Homework 7

(1) Because the $4 \times 4$ generators of the Lorentz group satisfy

$$\omega^T = -\eta \omega \eta$$

where

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

show that under transposition the space-space and time-time components of $\omega$ change sign but that the time-space and space-time components of $\omega$ don’t change sign. That is, show that $\omega_{00} = -\omega_{00}$ and $\omega_{ij} = -\omega_{ji}$ and $\omega_{0i} = \omega_{i0}$.

(2) It follows that the tiny matrix $\omega$ must be for infinitesimal $\theta$ and $\lambda$ a linear combination

$$\omega = \theta \cdot R + \lambda \cdot B$$

of the six matrices

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which satisfy condition [1]. The three $R_j$ generate rotations; the three $B_j$ generate Lorentz boosts.

If we write $L = I + \omega$ as

$$L = I - i\theta_\ell iR_\ell - i\lambda_j iB_j \equiv I - i\theta_\ell J_\ell - i\lambda_j K_j$$

then the three matrices $J_\ell = iR_\ell$ are imaginary and antisymmetric, and therefore hermitian. But the three matrices $K_j = iB_j$ are imaginary and
symmetric, and so are antihermitian. Thus, the $4 \times 4$ matrix $L$ is not unitary. The reason is that the Lorentz group is not compact.

As your second homework problem, show that the six generators $J_\ell$ and $K_j$ satisfy three sets of commutation relations:

\begin{align*}
[J_i, J_j] &= i\epsilon_{ijk} J_k \\
[J_i, K_j] &= i\epsilon_{ijk} K_k \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k.
\end{align*}

This homework is due on 7 xi 2012.