Homework 7

(1) Because the 4×4 generators of the Lorentz group satisfy

$$\omega^{\mathsf{T}} = -\eta \omega \eta \tag{1}$$

where

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

show that under transposition the space-space and time-time components of ω change sign but that the time-space and space-time components of ω don't change sign. That is, show that $\omega_{00} = -\omega_{00}$ and $\omega_{ij} = -\omega_{ji}$ and $\omega_{0i} = \omega_{i0}$.

(2) It follows that the tiny matrix ω must be for infinitesimal $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ a linear combination

$$\omega = \boldsymbol{\theta} \cdot \boldsymbol{R} + \boldsymbol{\lambda} \cdot \boldsymbol{B} \tag{3}$$

of the six matrices

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

and

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

which satisfy condition (1). The three R_j generate rotations; the three B_j generate Lorentz boosts.

If we write $L = I + \omega$ as

$$L = I - i\theta_\ell i R_\ell - i\lambda_j i B_j \equiv I - i\theta_\ell J_\ell - i\lambda_j K_j \tag{6}$$

then the three matrices $J_\ell = iR_\ell$ are imaginary and antisymmetric, and therefore hermitian. But the three matrices $K_j = iB_j$ are imaginary and symmetric, and so are antihermitian. Thus, the 4×4 matrix L is **not unitary**. The reason is that the Lorentz group is **not compact**.

As your second homework problem, show that the six generators J_ℓ and K_j satisfy three sets of commutation relations:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k && (7) \\ [J_i, K_j] &= i\epsilon_{ijk} K_k && (8) \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k. && (9) \end{aligned}$$

This homework is due on 7 xi 2012.