Fourth homework assignment

(1) Consider a scalar field for a particle of mass $m$ like

$$\phi(x) = \int \left[ a(\boldsymbol{k}) e^{ikx} + a^\dagger(\boldsymbol{k}) e^{-ikx} \right] \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \tag{1}$$

in which $kx = \boldsymbol{k} \cdot \boldsymbol{x} - k^0 t = \boldsymbol{k} \cdot \boldsymbol{x} - \omega t$. The annihilation $a(\boldsymbol{k})$ and creation $a^\dagger(\boldsymbol{k})$ operators satisfy the commutation relations

$$\begin{aligned} [a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] &= \delta^3(\boldsymbol{k} - \boldsymbol{k}') \\ [a(\boldsymbol{k}), a(\boldsymbol{k}')] &= 0 = [a^\dagger(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')]. \end{aligned} \tag{2}$$

We saw in class that the field $\phi$ and its conjugate momentum $\pi = \dot{\phi}$ satisfy the equal-time commutation relation

$$[\phi(\boldsymbol{x}, t), \pi(\boldsymbol{y}, t)] = i\delta^3(\boldsymbol{x} - \boldsymbol{y}). \tag{3}$$

Show that

$$[\phi(\boldsymbol{x}, t), \phi(\boldsymbol{y}, t)] = 0 = [\pi(\boldsymbol{x}, t), \pi(\boldsymbol{y}, t)]\,. \tag{4}$$

(2) With each index $\lambda = \pm$, we associate the polarization vectors

$$\boldsymbol{\epsilon}_\pm(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\, R(\boldsymbol{k}) \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \tag{5}$$

in which $R(\boldsymbol{k})$ is a standard $3 \times 3$ matrix that rotates the vector $k\hat{z}$ into $\boldsymbol{k}$. In terms of these vectors and the photon annihilation $a_\lambda(\boldsymbol{k})$ and creation $a_\lambda^\dagger(\boldsymbol{k})$ operators which satisfy the commutation relations

$$\begin{aligned} [a_\lambda(\boldsymbol{k}), a_{\lambda'}^\dagger(\boldsymbol{k}')] &= \delta_{\lambda,\lambda'}\, \delta^3(\boldsymbol{k} - \boldsymbol{k}') \\ [a_\lambda(\boldsymbol{k}), a_{\lambda'}(\boldsymbol{k}')] &= 0 = [a_\lambda^\dagger(\boldsymbol{k}), a_{\lambda'}^\dagger(\boldsymbol{k}')] \end{aligned} \tag{6}$$

the field $A^i$ is for $i = 1$, 2 , and 3

$$A^i(x) = \sum_{\lambda=\pm} \int \left[ \boldsymbol{\epsilon}_\lambda^i(\boldsymbol{k}) a_\lambda(\boldsymbol{k}) e^{ikx} + \boldsymbol{\epsilon}_\lambda^{*i}(\boldsymbol{k}) a_\lambda^\dagger(\boldsymbol{k}) e^{-ikx} \right] \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \tag{7}$$

in which $kx = \boldsymbol{k} \cdot \boldsymbol{x} - k^0 t$. Use the relation

$$\sum_{\lambda=\pm} \epsilon^i_\lambda(\boldsymbol{k}) \epsilon^{*j}_\lambda(\boldsymbol{k}) = \delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2} \tag{8}$$

to show that

$$[A^i(t, \boldsymbol{x}), A^j(t, \boldsymbol{y})] = 0 = [\dot{A}^i(t, \boldsymbol{x}), \dot{A}^j(t, \boldsymbol{y})]. \tag{9}$$

Here $i$ and $j$ are spatial indices, so whether they are up or down makes no difference in SW's metric.