Second homework assignment

(1) Suppose the action functional is

$$S[q] = \int_0^T \left[\tfrac{1}{2} m \dot{q}^2(t') - g q^4(t')\right] dt'. \tag{1}$$

(a) Compute the functional derivative

$$\delta S[q][h] \tag{2}$$

(b) Find an equation that $q(t)$ must satisfy if its action is to be stationary. That is, what equation must $q(t)$ obey if the functional derivative

$$\delta S[q][h] = 0 \tag{3}$$

is to vanish for all functions $h(t)$ that are zero at the end points, $h(0) = h(T) = 0$.
(c) Evaluate the physicist's functional differential equation

$$\frac{\delta S[A]}{\delta q(t)} \equiv \delta S[q][h] = 0 \tag{4}$$

where $h(t') = \delta(t' - t)$. You don't need to solve it.

(2) We have seen in class that for the free quadratic theory described by the hamiltonian

$$H_0 = \int \tfrac{1}{2} \left[\pi^2(x) + (\nabla\phi(x))^2 + m^2\phi^2(x)\right] d^3x. \tag{5}$$

the mean value in the vacuum (ground state) of the time-ordered exponential of the space-time integral of the product of an external (classical) current $j(x)$ and the field $\phi(x)$ is

$$\begin{aligned} Z_0[j] &= \langle 0|\,\mathcal{T}\left\{\exp\left[\int j(x)\,\phi(x)\,d^4x\right]\right\}|0\rangle \\ &= \exp\left[\frac{1}{2}\int j(x)\,\Delta(x-x')\,j(x')\,d^4x\,d^4x'\right]. \end{aligned} \tag{6}$$

(a) Compute the functional derivative

$$\delta Z_0[j][h] = \frac{d}{d\epsilon} Z_0[j+\epsilon h]\Big|_{\epsilon=0} \tag{7}$$

twice by differentiating both of these formulas (6) for $Z_0[j]$ and so get an equation relating the two functional derivatives. Use the fact that $\Delta(x)$ is an even function of each of the four components $x = (x^0, x^1, x^2, x^3)$

$$\Delta(x) = \int \frac{e^{ipx}}{p^2 + m^2} \frac{d^4p}{(2\pi)^4} = \Delta(-x). \tag{8}$$

(b) Repeat part (a) in physics notation

$$\frac{\delta Z_0[j]}{\delta j(y)} \equiv \delta Z_0[j][\delta_y] = \left. \frac{dZ_0[j + \epsilon\delta_y]}{d\epsilon} \right|_{\epsilon=0} \tag{9}$$

where $\delta_y(x) = \delta(x - y)$.

(c) Now compute the second functional (or variational) derivatives of the two formulas (6) for $Z_0[j]$ at $j(x) = 0$ using the notation

$$\left. \frac{\delta^2 Z_0[j]}{\delta j(y) \delta j(z)} \right|_{j=0} \equiv \left. \left. \frac{\partial^2 Z_0[j + \epsilon\delta_y + \epsilon'\delta_z]}{\partial\epsilon \, \partial\epsilon'} \right|_{\epsilon=\epsilon'=0} \right|_{j=0} \tag{10}$$

and so obtain an expression for the euclidian time-ordered product

$$\langle 0 | \mathcal{T} [\phi(y)\phi(z)] | 0 \rangle. \tag{11}$$

(3) Consider the simple functional

$$G[f] = \int f^3(x) \, dx \tag{12}$$

and check the Taylor series

$$e^{\delta} G[f][h] = \sum_{n=0}^{\infty} \frac{\delta^n}{n!} G[f][h] = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n}{d\epsilon^n} G[f + \epsilon h] \right|_{\epsilon=0} = G[f + h] \tag{13}$$

explicitly.