$$
\begin{aligned}
\langle q_3|e^{-3i\epsilon H}|q_0\rangle &= \iint_{-\infty}^{\infty} \langle q_3|e^{-i\epsilon H}|q_2\rangle\langle q_2|e^{-i\epsilon H}|q_1\rangle\langle q_1|e^{-i\epsilon H}|q_0\rangle \, dq_1 dq_2 \\
&= \left(\frac{m}{2\pi i\epsilon}\right)^{3/2} \iint_{-\infty}^{\infty} \exp\left\{ i\epsilon \sum_{j=0}^{2} \left[\tfrac{1}{2}\, m\, \dot{q}_j^2 - V(q_j) \right] \right\} dq_1 dq_2 .
\end{aligned} \tag{1}
$$

Let's set the potential $V(q)$ to $V(q) = \frac{1}{2}m\omega^2 q^2$ and integrate over q_1

$$
\begin{aligned}
I_1 &= \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{ i\frac{m}{2\epsilon} \left[(q_2 - q_1)^2 + (q_1 - q_0)^2 - \epsilon^2\omega^2 q_1^2 \right] \right\} dq_1 \\
&= \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{ i\frac{m}{2\epsilon} \left[\left(2 - \epsilon^2\omega^2\right) q_1^2 - 2(q_2 + q_0)q_1 + q_2^2 + q_0^2 \right] \right\} dq_1 .
\end{aligned} \tag{2}
$$

Using the formula

$$
\int_{-\infty}^{\infty} \exp\left(iax^2 + ibx\right) \, dx = \sqrt{\frac{i\pi}{a}} \, \exp\left(-i\frac{b^2}{4a}\right) \tag{3}
$$

with $a = m(2 - \epsilon^2\omega^2)/2\epsilon$ and $b = -m(q_2 + q_0)/\epsilon$, we get

$$
\begin{aligned}
I_1 &= \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \sqrt{\frac{2\pi i\epsilon}{m(2 - \epsilon^2\omega^2)}} \exp\left[-i\frac{m(q_2 + q_0)^2}{2\epsilon(2 - \epsilon^2\omega^2)} + i\frac{m(q_2^2 + q_0^2)}{2\epsilon} \right] \\
&= \sqrt{\frac{1}{2 - \epsilon^2\omega^2}} \exp\left\{ i\frac{m}{2\epsilon} \left[q_2^2 + q_0^2 - \frac{(q_2 + q_0)^2}{2 - \epsilon^2\omega^2} \right] \right\} \\
&= \sqrt{\frac{1}{2 - \epsilon^2\omega^2}} \exp\left\{ i\frac{m}{2\epsilon} \left[\frac{(1 - \epsilon^2\omega^2)(q_2^2 + q_0^2) - 2q_2 q_0}{2 - \epsilon^2\omega^2} \right] \right\} .
\end{aligned} \tag{4}
$$

With

$$
f = \frac{m}{2\pi i\epsilon} \sqrt{\frac{1}{2 - \epsilon^2\omega^2}} \exp\left\{ i\frac{m}{2\epsilon} \left[\frac{(1 - \epsilon^2\omega^2)q_0^2}{2 - \epsilon^2\omega^2} + q_3^2 - \epsilon^2\omega^2 q_0^2 \right] \right\} \tag{5}
$$

the amplitude now is

$$
\begin{aligned}
\langle q_3|e^{-3i\epsilon H}|q_0\rangle &= f\int_{-\infty}^{\infty}\exp\left\{i\frac{m}{2\epsilon}\left[q_2^2 - 2q_3q_2 - \epsilon^2\omega^2q_2^2 + \frac{(1-\epsilon^2\omega^2)q_2^2 - 2q_0q_2}{2-\epsilon^2\omega^2}\right]\right\}dq_2 \\
&= f\int_{-\infty}^{\infty}\exp\left\{i\frac{m}{2\epsilon}\left[q_2^2\frac{(3-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)}{2-\epsilon^2\omega^2} + q_2\left(-2q_3 - \frac{2q_0}{2-\epsilon^2\omega^2}\right)\right]\right\}dq_2 \\
&= f\int_{-\infty}^{\infty}\exp\left\{i\frac{m}{2\epsilon}\left[q_2^2\frac{(3-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)}{2-\epsilon^2\omega^2} - 2q_2\frac{q_3(2-\epsilon^2\omega^2)+q_0}{2-\epsilon^2\omega^2}\right]\right\}dq_2 \\
&= f\int_{-\infty}^{\infty}\exp\left[iaq_2^2 + ibq_2\right]dq_2 \\
&= f\sqrt{\frac{i\pi}{a}}\,\exp\left(-i\frac{b^2}{4a}\right)
\end{aligned} \tag{6}
$$

in which

$$a = \frac{m}{2\epsilon}\frac{(3-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)}{2-\epsilon^2\omega^2} \tag{7}$$

and

$$b = -\frac{m}{\epsilon}\frac{q_3(2-\epsilon^2\omega^2)+q_0}{2-\epsilon^2\omega^2}. \tag{8}$$

So

$$
\begin{aligned}
-\frac{b^2}{4a} &= -\frac{m}{2\epsilon}\left(\frac{q_3(2-\epsilon^2\omega^2)+q_0}{2-\epsilon^2\omega^2}\right)^2\frac{2-\epsilon^2\omega^2}{(3-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)} \\
&= -\frac{m}{2\epsilon}\frac{(q_3(2-\epsilon^2\omega^2)+q_0)^2}{(3-\epsilon^2\omega^2)(2-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)}.
\end{aligned} \tag{9}
$$

So the amplitude $\langle q_3|e^{-3i\epsilon H}|q_0\rangle$ is

$$\begin{aligned}
&= f\sqrt{\frac{i\pi}{a}}\,\exp\left(-i\frac{b^2}{4a}\right)\\
&= t\exp\left\{i\frac{m}{2\epsilon}\left[\frac{(1-\epsilon^2\omega^2)q_0^2}{2-\epsilon^2\omega^2}+q_3^2-\epsilon^2\omega^2q_0^2-\frac{(q_3(2-\epsilon^2\omega^2)+q_0)^2}{(3-\epsilon^2\omega^2)(2-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)}\right]\right\}\\
&= t\exp\left\{i\frac{m}{2\epsilon}\left[\frac{Aq_0^2+Bq_3^2-2(2-\epsilon^2\omega^2)q_0q_3}{(3-\epsilon^2\omega^2)(2-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)}\right]\right\}
\end{aligned} \tag{10}$$

in which

$$\begin{aligned}
A &= (1-\epsilon^2\omega^2)^2(3-\epsilon^2\omega^2)-\epsilon^2\omega^2(3-\epsilon^2\omega^2)(2-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)-1\\
&= (1-\epsilon^2\omega^2)(3-\epsilon^2\omega^2)\left[(1-\epsilon^2\omega^2)-\epsilon^2\omega^2(2-\epsilon^2\omega^2)\right]-1\\
&= (1-\epsilon^2\omega^2)(3-\epsilon^2\omega^2)\left(1-3\epsilon^2\omega^2+\epsilon^4\omega^4\right)-1\\
&= 2-13\epsilon^2\omega^2+16\epsilon^4\omega^4-7\epsilon^6\omega^6+\epsilon^8\omega^8
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
B &= (3-\epsilon^2\omega^2)(2-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)-(2-\epsilon^2\omega^2)^2\\
&= (2-\epsilon^2\omega^2)\left[(3-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)-(2-\epsilon^2\omega^2)\right]\\
&= 2-7\epsilon^2\omega^2+5\epsilon^4\omega^4-\epsilon^6\omega^6.
\end{aligned} \tag{12}$$

Thus

$$= t\exp\left\{i\frac{m}{2\epsilon}\left[\frac{(2-13\epsilon^2\omega^2+16\epsilon^4\omega^4-7\epsilon^6\omega^6+\epsilon^8\omega^8)q_0^2+(2-7\epsilon^2\omega^2+5\epsilon^4\omega^4-\epsilon^6\omega^6)q_3^2-2q_3q_0(2-\epsilon^2\omega^2)}{(3-\epsilon^2\omega^2)(2-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)}\right]\right\} \tag{13}$$

in which

$$t=\frac{m}{2\pi i\epsilon}\sqrt{\frac{1}{2-\epsilon^2\omega^2}}\sqrt{\frac{2\pi i\epsilon(2-\epsilon^2\omega^2)}{m(3-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)}}=\sqrt{\frac{m}{2\pi i\epsilon(3-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)}}. \tag{14}$$

So finally,

$$\langle q_3|e^{-3i\epsilon H}|q_0\rangle = t\exp\left\{i\frac{m}{\epsilon}\left[\frac{(1-13\epsilon^2\omega^2/2+8\epsilon^4\omega^4-7\epsilon^6\omega^6/2+\epsilon^8\omega^8/2)q_0^2+(1-7\epsilon^2\omega^2/2+5\epsilon^4\omega^4/2-\epsilon^6\omega^6)q_3^2-2q_3q_0(1-\epsilon^2\omega^2/2)}{(3-\epsilon^2\omega^2)(2-\epsilon^2\omega^2)(1-\epsilon^2\omega^2)}\right]\right\} \tag{15}$$

which for $\omega = 0$ gives us back the answer to the first problem

$$\langle q_3|e^{-3i\epsilon H}|q_0\rangle = \sqrt{\frac{m}{2\pi i3\epsilon}}\,\exp\left(i\frac{(q_3-q_0)^2}{2(3\epsilon)}\right). \tag{16}$$