Lattice Gauge Theory

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1 Glauber's identity

First we define

$$f(z) = e^{zA} B e^{-zA} \tag{1}$$

whose derivative is

$$f'(z) = e^{zA} [A, B] e^{-zA}. \tag{2}$$

Now we assume that

$$[A, [A, B]] = 0 = [B, [A, B]]. \tag{3}$$

Then

$$f'(z) = e^{zA} [A, B] e^{-zA} = [A, B]. \tag{4}$$

So integrating, we find

$$\begin{aligned} f(1) = f(0) + \int_0^1 f'(z)\, dz \\ e^A B e^{-A} = B + [A, B]. \end{aligned} \tag{5}$$

Now we define

$$g(y) = e^{yA} e^{yB}. \tag{6}$$

Its derivative is

$$g'(y) = Ae^{yA}e^{yB} + e^{yA}Be^{yB}. \tag{7}$$

But

$$e^{yA}Be^{-yA} = B + y[A,B] \tag{8}$$

or

$$e^{yA}B = (B + y[A,B])e^{yA}. \tag{9}$$

So

$$g'(y) = Ae^{yA}e^{yB} + e^{yA}Be^{yB} = (A + B + y[A,B])e^{yA}e^{yB} = (A + B + y[A,B])\, g(y). \tag{10}$$

The key point now is that all the operators commute with each other. So we can just integrate this equation

$$\begin{aligned} \int \frac{dg}{g} = \ln g &= \int (A + B + y[A,B])\, dy \\ \ln g(y) &= (A + B)\, y + \frac{1}{2}[A,B]\, y^2 \end{aligned} \tag{11}$$

to find

$$g(y) = e^{(A+B)y} e^{\frac{1}{2}[A,B]y^2}. \tag{12}$$

Setting $y = 1$ gives

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{13}$$

or

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}. \tag{14}$$

One also has

$$e^A e^B = e^B e^A e^{[A,B]}. \tag{15}$$

For arbitrary A and B, the more general Baker-Campbell-Hausdorff-Zassenhaus identity is

$$e^{t(X+Y)} = e^{tX}\ e^{tY}\ e^{-\frac{t^2}{2}[X,Y]}\ e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\ e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])} \cdots. \tag{16}$$

2 Wilson's action

Consider a square of side a, called the lattice spacing, in the i–j plane with vertices at

$$\begin{aligned} v_1 &= x - a\,\tfrac{1}{2}\,(\hat{\imath} + \hat{\jmath}) \\ v_2 &= x + a\,\tfrac{1}{2}\,(\hat{\imath} - \hat{\jmath}) \\ v_3 &= x + a\,\tfrac{1}{2}\,(\hat{\imath} + \hat{\jmath}) \\ v_4 &= x - a\,\tfrac{1}{2}\,(\hat{\imath} - \hat{\jmath}). \end{aligned} \tag{17}$$

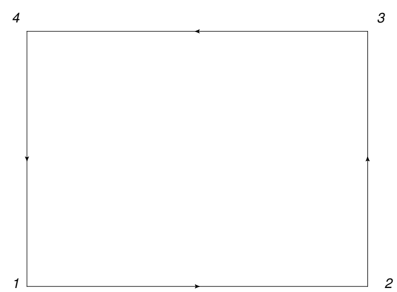

Figure 1: default

Put gauge-field matrices on the four sides of the little square, called a **plaquette**,

$$\begin{aligned} A_{12} &= iA_i^b(x - a\,\tfrac{1}{2}\,\hat{\jmath})\,t_b \\ A_{23} &= iA_j^b(x + a\,\tfrac{1}{2}\,\hat{\imath})\,t_b \\ A_{43} &= iA_i^b(x + a\,\tfrac{1}{2}\,\hat{\jmath})\,t_b = -A_{43} \\ A_{14} &= iA_j^b(x - a\,\tfrac{1}{2}\,\hat{\imath})\,t_b = -A_{14} \end{aligned} \tag{18}$$

in which the matrices t_b are the $n \times n$ generators of the gauge group. They obey the commutation relations

$$[t_a, t_b] = i\, f_{abc}\, t_c \tag{19}$$

in which the f_{abc} are the totally antisymmetric **structure constants**.

We now form the product

$$\begin{aligned} U &= e^{aA_{12}}\, e^{aA_{23}}\, e^{-aA_{43}}\, e^{-aA_{14}} \\ &\approx e^{aA_{12}+aA_{23}+\frac{1}{2}a^2[A_{12},A_{23}]}\, e^{-aA_{43}-aA_{14}+\frac{1}{2}a^2[A_{43},A_{14}]} \end{aligned} \tag{20}$$

in which we dropped from equalities to approximations that are valid to order a^2. Now

$$\begin{aligned} A_{12} &= iA_i^b(x - a\,\tfrac{1}{2}\,\hat{j})\,t_b \approx iA_i^b(x)\,t_b - \tfrac{1}{2}\,i\,a\,\partial_j A_i^b(x)\,t_b \\ A_{23} &= iA_j^b(x + a\,\tfrac{1}{2}\,\hat{i})\,t_b \approx iA_j^b(x)\,t_b + \tfrac{1}{2}\,i\,a\,\partial_i A_j^b(x)\,t_b \\ A_{43} &= iA_i^b(x + a\,\tfrac{1}{2}\,\hat{j})\,t_b \approx iA_i^b(x)\,t_b + \tfrac{1}{2}\,i\,a\,\partial_j A_i^b(x)\,t_b \\ A_{14} &= iA_j^b(x - a\,\tfrac{1}{2}\,\hat{i})\,t_b \approx iA_j^b(x)\,t_b - \tfrac{1}{2}\,i\,a\,\partial_i A_j^b(x)\,t_b \end{aligned} \tag{21}$$

to first order in the lattice spacing a. So to order a^2 we have

$$\begin{aligned} a^2[A_{12}, A_{23}] &\approx a^2[iA_i^b(x)\,t_b, iA_j^b(x)\,t_b] \\ a^2[A_{43}, A_{14}] &\approx a^2[iA_i^b(x)\,t_b, iA_j^b(x)\,t_b]. \end{aligned} \tag{22}$$

That is,

$$a^2[A_{12}, A_{23}] \approx a^2[A_{43}, A_{14}] \approx a^2[iA_i^b(x)\,t_b, iA_j^b(x)\,t_b]. \tag{23}$$

Again to order a^2, we have

$$\begin{aligned} aA_{12} + aA_{23} &\approx iaA_i^b(x)\,t_b - \tfrac{1}{2}\,i\,a^2\,\partial_j A_i^b(x)\,t_b + iaA_j^b(x)\,t_b + \tfrac{1}{2}\,i\,a^2\,\partial_i A_j^b(x)\,t_b \\ -aA_{43} - aA_{14} &\approx\, - iaA_i^b(x)\,t_b - \tfrac{1}{2}\,i\,a^2\,\partial_j A_i^b(x)\,t_b - iaA_j^b(x)\,t_b + \tfrac{1}{2}\,i\,a^2\,\partial_i A_j^b(x)\,t_b, \end{aligned} \tag{24}$$

and so to order a^2 the commutator

$$[aA_{12} + aA_{23}, -aA_{43} - aA_{14}] \approx [iaA_i^b(x)\,t_b + iaA_j^b(x)\,t_b, -iaA_i^b(x)\,t_b - iaA_j^b(x)\,t_b] = 0 \tag{25}$$

vanishes. So when we combine the two exponentials is our formula (20), we get no new commutators to order a^2, and the term of order a in the exponential vanishes:

$$aA_{12} + aA_{23} - aA_{43} - aA_{14} \approx -i\,a^2\,\partial_j A_i^b(x)\,t_b + i\,a^2\,\partial_i A_j^b(x)\,t_b. \tag{26}$$

Thus, again to order a^2, the product of exponentials around the plaquette is

$$\begin{aligned} U &\approx e^{aA_{12}+aA_{23}+a^2[A_{12},A_{23}]-aA_{43}-aA_{14}+a^2[A_{43},A_{14}]} \\ &= e^{-i\,a^2\,\partial_j A_i^b(x)\,t_b + i\,a^2\,\partial_i A_j^b(x)\,t_b + a^2[iA_i^b(x)\,t_b,\,iA_j^b(x)\,t_b]} \\ &= \exp\left\{a^2[\partial_i + iA_i^b(x)\,t_b, \partial_j + iA_j^b(x)\,t_b]\right\} = e^{a^2 F_{ij}(x)} \end{aligned} \tag{27}$$

in which

$$\begin{aligned} F_{ij}(x) &= i\partial_i A_j^b(x)\,t_b - i\partial_j A_i^c(x)\,t_c - A_j^b(x)\,A_i^c(x)\,[t_b, t_c] \\ &= i\partial_i A_j^b(x)\,t_b - i\partial_j A_i^c(x)\,t_c - A_j^b(x)\,A_i^c(x)\,i f_{bca} t_a \\ &= i\partial_i A_j^a(x)\,t_a - i\partial_j A_i^a(x)\,t_a - A_j^b(x)\,A_i^c(x)\,i f_{bca} t_a \\ &= i\left[\partial_i A_j^a(x) - \partial_j A_i^a(x) - f_{abc}\,A_j^b(x)\,A_i^c(x)\right]\,t_a = i\,F_{ij}^a\,t_a. \end{aligned} \tag{28}$$

Interest focuses on **semi-simple** gauge groups, that is, on groups that have no invariant abelian subgroups. The generators t_a of semi-simple gauge groups are traceless. Thus, the trace of the Faraday tensor

$$\mathrm{Tr} F_{ij} = \mathrm{Tr} i\,F_{ij}^a\,t_a = i\,F_{ij}^a\,\mathrm{Tr} t_a = 0 \tag{29}$$

vanishes for any semi-simple gauge group, such as $SU(N)$ or $SO(N)$.h The generators are $n \times n$ hermitian matrices that can be chosen to obey the trace relation

$$\mathrm{Tr}(t_a t_b) = k\delta_{ab} \tag{30}$$

where k is a positive constant that (unlike the structure constants f_{abc}) depends upon the representation of the group. It follows that to lowest order in the lattice spacing a

$$\begin{aligned} \mathrm{Tr} U = \mathrm{Tr} e^{a^2\,F_{ij}(x)} &= \mathrm{Tr}\left[1 + a^2\,F_{ij}(x) + \tfrac{1}{2} a^4\,F_{ij}^2(x)\right] = n - \tfrac{1}{2}\,a^4\,\mathrm{Tr}\left[F_{ij}^a(x) t_a F_{ij}^b t_b\right] \\ &= n - \tfrac{1}{2}\,a^4\,F_{ij}^a(x)\,F_{ij}^b\,\mathrm{Tr}\,[t_a t_b] = n - \tfrac{1}{2}\,a^4\,F_{ij}^a(x)\,F_{ij}^b\,k\,\delta_{ab} = n - \frac{ka^4}{2}\left[F_{ij}^a(x)\right]^2. \end{aligned} \tag{31}$$

The euclidian action of the continuum (pure) gauge theory is

$$\frac{1}{2}\int_{i<j}\left[F^a_{ij}(x)\right]^2\, d^4x = \frac{1}{4}\int\left[F^a_{ij}(x)\right]^2\, d^4x. \tag{32}$$

So when we sum over all the plaquettes of a space-time lattice, $a^4 = d^4x$, and we count each plaquette once. But each plaquette bounds two ($d = 4$) hypercubes, and the action of each hypercube is the sum of the averages of the plaquette actions of opposite faces of the hypercube. So the continuum action is approximately the sum over all plaquettes

$$\sum_{\Box} S_{\Box} = \sum_{\Box}\frac{1}{k}\left(n - \mathrm{Tr}U\right) = \sum_{\Box}\frac{a^4}{2}\left[F^a_{ij}(x)\right]^2 = \frac{1}{2}\int_{i<j}\left[F^a_{ij}(x)\right]^2\, d^4x. \tag{33}$$

Usually, one makes a coupling constant g explicit by replacing A by gA. The actions then are

$$\frac{1}{g^2}\sum_{\Box} S_{\Box} = \sum_{\Box}\frac{1}{g^2 k}\left(n - \mathrm{Tr}U\right) = \sum_{\Box}\frac{a^4}{2g^2}\left[F^a_{ij}(x)\right]^2 = \frac{1}{2g^2}\int_{i<j}\left[F^a_{ij}(x)\right]^2\, d^4x = \frac{1}{4g^2}\int\left[F^a_{ij}(x)\right]^2\, d^4x. \tag{34}$$