}^2 + m_\mu^2}$. The amplitude for an electron of momentum $\vec{p}$ and spin s and a positron of momentum $\vec{q}$ and spin t to turn into a muon of momentum $\vec{p}'$ and spin s' and an antimuon of momentum $\vec{q}'$ and spin t' is given by equation (35) but in which the final-state spinors are built with the mass of the muon not that of the electron

$$
\langle p', s'; q', t'|U|p, s; q, t\rangle = \frac{-ie^2}{(2\pi)^2}\overline{u}_\mu(p',s')\gamma^a v_\mu(q',t')\overline{v}_e(q,t)\gamma_a u_e(p,s)\frac{\delta(p'+q'-p-q)}{(p+q)^2-i\epsilon} \tag{38}
$$

in which, of course, we sum over the index a from 0 to 3.

The probability of such a process is the absolute value squared

$$
\begin{aligned}
|\langle p', s'; q', t'|U|p, s; q, t\rangle|^2 &= \frac{e^4}{(2\pi)^4} \left| \overline{u}_\mu(p', s')\gamma^a v_\mu(q', t')\overline{v}_e(q, t)\gamma_a u_e(p, s) \frac{\delta(p' + q' - p - q)}{(p + q)^2} \right|^2 \\
&= \frac{e^4}{(2\pi)^4} \sum_{a,b=0}^{3} \overline{u}_\mu(p', s')\gamma^a v_\mu(q', t') \left(\overline{u}_\mu(p', s')\gamma^b v_\mu(q', t')\right)^* \\
&\quad \times \overline{v}_e(q, t)\gamma_a u_e(p, s) \left(\overline{v}_e(q, t)\gamma_b u_e(p, s)\right)^* \frac{\delta^2(p' + q' - p - q)}{(p + q)^4}.
\end{aligned} \tag{39}
$$

This probability is much simpler if we average over the initial spins and sum over the final spins. Let's look at various parts of such an expression. The final spinors give

$$
\overline{u}\gamma^a v \left(\overline{u}\gamma^b v\right)^* = \overline{u}\gamma^a v(-i)v^\dagger(\gamma^b)^\dagger(\gamma^0)^\dagger u. \tag{40}
$$

Since the gamma matrices are (10.286 of notes on group theory)

$$
\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \vec{\gamma} = -i \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \tag{41}
$$

$(\gamma^0)^\dagger = -\gamma^0$ and $(\vec{\gamma})^\dagger = \vec{\gamma}$. But the gamma matrices anticommute, so

$$
\overline{u}\gamma^a v(-i)v^\dagger(\gamma^b)^\dagger(\gamma^0)^\dagger u = \overline{u}\gamma^a viv^\dagger(\gamma^b)^\dagger\gamma^0 u = -\overline{u}\gamma^a viv^\dagger\gamma^0\gamma^b u = -\overline{u}\gamma^a v\,\overline{v}\gamma^b u. \tag{42}
$$

When we sum over the final spins, we get

$$
\begin{aligned}
\sum_{s',t'} \overline{u}(p', s')\gamma^a v(q', t') \left(\overline{u}(p', s')\gamma^b v(q', t')\right)^* &= -\sum_{s',t'} \overline{u}_b\gamma^a_{bc}v_c\,\overline{v}_d\gamma^b_{de}u_e = -\sum_{s',t'} v_c\,\overline{v}_d\gamma^a_{de}u_e\overline{u}_b\gamma^b_{bc} \\
&= -\sum_{s',t'} \mathrm{Tr}\left[v\overline{v}\gamma^a u\overline{u}\gamma^b\right].
\end{aligned} \tag{43}
$$

Our spin-sum formulas (27 of the notes on Dirac spinors) now give

$$\sum_{t'} v(q',t')\overline{v}(q',t') = \frac{1}{2q'^0}(-i\not{q}' - m_\mu) \quad \text{and} \quad \sum_{s'} u(p',s')\overline{u}(p',s') = \frac{1}{2p'^0}(-i\not{p}' + m_\mu). \tag{44}$$

So we have

$$\sum_{s',t'} \overline{u}(p',s')\gamma^a v(q',t') \left(\overline{u}(p',s')\gamma^b v(q',t')\right)^* = -\frac{1}{2q'^0 2p'^0} \operatorname{Tr}\left[(-i\not{q}' - m_\mu)\gamma^a(-i\not{p}' + m_\mu)\gamma^b\right] \tag{45}$$

Similarly, the average over the spins of the initial spinors is

$$\tfrac{1}{4}\overline{v}\gamma_a u \; (\overline{v}\gamma_b u)^* = -\tfrac{1}{4}\sum_{s,t} \overline{v}\gamma_a u \overline{u}\gamma_b v = -\frac{1}{4}\frac{1}{2q^0 2p^0} \operatorname{Tr}\left[(-i\not{q} - m_e)\gamma_a(-i\not{p} + m_e)\gamma_b\right]. \tag{46}$$

So the probability is

$$\begin{aligned} P = \frac{1}{4}\sum_{s,t,s',t'} |\langle p',s';q',t'|U|p,s;q,t\rangle|^2 &= \frac{e^4}{(2\pi)^4}\left|\overline{u}_\mu(p',s')\gamma^a v_\mu(q',t')\overline{v}_e(q,t)\gamma_a u_e(p,s)\frac{\delta(p'+q'-p-q)}{(p+q)^2}\right|^2 \\ &= \frac{e^4}{4(2\pi)^4}\frac{\operatorname{Tr}\left[(-i\not{q}' - m_\mu)\gamma^a(-i\not{p}' + m_\mu)\gamma^b\right]\operatorname{Tr}\left[(-i\not{q} - m_e)\gamma_a(-i\not{p} + m_e)\gamma_b\right]}{2p^0 2q^0 2p'^0 2q'^0 (p+q)^4}\delta^{(4)}(p'+q'-p-q)\delta^{(4)}(0). \end{aligned} \tag{47}$$

The square of the delta function plays a role in Fermi's golden rule. The time delta function $\delta(0) = T/2\pi$ is the time of the process divided by 2π. So the rate is

$$R = \frac{P}{T} = \frac{e^4}{4(2\pi)^5}\frac{\operatorname{Tr}\left[(-i\not{q}' - m_\mu)\gamma^a(-i\not{p}' + m_\mu)\gamma^b\right]\operatorname{Tr}\left[(-i\not{q} - m_e)\gamma_a(-i\not{p} + m_e)\gamma_b\right]}{2p^0 2q^0 2p'^0 2q'^0 (p+q)^4}\delta^{(4)}(p'+q'-p-q)\delta^{(3)}(\vec{0}). \tag{48}$$

5 Compton Scattering

The scattering of a photon off an electron is called Compton scattering. In computing the Feynman diagrams for this process, we'll need to recall that for Fermi fields the time-ordered product has an extra minus sign

$$\langle 0|\mathcal{T}[\psi_\ell(x)\psi_n^\dagger(y)]|0\rangle \equiv \langle 0|\theta(x^0-y^0)\psi_\ell(x)\psi_n^\dagger(y) - \theta(y^0-x^0)\psi_n^\dagger(y)\psi_\ell(x)|0\rangle. \tag{49}$$

Our formula (39) of the notes on Dirac spinors for the mean value in the vacuum (or ground state of H_0) of the time-ordered product of $\psi_\ell(x)$ and $\psi_n^\dagger(y)$ is

$$\begin{aligned} \langle 0|\mathcal{T}[\psi_\ell(x)\psi_n^\dagger(y)]|0\rangle &= \left[(-\gamma^a\partial_a + m)\,\gamma^0\right]_{\ell n} \Delta_F(x-y) \\ &= \left[(-\gamma^a\partial_a + m)\,\gamma^0\right]_{\ell n} \int \frac{d^4q}{(2\pi)^4}\frac{\exp(iq(x-y))}{q^2+m^2-i\epsilon} \\ &= \int \frac{d^4q}{(2\pi)^4}\frac{[(-i\gamma^a q_a + m)\,\gamma^0]_{\ell n}}{q^2+m^2-i\epsilon}\,e^{iq(x-y)}. \end{aligned} \tag{50}$$

Since $\overline{\psi} = i\psi^\dagger\gamma^0$, we can write this somewhat more simply as

$$\langle 0|\mathcal{T}[\psi_\ell(x)\overline{\psi}_n(y)]|0\rangle = \int \frac{d^4q}{(2\pi)^4}\frac{\left(-\not{q} - im\right)_{\ell n}}{q^2+m^2-i\epsilon}\,e^{iq(x-y)}. \tag{51}$$

Let's agree that the incoming electron is absorbed at x. Then the amplitude for the process in which the incoming photon also is absorbed at x is

$$\begin{aligned}
\langle p',s';k',t'|U|p,s;k,t\rangle_1 &= \langle p',s';k',t'|\, e^2 \int \mathcal{T}\left[\overline{\psi}^-(y)\not{A}^-(y)\psi(y)\,\overline{\psi}(x)\not{A}^+(x)\psi^+(x)\right]\, d^4x d^4y\,|p,s;k,t\rangle \\
&= \frac{e^2}{(2\pi)^3}\langle k',t'| \int \mathcal{T}\left[\overline{u}(p',s')\gamma^a A_a^-(y)\psi(y)\,\overline{\psi}(x)\gamma^b A_b^+(x)u(p,s)e^{i(px-p'y)}\right]\, d^4x d^4y\,|k,t\rangle \\
&= \frac{e^2}{(2\pi)^{10}\sqrt{4k^0k'^0}} \int \overline{u}(p',s')\gamma^a \frac{\left(-\not{q}-im\right)}{q^2+m^2-i\epsilon} e^{iq(y-x)}\gamma^b e^{i(kx-k'y)}u(p,s)e^{i(px-p'y)}\epsilon_a^*(k',t')\epsilon_b(k,t)\, d^4x d^4y d^4q \\
&= \frac{e^2}{(2\pi)^2\sqrt{4k^0k'^0}} \int \overline{u}(p',s')\gamma^a \frac{\left(-\not{q}-im\right)}{q^2+m^2-i\epsilon}\gamma^b u(p,s)\epsilon_a^*(k',t')\epsilon_b(k,t)\delta(q-k'-p')\delta(-q+k+p)\, d^4q \\
&= \frac{e^2\delta(k+p-k'-p')}{(2\pi)^2\sqrt{4k^0k'^0}}\,\overline{u}(p',s')\gamma^a \frac{\left(-\not{p}-\not{k}-im\right)}{(p+k)^2+m^2-i\epsilon}\gamma^b u(p,s)\epsilon_a^*(k',t')\epsilon_b(k,t).
\end{aligned} \tag{52}$$

The other process, in which the photon is emitted at x where the electron is absorbed, differs only trivially from this amplitude

$$\begin{aligned}
\langle p',s';k',t'|U|p,s;k,t\rangle_2 &= \langle p',s';k',t'|\, e^2 \int \mathcal{T}\left[\overline{\psi}^-(y)\not{A}^+(y)\psi(y)\,\overline{\psi}(x)\not{A}^-(x)\psi^+(x)\right]\, d^4x d^4y\,|p,s;k,t\rangle \\
&= \frac{e^2}{(2\pi)^3}\langle k',t'| \int \mathcal{T}\left[\overline{u}(p',s')\gamma^a A_a^+(y)\psi(y)\,\overline{\psi}(x)\gamma^b A_b^-(x)u(p,s)e^{i(px-p'y)}\right]\, d^4x d^4y\,|k,t\rangle \\
&= \frac{e^2}{(2\pi)^{10}\sqrt{4k^0k'^0}} \int \overline{u}(p',s')\gamma^a \frac{\left(-\not{q}-im\right)}{q^2+m^2-i\epsilon} e^{iq(y-x)}\gamma^b e^{i(ky-k'x)}u(p,s)e^{i(px-p'y)}\epsilon_a(k,t)\epsilon_b^*(k',t')\, d^4x d^4y d^4q \\
&= \frac{e^2}{(2\pi)^2\sqrt{4k^0k'^0}} \int \overline{u}(p',s')\gamma^a \frac{\left(-\not{q}-im\right)}{q^2+m^2-i\epsilon}\gamma^b u(p,s)\epsilon_a(k,t)\epsilon_b^*(k',t')\delta(q+k-p')\delta(-q-k'+p)\, d^4q \\
&= \frac{e^2\delta(k+p-k'-p')}{(2\pi)^2\sqrt{4k^0k'^0}}\,\overline{u}(p',s')\gamma^a \frac{\left(-\not{p}+\not{k}'-im\right)}{(p-k')^2+m^2-i\epsilon}\gamma^b u(p,s)\epsilon_a(k,t)\epsilon_b^*(k',t').
\end{aligned} \tag{53}$$

The full amplitude for Compton scattering is the sum of the amplitudes for the two processes

$$\begin{aligned}\langle p', s'; k', t'|U|p, s; k, t\rangle &= \frac{e^2\delta(k+p-k'-p')}{(2\pi)^2\sqrt{4k^0k'^0}}\overline{u}(p', s')\left[\gamma^a \frac{(-\not{p}-\not{k}-im)}{(p+k)^2+m^2-i\epsilon}\gamma^b + \gamma^b \frac{(-\not{p}+\not{k}'-im)}{(p-k')^2+m^2-i\epsilon}\gamma^a\right] u(p,s)\epsilon_a^*(k',t')\epsilon_b(k,t) \\ &= \frac{e^2\delta(k+p-k'-p')}{(2\pi)^2\sqrt{4k^0k'^0}}\overline{u}(p', s')\left[\not{\epsilon}^*(k',t') \frac{(-\not{p}-\not{k}-im)}{(p+k)^2+m^2-i\epsilon}\not{\epsilon}(k,t) + \not{\epsilon}(k,t) \frac{(-\not{p}+\not{k}'-im)}{(p-k')^2+m^2-i\epsilon}\not{\epsilon}^*(k',t')\right] u(p,s)\end{aligned} \tag{54}$$

in which $\not{\epsilon}^*(k', t') = \gamma^a \epsilon_a^*(k', t')$, not $\not{\epsilon}^{\,*}(k', t')$.