Third homework assignment

(1) With each index λ, we associate the polarization vectors

$$\boldsymbol{\epsilon}_\pm(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\, R(\boldsymbol{k}) \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \tag{1}$$

in which $R(\boldsymbol{k})$ is a standard 3×3 matrix that rotates the vector $k\hat{z}$ into $\boldsymbol{k}$. In terms of these vectors and the photon annihilation $a_\lambda(\boldsymbol{k})$ and creation $a^\dagger_\lambda(\boldsymbol{k})$ operators which satisfy the commutation relations

$$\begin{aligned} [a_\lambda(\boldsymbol{k}), a^\dagger_{\lambda'}(\boldsymbol{k}')] &= \delta_{\lambda,\lambda'}\, \delta^3(\boldsymbol{k} - \boldsymbol{k}') \\ [a_\lambda(\boldsymbol{k}), a_{\lambda'}(\boldsymbol{k}')] &= 0 = [a^\dagger_\lambda(\boldsymbol{k}), a^\dagger_{\lambda'}(\boldsymbol{k}')] \end{aligned} \tag{2}$$

the field A_i is

$$\boldsymbol{A}(x) = \sum_{\lambda=\pm} \int \left[\boldsymbol{\epsilon}_\lambda(\boldsymbol{k}) a_\lambda(\boldsymbol{k}) e^{ikx} + \boldsymbol{\epsilon}^*_\lambda(\boldsymbol{k}) a^\dagger_\lambda(\boldsymbol{k}) e^{-ikx} \right] \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \tag{3}$$

in which $kx = \boldsymbol{k} \cdot \boldsymbol{x} - k^0 t$. Show that this field is transverse, that is, that it satisfies the Coulomb-gauge condition $\boldsymbol{\nabla} \cdot \boldsymbol{A}(x) = 0$.

(2) A coherent state of argument α is given by the sum

$$\begin{aligned} |\alpha\rangle &= e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!} |0\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \end{aligned} \tag{4}$$

Show that it is an eigenstate of the annihilation operator a with eigenvalue α

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{5}$$

(3) Show that the coherent states provide for the identity operator the expression

$$I = \int |\alpha\rangle\langle\alpha|\, \frac{d^2\alpha}{\pi} \tag{6}$$

in which $d^2\alpha = d\mathrm{Re}\,\alpha\, d\mathrm{Im}\,\alpha$.

(4) To show that they are complete, derive a formula that expresses an arbitrary state $|\psi\rangle$ (in the space spanned by the states $|n\rangle$) in terms of the coherent states.

(5) Show that the inner product of two coherent states is

$$\langle\beta|\alpha\rangle = e^{\beta^*\alpha-(|\alpha|^2+|\beta|^2)/2}. \tag{7}$$

(6) Show that the coherent state $|\alpha\rangle$ is normalized. What is its norm?