I.3 From Mattress to Field

The mattress in the continuum limit

The path integral representation

$$Z \equiv \langle 0| e^{-iHT} |0\rangle = \int Dq(t)\, e^{i \int_0^T dt[\frac{1}{2}m\dot{q}^2 - V(q)]} \tag{1}$$

(we suppress the factor $\langle 0| q_f\rangle\langle q_I |0\rangle$; we will come back to this issue later in this chapter) which we derived for the quantum mechanics of a single particle, can be generalized almost immediately to the case of N particles with the Hamiltonian

$$H = \sum_a \frac{1}{2m_a}\hat{p}_a^2 + V(\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_N). \tag{2}$$

We simply keep track mentally of the position of the particles q_a with $a = 1, 2, \cdots, N$. Going through the same steps as before, we obtain

$$Z \equiv \langle 0| e^{-iHT} |0\rangle = \int Dq(t)\, e^{iS(q)} \tag{3}$$

with the action

$$S(q) = \int_0^T dt\Big(\sum_a \tfrac{1}{2}m_a\dot{q}_a^2 - V[q_1, q_2, \cdots, q_N]\Big).$$

The potential energy $V(q_1, q_2, \ldots, q_N)$ now includes interaction energy between particles, namely terms of the form $v(q_a - q_b)$, as well as the energy due to an external potential, namely terms of the form $w(q_a)$. In particular, let us now write the path integral description of the quantum dynamics of the mattress described in chapter I.1, with the potential

$$V(q_1, q_2, \ldots, q_N) = \sum_{ab} \frac{1}{2}k_{ab}(q_a - q_b)^2 + \cdots$$

We are now just a short hop and skip away from a quantum field theory! Suppose we are only interested in phenomena on length scales much greater than the lattice spacing l (see fig. I.1.1). Mathematically, we take the continuum limit $l \to 0$. In this limit, we can replace the label a on the particles by a two-dimensional position vector $\vec{x}$, and so we write $q(t, \vec{x})$ instead of $q_a(t)$. It is traditional to replace the Latin letter q by the Greek letter φ. The function $\varphi(t, \vec{x})$ is called a field.

The kinetic energy $\sum_a \frac{1}{2} m_a \dot{q}_a^2$ now becomes $\int d^2x \frac{1}{2}\sigma(\partial\varphi/\partial t)^2$. We replace $\sum_a$ by $\int d^2x/l^2$ and denote the mass per unit area m_a/l^2 by σ. We take all the m_a's to be equal; otherwise σ would be a function of $\vec{x}$, the system would be inhomogeneous, and we would have a hard time writing down a Lorentz-invariant action (see later).

We next focus on the first term in V. Assume for simplicity that k_{ab} connect only nearest neighbors on the lattice. For nearest-neighbor pairs $(q_a - q_b)^2 \simeq l^2(\partial\varphi/\partial x)^2 + \cdots$ in the continuum limit; the derivative is obviously taken in the direction that joins the lattice sites a and b.

Putting it together then, we have

$$\begin{aligned} S(q) \to S(\varphi) &\equiv \int_0^T dt \int d^2x \mathcal{L}(\varphi) \\ &= \int_0^T dt \int d^2x \frac{1}{2}\left\{\sigma\left(\frac{\partial\varphi}{\partial t}\right)^2 - \rho\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] - \tau\varphi^2 - \varsigma\varphi^4 + \cdots\right\} \end{aligned} \tag{4}$$

where the parameter ρ is determined by k_{ab} and l. The precise relations do not concern us.

Henceforth in this book, we will take the $T \to \infty$ limit so that we can integrate over all of spacetime in (4).

We can clean up a bit by writing $\rho = \sigma c^2$ and scaling $\varphi \to \varphi/\sqrt{\sigma}$, so that the combination $(\partial\varphi/\partial t)^2 - c^2[(\partial\varphi/\partial x)^2 + (\partial\varphi/\partial y)^2]$ appears in the Lagrangian. The parameter c evidently has the dimension of a velocity and defines the phase velocity of the waves on our mattress.

We started with a mattress for pedagogical reasons. Of course nobody believes that the fields observed in Nature, such as the meson field or the photon field, are actually constructed of point masses tied together with springs. The modern view, which I will call Landau-Ginzburg, is that we start with the desired symmetry, say Lorentz invariance if we want to do particle physics, decide on the fields we want by specifying how they transform under the symmetry (in this case we decided on a scalar field φ), and then write down the action involving no more than two time derivatives (because we don't know how to quantize actions with more than two time derivatives).

We end up with a Lorentz-invariant action (setting $c = 1$)

$$S = \int d^dx \left[\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4 + \cdots\right] \tag{5}$$

where various numerical factors are put in for later convenience. The relativistic notation $(\partial\varphi)^2 \equiv \partial_\mu\varphi\partial^\mu\varphi = (\partial\varphi/\partial t)^2 - (\partial\varphi/\partial x)^2 - (\partial\varphi/\partial y)^2$ was explained in the note on convention. The dimension of spacetime, d, clearly can be any integer, even though in our mattress model it was actually 3. We often write $d = D + 1$ and speak of a $(D + 1)$-dimensional spacetime.

We see here the power of imposing a symmetry. Lorentz invariance together with the insistence that the Lagrangian involve only at most two powers of $\partial/\partial t$ immediately tells us that the Lagrangian can only have the form[1] $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi)$ with V some function of φ. For simplicity, we now restrict V to be a polynomial in φ, although much of the present discussion will not depend on this restriction. We will have a great deal more to say about symmetry later. Here we note that, for example, we could insist that physics is symmetric under $\varphi \to -\varphi$, in which case $V(\varphi)$ would have to be an even polynomial.

Now that you know what a quantum field theory is, you realize why I used the letter q to label the position of the particle in the previous chapter and not the more common $\vec{x}$. In quantum field theory, $\vec{x}$ is a label, not a dynamical variable. The $\vec{x}$ appearing in $\varphi(t, \vec{x})$ corresponds to the label a in $q_a(t)$ in quantum mechanics. The dynamical variable in field theory is not position, but the field φ. The variable $\vec{x}$ simply specifies which field variable we are talking about. I belabor this point because upon first exposure to quantum field theory some students, used to thinking of $\vec{x}$ as a dynamical operator in quantum mechanics, are confused by its role here.

In summary, we have the table

$q \to \varphi$
$a \to \vec{x}$
$q_a(t) \to \varphi(t, \vec{x}) = \varphi(x)$
$\sum_a \to \int d^D x$

(6)

Thus we finally have the path integral defining a scalar field theory in $d = (D + 1)$ dimensional spacetime:

$$Z = \int D\varphi e^{i \int d^d x (\frac{1}{2}(\partial\varphi)^2 - V(\varphi))} \tag{7}$$

Note that a $(0 + 1)$-dimensional quantum field theory is just quantum mechanics.

The classical limit

As I have already remarked, the path integral formalism is particularly convenient for taking the classical limit. Remembering that Planck's constant $\hbar$ has the dimension of energy multiplied by time, we see that it appears in the unitary evolution operator $e^{(-i/\hbar)HT}$. Tracing through the derivation of the path integral, we see that we simply divide the overall factor i by $\hbar$ to get

$$Z = \int D\varphi e^{(i/\hbar) \int d^4 x \mathcal{L}(\varphi)} \tag{8}$$

[1] Strictly speaking, a term of the form $U(\varphi)(\partial\varphi)^2$ is also possible. In quantum mechanics, a term such as $U(q)(dq/dt)^2$ in the Lagrangian would describe a particle whose mass depends on position. We will not consider such "nasty" terms until much later.

In the limit $\hbar$ much smaller than the relevant action we are considering, we can evaluate the path integral using the stationary phase (or steepest descent) approximation, as I explained in the previous chapter in the context of quantum mechanics. We simply determine the extremum of $\int d^4x \mathcal{L}(\varphi)$. According to the usual Euler-Lagrange variational procedure, this leads to the equation

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} - \frac{\delta \mathcal{L}}{\delta \varphi} = 0 \tag{9}$$

We thus recover the classical field equation, exactly as we should, which in our scalar field theory reads

$$(\partial^2 + m^2)\varphi(x) + \frac{g}{2}\varphi(x)^2 + \frac{\lambda}{6}\varphi(x)^3 + \cdots = 0 \tag{10}$$

The vacuum

In the point particle quantum mechanics discussed in chapter I.2 we wrote the path integral for $\langle F| e^{-iHT} |I\rangle$, with some initial and final state, which we can choose at our pleasure. A convenient and particularly natural choice would be to take $|I\rangle = |F\rangle$ to be the ground state. In quantum field theory what should we choose for the initial and final states? A standard choice for the initial and final states is the ground state or the vacuum state of the system, denoted by $|0\rangle$, in which, speaking colloquially, nothing is happening. In other words, we would calculate the quantum transition amplitude from the vacuum to the vacuum, which would enable us to determine the energy of the ground state. But this is not a particularly interesting quantity, because in quantum field theory we would like to measure all energies relative to the vacuum and so, by convention, would set the energy of the vacuum to zero (possibly by having to subtract a constant from the Lagrangian). Incidentally, the vacuum in quantum field theory is a stormy sea of quantum fluctuations, but for this initial pass at quantum field theory, we will not examine it in any detail. We will certainly come back to the vacuum in later chapters.

Disturbing the vacuum

We might enjoy doing something more exciting than watching a boiling sea of quantum fluctuations. We might want to disturb the vacuum. Somewhere in space, at some instant in time, we would like to create a particle, watch it propagate for a while, and then annihilate it somewhere else in space, at some later instant in time. In other words, we want to set up a source and a sink (sometimes referred to collectively as sources) at which particles can be created and annihilated.

To see how to do this, let us go back to the mattress. Bounce up and down on it to create some excitations. Obviously, pushing on the mass labeled by a in the mattress corresponds to adding a term such as $J_a(t)q_a$ to the potential $V(q_1, q_2, \cdots, q_N)$. More generally,

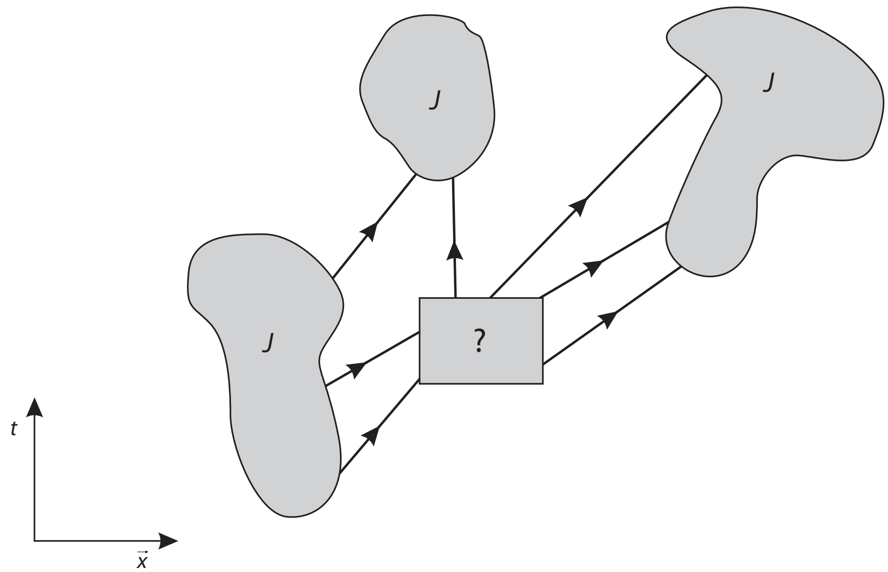

Figure I.3.1

we can add $\sum_a J_a(t)q_a$. When we go to field theory this added term gets promoted to $\int d^D x J(x)\varphi(x)$ in the field theory Lagrangian, according to the promotion table (6).

This so-called source function $J(t, \vec{x})$ describes how the mattress is being disturbed. We can choose whatever function we like, corresponding to our freedom to push on the mattress wherever and whenever we like. In particular, $J(x)$ can vanish everywhere in spacetime except in some localized regions.

By bouncing up and down on the mattress we can get wave packets going off here and there (fig. I.3.1). This corresponds precisely to sources (and sinks) for particles. Thus, we really want the path integral

$$Z = \int D\varphi e^{i\int d^4x[\frac{1}{2}(\partial\varphi)^2 - V(\varphi) + J(x)\varphi(x)]} \tag{11}$$

Free field theory

The functional integral in (11) is impossible to do except when

$$\mathcal{L}(\varphi) = \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] \tag{12}$$

The corresponding theory is called the free or Gaussian theory. The equation of motion (9) works out to be $(\partial^2 + m^2)\varphi = 0$, known as the Klein-Gordon equation.[2] Being linear, it can be solved immediately to give $\varphi(\vec{x}, t) = e^{i(\omega t - \vec{k}\cdot\vec{x})}$ with

$$\omega^2 = \vec{k}^2 + m^2 \tag{13}$$

[2] The Klein-Gordon equation was actually discovered by Schrödinger before he found the equation that now bears his name. Later, in 1926, it was written down independently by Klein, Gordon, Fock, Kudar, de Donder, and Van Dungen.

In the natural units we are using, $\hbar = 1$ and so frequency ω is the same as energy $\hbar\omega$ and wave vector $\vec{k}$ is the same as momentum $\hbar\vec{k}$. Thus, we recognize (13) as the energy-momentum relation for a particle of mass m, namely the sophisticate's version of the layperson's $E = mc^2$. We expect this field theory to describe a relativistic particle of mass m.

Let us now evaluate (11) in this special case:

$$Z = \int D\varphi e^{i\int d^4x\{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] + J\varphi\}} \tag{14}$$

Integrating by parts under the $\int d^4x$ and not worrying about the possible contribution of boundary terms at infinity (we implicitly assume that the fields we are integrating over fall off sufficiently rapidly), we write

$$Z = \int D\varphi e^{i\int d^4x[-\frac{1}{2}\varphi(\partial^2 + m^2)\varphi + J\varphi]} \tag{15}$$

You will encounter functional integrals like this again and again in your study of field theory. The trick is to imagine discretizing spacetime. You don't actually have to do it: Just imagine doing it. Let me sketch how this goes. Replace the function $\varphi(x)$ by the vector $\varphi_i = \varphi(ia)$ with i an integer and a the lattice spacing. (For simplicity, I am writing things as if we were in 1-dimensional spacetime. More generally, just let the index i enumerate the lattice points in some way.) Then differential operators become matrices. For example, $\partial\varphi(ia) \to (1/a)\ (\varphi_{i+1} - \varphi_i) \equiv \sum_j M_{ij}\varphi_j$, with some appropriate matrix M. Integrals become sums. For example, $\int d^4x\, J(x)\varphi(x) \to a^4 \sum_i J_i\varphi_i$.

Now, lo and behold, the integral (15) is just the integral we did in (I.2.23)

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dq_1 dq_2 \cdots dq_N\, e^{(i/2)q\cdot A\cdot q + iJ\cdot q}$$
$$= \left(\frac{(2\pi i)^N}{\det[A]}\right)^{\frac{1}{2}} e^{-(i/2)J\cdot A^{-1}\cdot J} \tag{16}$$

The role of A in (16) is played in (15) by the differential operator $-(\partial^2 + m^2)$. The defining equation for the inverse, $A \cdot A^{-1} = I$ or $A_{ij}A^{-1}_{jk} = \delta_{ik}$, becomes in the continuum limit

$$-(\partial^2 + m^2)D(x - y) = \delta^{(4)}(x - y) \tag{17}$$

We denote the continuum limit of A^{-1}_{jk} by $D(x - y)$ (which we know must be a function of $x - y$, and not of x and y separately, since no point in spacetime is special). Note that in going from the lattice to the continuum Kronecker is replaced by Dirac. It is very useful to be able to go back and forth mentally between the lattice and the continuum.

Our final result is

$$Z(J) = \mathcal{C}e^{-(i/2)\iint d^4x d^4y J(x)D(x-y)J(y)} \equiv \mathcal{C}e^{iW(J)} \tag{18}$$

with $D(x)$ determined by solving (17). The overall factor $\mathcal{C}$, which corresponds to the overall factor with the determinant in (16), does not depend on J and, as will become clear in the discussion to follow, is often of no interest to us. I will often omit writing $\mathcal{C}$ altogether. Clearly, $\mathcal{C} = Z(J = 0)$ so that $W(J)$ is defined by

$$Z(J) \equiv Z(J = 0)e^{iW(J)} \tag{19}$$

Observe that

$$W(J) = -\frac{1}{2}\iint d^4x d^4y J(x) D(x-y) J(y) \tag{20}$$

is a simple quadratic functional of J. In contrast, $Z(J)$ depends on arbitrarily high powers of J. This fact will be of importance in chapter I.7.

Free propagator

The function $D(x)$, known as the propagator, plays an essential role in quantum field theory. As the inverse of a differential operator it is clearly closely related to the Green's function you encountered in a course on electromagnetism.

Physicists are sloppy about mathematical rigor, but even so, they have to be careful once in a while to make sure that what they are doing actually makes sense. For the integral in (15) to converge for large φ we replace $m^2 \to m^2 - i\varepsilon$ so that the integrand contains a factor $e^{-\varepsilon \int d^4x \varphi^2}$, where ε is a positive infinitesimal we will let tend to zero.[3]

We can solve (17) easily by going to momentum space and multiplying together four copies of the representation (I.2.11) of the Dirac delta function

$$\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \tag{21}$$

The solution is

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\varepsilon} \tag{22}$$

which you can check by plugging into (17):

$$-(\partial^2 + m^2) D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{k^2 - m^2}{k^2 - m^2 + i\varepsilon} e^{ik(x-y)} = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} = \delta^{(4)}(x-y) \text{ as } \varepsilon \to 0.$$

Note that the so-called $i\varepsilon$ prescription we just mentioned is essential; otherwise the integral giving $D(x)$ would hit a pole. The magnitude of ε is not important as long as it is infinitesimal, but the positive sign of ε is crucial as we will see presently. (More on this in chapter III.8.) Also, note that the sign of k in the exponential does not matter here by the symmetry $k \to -k$.

To evaluate $D(x)$ we first integrate over k^0 by the method of contours. Define $\omega_k \equiv +\sqrt{\vec{k}^2 + m^2}$ with a plus sign. The integrand has two poles in the complex k^0 plane, at $\pm\sqrt{\omega_k^2 - i\varepsilon}$, which in the $\varepsilon \to 0$ limit are equal to $+\omega_k - i\varepsilon$ and $-\omega_k + i\varepsilon$. Thus for ε positive, one pole is in the lower half-plane and the other in the upper half plane, and so as we go along the real k^0 axis from $-\infty$ to $+\infty$ we do not run into the poles. The issue is how to close the integration contour.

For x^0 positive, the factor $e^{ik^0x^0}$ is exponentially damped for k^0 in the upper half-plane. Hence we should extend the integration contour extending from $-\infty$ to $+\infty$ on the real

[3] As is customary, ε is treated as generic, so that ε multiplied by any positive number is still ε.

axis to include the infinite semicircle in the upper half-plane, thus enclosing the pole at $-\omega_k + i\varepsilon$ and giving $-i \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-i(\omega_k t - \vec{k}\cdot\vec{x})}$. Again, note that we are free to flip the sign of $\vec{k}$. Also, as is conventional, we use x^0 and t interchangeably. (In view of some reader confusion here in the first edition, I might add that I generally use x^0 with k^0 and t with ω_k; k^0 is a variable that can take on either sign but ω_k is a positive function of $\vec{k}$.)

For x^0 negative, we do the opposite and close the contour in the lower half-plane, thus picking up the pole at $+\omega_k - i\varepsilon$. We now obtain $-i \int (d^3k/(2\pi)^3 2\omega_k) e^{+i(\omega_k t - \vec{k}\cdot\vec{x})}$.

Recall that the Heaviside (we will meet this great and aptly named physicist in chapter IV.4) step function $\theta(t)$ is defined to be equal to 0 for $t < 0$ and equal to 1 for $t > 0$. As for what $\theta(0)$ should be, the answer is that since we are proud physicists and not nitpicking mathematicians we will just wing it when the need arises. The step function allows us to package our two integration results together as

$$D(x) = -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} [e^{-i(\omega_k t - \vec{k}\cdot\vec{x})}\theta(x^0) + e^{i(\omega_k t - \vec{k}\cdot\vec{x})}\theta(-x^0)] \tag{23}$$

Physically, $D(x)$ describes the amplitude for a disturbance in the field to propagate from the origin to x. Lorentz invariance tells us that it is a function of x^2 and the sign of x^0 (since these are the quantities that do not change under a Lorentz transformation). We thus expect drastically different behavior depending on whether x is inside or outside the lightcone defined by $x^2 = (x^0)^2 - \vec{x}^2 = 0$. Without evaluating the d^3k integral we can see roughly how things go. Let us look at some cases.

In the future cone, $x = (t, 0)$ with $t > 0$, $D(x) = -i \int (d^3k/(2\pi)^3 2\omega_k) e^{-i\omega_k t}$ a superposition of plane waves and thus $D(x)$ oscillates. In the past cone, $x = (t, 0)$ with $t < 0$, $D(x) = -i \int (d^3k/(2\pi)^3 2\omega_k) e^{+i\omega_k t}$ oscillates with the opposite phase.

In contrast, for x spacelike rather than timelike, $x^0 = 0$, we have, upon interpreting $\theta(0) = \frac{1}{2}$ (the obvious choice; imagine smoothing out the step function), $D(x) = -i \int (d^3k/(2\pi)^3 2\sqrt{\vec{k}^2 + m^2}) e^{-i\vec{k}\cdot\vec{x}}$. The square root cut starting at $\pm im$ tells us that the characteristic value of $|\vec{k}|$ in the integral is of order m, leading to an exponential decay $\sim e^{-m|\vec{x}|}$, as we would expect. Classically, a particle cannot get outside the lightcone, but a quantum field can "leak" out over a distance of order m^{-1} by the Heisenberg uncertainty principle.

Exercises

I.3.1 Verify that $D(x)$ decays exponentially for spacelike separation.

I.3.2 Work out the propagator $D(x)$ for a free field theory in $(1+1)$-dimensional spacetime and study the large x^1 behavior for $x^0 = 0$.

I.3.3 Show that the advanced propagator defined by

$$D_{adv}(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 - i\,\mathrm{sgn}(k_0)\varepsilon}$$

(where the sign function is defined by $\text{sgn}(k_0) = +1$ if $k_0 > 0$ and $\text{sgn}(k_0) = -1$ if $k_0 < 0$) is nonzero only if $x^0 > y^0$. In other words, it only propagates into the future. [Hint: both poles of the integrand are now in the upper half of the k_0-plane.] Incidentally, some authors prefer to write $(k_0 - ie)^2 - \vec{k}^2 - m^2$ instead of $k^2 - m^2 - i\ \text{sgn}(k_0)\varepsilon$ in the integrand. Similarly, show that the retarded propagator

$$D_{ret}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\ \text{sgn}(k_0)\varepsilon}$$

propagates into the past.