$$
\begin{aligned}
\langle q_3|e^{-3\epsilon H}|q_0\rangle &= \iint_{-\infty}^{\infty} \langle q_3|e^{-\epsilon H}|q_2\rangle\langle q_2|e^{-\epsilon H}|q_1\rangle\langle q_1|e^{-\epsilon H}|q_0\rangle \, dq_1 dq_2 \\
&= \left(\frac{m}{2\pi\epsilon}\right)^{3/2} \iint_{-\infty}^{\infty} \exp\left\{ -\epsilon \sum_{j=0}^{2} \left[\tfrac{1}{2}\, m\, \dot{q}_j^2 + V(q_j) \right] \right\} dq_1 dq_2.
\end{aligned} \tag{1}
$$

Let's ignore the potential $V(q)$ and integrate over q_1

$$
\begin{aligned}
I_1 &= \left(\frac{m}{2\pi\epsilon}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{ -\frac{m}{2\epsilon}\left[(q_2-q_1)^2 + (q_1-q_0)^2\right]\right\} dq_1 \\
&= \left(\frac{m}{2\pi\epsilon}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{ -\frac{m}{\epsilon}\left[q_1^2 - (q_2+q_0)q_1 + \tfrac{1}{2}(q_2^2+q_0^2)\right]\right\} dq_1.
\end{aligned} \tag{2}
$$

Using the formula

$$
\int_{-\infty}^{\infty} \exp\left(-rx^2 + cx\right) dx = \sqrt{\frac{\pi}{r}} \exp\left(\frac{c^2}{4r}\right) \tag{3}
$$

with $r = m/\epsilon$ and $c = m(q_2+q_0)/\epsilon$, we get

$$
\begin{aligned}
I_1 &= \left(\frac{m}{2\pi\epsilon}\right)^{1/2} \sqrt{\frac{\pi\epsilon}{m}} \exp\left[m(q_2+q_0)^2/4\epsilon - m(q_2^2+q_0^2)/2\epsilon\right] \\
&= \frac{1}{\sqrt{2}} e^{-m(q_0-q_2)^2/4\epsilon}.
\end{aligned} \tag{4}
$$

The amplitude now is

$$
\begin{aligned}
\langle q_3|e^{-3\epsilon H}|q_0\rangle &= \frac{m}{2\sqrt{2}\pi\epsilon} \int_{-\infty}^{\infty} \exp\left[-\frac{m}{2\epsilon}\left[(q_3-q_2)^2 + \tfrac{1}{2}(q_2-q_0)^2\right]\right] dq_2 \\
&= \frac{m}{2\sqrt{2}\pi\epsilon} \int_{-\infty}^{\infty} \exp\left[-\frac{m}{2\epsilon}\left[\frac{3}{2}q_2^2 - (2q_3+q_0)q_2 + q_3^2 + \tfrac{1}{2}q_0^2\right]\right] dq_2
\end{aligned} \tag{5}
$$

Now we do the q_2 integral using (3) with $r = 3m/4\epsilon$ and $c = m(2q_3+q_0)/2\epsilon$. We find

$$
\langle q_3|e^{-3\epsilon H}|q_0\rangle = \left(\frac{m}{6\pi\epsilon}\right)^{1/2} e^{-m(q_3-q_0)^2/6\epsilon}. \tag{6}
$$

We now leap from 3 to n and replace 3ϵ with $n\epsilon = \beta$ so as to find that

$$
\langle q_\beta|e^{-\beta H}|q_0\rangle = \left(\frac{m}{2\pi\beta}\right)^{1/2} e^{-m(q_\beta-q_0)^2/2\beta}. \tag{7}
$$