We start from the last equations of section 16.10 of the online notes on path integrals: The mean value in a system described by a stationary density operator $\rho$—one that commutes with the hamiltonian—is the ratio
\[
\langle T [\phi_e(x_1)\phi_e(x_2)] \rangle = \frac{\text{Tr}\{\rho T [\phi_e(x_1)\phi_e(x_2)]\}}{\text{Tr}\{e^{-\beta H}\}} \tag{1}
\]
in which we integrate over the field $\phi_0(x)$ and over all fields that run from $\phi_0$ at (euclidian) time minus infinity to the same $\phi_0$ at (euclidian) time plus infinity.

In the zero-temperature ($\beta \to \infty$) limit, the density operator $\rho$ becomes the projection operator $|0\rangle \langle 0|$ on the ground state, and mean-value formulas like (1) become
\[
\langle 0 | T [\phi_e(x_1) \ldots \phi_e(x_n)] | 0 \rangle = \frac{\int \phi_0(x_1) \ldots \phi_0(x_n) \exp \left[-\int_0^\beta \mathcal{H}(\phi) \, d^4 x \, d t\right] D\phi}{\int \exp \left[-\int_0^\beta \mathcal{H}(\phi) \, d^4 x \, d t\right] D\phi} \tag{2}
\]
in which Hamilton’s density $\mathcal{H}(\phi)$ is integrated over all of euclidean space-time and over all fields that are periodic on the infinite time interval. Statistical field theory and lattice gauge theory are based upon such formulas.

To keep the notation simple, let’s restrict ourselves to a hamiltonian that is quadratic in the fields like
\[
H_0 = \int \frac{1}{2} \left[ \pi^2(x) + (\nabla \phi(x))^2 + m^2 \phi^2(x) \right] d^3 x. \tag{3}
\]
Integrating by parts and dropping surface terms, we write the euclidian action
(time integral of the Hamiltonian) as

\[
S_0[\phi] = \int \frac{1}{2} \left[ \dot{\phi}(x)^2 + (\nabla \phi(x))^2 + m^2 \phi^2(x) \right] d^4x
\]

\[
= \int \frac{1}{2} \left[ \partial_a \phi(x) \partial_a \phi(x) + m^2 \phi^2(x) \right] d^4x 
\]

\[
= \int \frac{1}{2} \left[ \phi(x) \left( -\partial_a^2 + m^2 \right) \phi(x) \right] d^4x .
\]

The Fourier transforms

\[
\tilde{\phi}(p) = \int e^{-ipx} \phi(x) d^4x \quad \text{and} \quad \phi(x) = \int e^{ipx} \tilde{\phi}(p) \frac{d^4p}{(2\pi)^4}
\]

turn the space-time derivatives in the action into a positive quadratic form

\[
S_0[\phi] = \frac{1}{2} \int \left| \tilde{\phi}(p) \right|^2 \left( p^2 + m^2 \right) \frac{d^4p}{(2\pi)^4}
\]

in which \( p^2 = p^2 + p^0^2 \), and \( \tilde{\phi}(-p) = \tilde{\phi}^*(p) \) since the field \( \phi \) is real. This is why such path integrals are called *Euclidian*.

We can use the formula (2) to express the mean value in the vacuum \(|0\rangle\) of the time-ordered exponential of a space-time integral of a classical (c-number, external) current \( j(x) \) as the ratio

\[
Z_0[j] \equiv \langle 0 | \mathcal{T} \left\{ \exp \left[ \int j(x) \phi(x) d^4x \right] \right\} |0 \rangle 
\]

\[
= \frac{\int \exp \left[ \int j(x) \phi(x) d^4x \right] e^{-S_0[\phi]} D\phi}{\int e^{-S_0[\phi]} D\phi} .
\]

Since the state \(|0\rangle\) is normalized, the mean value \( Z_0[0] \) is unity,

\[
Z_0[0] = 1 .
\]

If we absorb the current into the action

\[
S_0[\phi,j] = S_0[\phi] - \int j(x) \phi(x) d^4x
\]
then in terms of the current’s Fourier transform

\[ \tilde{j}(p) = \int e^{-ipx} j(x) \, d^4x \]  

(10)

the modified action \( S_0[\phi, j] \) is

\[ S_0[\phi, j] = \frac{1}{2} \int \left[ \frac{|\tilde{\phi}(p)|^2 (p^2 + m^2) - \tilde{j}^*(p)\tilde{\phi}(p) - \tilde{\phi}^*(p)\tilde{j}(p)}{(p^2 + m^2)} \right] \frac{d^4p}{(2\pi)^4}. \]  

(11)

Changing variables to

\[ \tilde{\psi}(p) = \tilde{\phi}(p) + \tilde{j}(p)/(p^2 + m^2) \]  

(12)

we write the action \( S_0[\phi, j] \) as

\[ S_0[\phi, j] = \frac{1}{2} \int \left[ \frac{|\tilde{\psi}(p)|^2 (p^2 + m^2) - \tilde{j}^*(p)\tilde{j}(p)}{(p^2 + m^2)} \right] \frac{d^4p}{(2\pi)^4} \]

\[ = S_0[\psi] - \frac{1}{2} \int \left[ \frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2 + m^2)} \right] \frac{d^4p}{(2\pi)^4}. \]  

(13)

And since \( D\phi = D\psi \), our formula (7) gives simply

\[ Z_0[j] = \exp \left[ \frac{1}{2} \int \frac{\tilde{j}(p)^2}{p^2 + m^2} \frac{d^4p}{(2\pi)^4} \right]. \]  

(14)

Going back to position space, one finds

\[ Z_0[j] = \exp \left[ \frac{1}{2} \int \frac{j(x) \Delta(x - x') j(x') \, d^4x \, d^4x'}{(2\pi)^4} \right] \]  

(15)

in which \( \Delta(x - x') \) is the euclidian version of Feynman’s propagator

\[ \Delta(x - x') = \int \frac{e^{ip(x-x')}}{p^2 + m^2} \frac{d^4p}{(2\pi)^4}. \]  

(16)

Let’s see how this works for a simple time-independent current

\[ j(x) = j_1(x) + j_2(x) = \delta(x - x_1) + \delta(x - x_2) \]  

(17)
which describes two stationary particles. The integral in $Z_0[j]$ has four terms. The 1-2 term involves the integral

\[
\int j_1(x) \Delta(x - x') j_2(x') \, d^4x \, d^4x' = \int \delta(x - x_1) \Delta(x - x') \delta(x - x_2) \, d^4x \, d^4x' = \int \Delta(x_1 - x_2) \, dx_0^1 \, dx_0^2
\]

\[
= \int \frac{e^{ip(x_1 - x_2)}}{p^2 + m^2} \frac{d^4p}{(2\pi)^4} \, dx_0^1 \, dx_0^2.
\]

(18)

In these integrals, the spatial parts of $x_1$ and $x_2$ are fixed by the delta functions, but we integrate over the time components from minus infinity to plus infinity. The integral over $x_0^1$ makes a delta function

\[
\int_{-\infty}^{\infty} e^{ip \cdot x_1^0} \, dx_1^0 = \delta(p^0).
\]

(19)

So the 1-2 term is now

\[
\int j_1(x) \Delta(x - x') j_2(x') \, d^4x \, d^4x' = \int \frac{e^{ip(x_1 - x_2)}}{p^2 + m^2} \frac{d^3p}{(2\pi)^3} \, dx_0^0.
\]

(20)

We did the integral over $p$ in class and got the Yukawa potential

\[
\int e^{ip(x_1 - x_2)} \frac{d^3p}{p^2 + m^2} \frac{1}{(2\pi)^3} \, e^{-m|x_1 - x_2|} = \frac{1}{4\pi r} \, e^{-mr}
\]

(21)

where $r = |x_1 - x_2|$. The 2-1 term gives the same result. We ignore the 1-1 and 2-2 terms because they are infinite self energies arising from the use of delta functions in the current $j$. The functional (15) then is

\[
Z_0[j] = \exp \left[ \frac{1}{2} \int j(x) \Delta(x - x') j(x') \, d^4x \, d^4x' \right]
\]

\[
= \exp \left[ \frac{1}{4\pi r} \, e^{-mr} \int_{-\infty}^{\infty} dx_0^0 \right]
\]

(22)

which we are to compare with

\[
Z_0[j] = \exp \left[ -E[r, j] \int_{-\infty}^{\infty} dx_0^0 \right].
\]

(23)
(I’ll explain why in a moment.) We conclude that the energy of the system due to the current \( j \) is

\[
E[r, j] = -\frac{1}{4\pi r} e^{-mr}
\]

where \( r = |\mathbf{x}_1 - \mathbf{x}_2| \) is the distance between the two points. This exercise shows that scalar fields produce attractive forces.

To see why the simple formula (23) is valid, let’s recall that the ratio (7) for \( Z_0[j] \) is by a zero-temperature limit of a ratio like (2). So the numerator is a path-integral representation of the (euclidian-) time-ordered product of the exponentials

\[
\exp \left[ -\epsilon \left( H_0 + \int j(x) \phi(x) \, d^3x \right) \right]
\]

in which \( j \) is explicitly time independent and \( \phi \) is the field operator at time (or inverse temperature) zero. So when \( j \) is time independent, the time ordering has no effect, and \( Z_0[j] \) is just the limit as \( \beta \to \infty \) of the ratio

\[
\frac{\text{Tr} \{ e^{-\beta H} \}}{\text{Tr} \{ e^{-\beta H_0} \}}
\]

in which

\[
H = H_0 + \int j(x) \phi(x) \, d^3x.
\]

So \( Z_0[j] \) is the limit as \( \beta \to \infty \) of the ratio of the exponentials

\[
Z_0[j] = \frac{e^{-\beta E}}{e^{-\beta E_0}} = e^{-\beta \Delta E}.
\]

Identifying \( \beta \) as \( \beta \to \infty \) with \( \int dt \), we get (23).

We can use the functional \( Z_0[j] \) to compute the mean value of the euclidianly time-ordered product of the field \( \phi \) at two points. We use the formula (15) and the definition (7) to write

\[
Z_0[j] = \langle 0 | T \left\{ \exp \left[ \int j(x) \phi(x) \, d^4x \right] \right\} | 0 \rangle
\]

\[
= \exp \left[ \frac{1}{2} \int j(x) \Delta(x - x') j(x') \, d^4x \, d^4x' \right].
\]

We now differentiate with respect to \( j(x) \). Before explaining how to do such a functional derivative, I will do a sloppy version. Imagine that the integrals
in these formulas for \( Z_0[j] \) are just sums over discrete (hyper)cubes in space-time. Then we differentiate with respect to \( j(x_1) \) and get

\[
\frac{dZ_0[j]}{dj(x_1)} = \langle 0 | \mathcal{T} \left\{ \phi(x_1) \exp \left[ \int j(x) \phi(x) \, d^4x \right] \right\} |0 \rangle
= \int \Delta(x_1 - x') \, j(x') \, d^4x \exp \left[ \frac{1}{2} \int j(x) \Delta(x - x') \, j(x') \, d^4x \, d^4x' \right]
\] (30)

in which we got a factor of 2 because either \( x \) or \( x' \) can be \( x_1 \). Now we differentiate with respect to \( j(x_2) \), and to clean things up we then set the current equal to zero. We get

\[
\langle 0 | \mathcal{T} \{ \phi(x_1)\phi(x_2) \} |0 \rangle = \Delta(x_1 - x_2).
\] (31)

For the sake of completeness, I include the calculation of the Yukawa potential carried out in momentum space. The Fourier transform of the current \( j(x) \) is

\[
\tilde{j}(p) = \int e^{-ipx} \left[ \delta(x - x_1) + \delta(x - x_2) \right] \, d^4x = 2\pi \delta(p^0) \left( e^{-ip\cdot x_1} + e^{-ip\cdot x_2} \right).
\] (32)

So now \( Z_0[j] \)

\[
Z_0[j] = \exp \left[ \frac{1}{2} \int \frac{[\tilde{j}(p)]^2 \, p^4p}{p^2 + m^2} \frac{d^4p}{(2\pi)^4} \right]
\] (33)

has four terms. The 1-2 term involves

\[
\frac{1}{2} \int \frac{(2\pi \delta(p^0))^2 e^{-ip\cdot(x_1-x_2)} \, p^4p}{p^2 + m^2} \frac{d^4p}{(2\pi)^4} = \frac{1}{2} 2\pi \delta(0) \int \frac{e^{-ip\cdot(x_1-x_2)} \, p^4p}{p^2 + m^2} \frac{d^3p}{(2\pi)^3}.
\] (34)

The 2-1 term is the same. The 1-1 and 2-2 terms are self-energy terms that we ignore. The quantity \( 2\pi \delta(0) \) is the huge time \( T \) or inverse temperature over which we are integrating \( 2\pi \delta(0) = T \). So the 1-2 and 2-1 parts give

\[
Z_0[j] = \exp \left[ T \int \frac{e^{-ip\cdot(x_1-x_2)} \, d^3p}{p^2 + m^2} \frac{1}{(2\pi)^3} \right].
\] (35)

Doing the integral, we get

\[
Z_0[j] = e^{-TE_j} = \exp \left( \frac{T e^{-mr}}{4\pi r} \right)
\] (36)
which means that the interaction energy is

\[ E_j = -\frac{e^{-mr}}{4\pi r}. \]  

(37)

The two points attract each other.