In electrodynamics and in MKSA (SI) units, the three-dimensional vector potential $\boldsymbol{A}$ and the scalar potential ϕ form a covariant 4-vector potential

$$A_i = \left(\frac{-\phi}{c}, \boldsymbol{A} \right). \tag{1}$$

The contravariant 4-vector potential is $A^i = (\phi/c, \boldsymbol{A})$. The **magnetic induction** is

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \text{or} \quad B_i = \epsilon_{ijk} \partial_j A_k \tag{2}$$

in which $\partial_j = \partial/\partial x^j$, the sum over the repeated indices j and k runs from 1 to 3, and ϵ_{ijk} is totally antisymmetric with $\epsilon_{123} = 1$. The electric field is

$$E_i = c \left(\frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial x^0} \right) = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} \tag{3}$$

where $x^0 = ct$. In 3-vector notation, $\boldsymbol{E}$ is given by the gradient of ϕ and the time-derivative of $\boldsymbol{A}$

$$\boldsymbol{E} = -\nabla\phi - \dot{\boldsymbol{A}}. \tag{4}$$

In terms of the second-rank, antisymmetric Faraday field-strength tensor

$$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} = -F_{ji} \tag{5}$$

the electric field is $E_i = c\, F_{i0}$ and the magnetic field B_i is

$$B_i = \frac{1}{2} \epsilon_{ijk}\, F_{jk} = \frac{1}{2} \epsilon_{ijk} \left(\frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} \right) = (\nabla \times \boldsymbol{A})_i \tag{6}$$

where the sum over repeated indices runs from 1 to 3. The inverse equation $F_{jk} = \epsilon_{jki} B_i$ for spatial j and k follows from the Levi-Civita identity (**??**)

$$\begin{aligned} \epsilon_{jki} B_i &= \frac{1}{2} \epsilon_{jki} \epsilon_{inm}\, F_{nm} = \frac{1}{2} \epsilon_{ijk} \epsilon_{inm}\, F_{nm} \\ &= \frac{1}{2} \left(\delta_{jn}\, \delta_{km} - \delta_{jm}\, \delta_{kn} \right) F_{nm} = \frac{1}{2} \left(F_{jk} - F_{kj} \right) = F_{jk}. \end{aligned} \tag{7}$$

In 3-vector notation and MKSA = SI units, Maxwell's equations are a ban on magnetic monopoles and **Faraday's law**, both homogeneous,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad \text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0 \tag{8}$$

and **Gauss's law** and the **Maxwell-Ampère law**, both inhomogeneous,

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f \quad \text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{j}_f + \dot{\boldsymbol{D}}. \tag{9}$$

Here ρ_f is the density of **free** charge and $\boldsymbol{j}_f$ is the **free current density**. By *free*, we understand charges and currents that do not arise from polarization and are not restrained by chemical bonds. The divergence of $\boldsymbol{\nabla} \times \boldsymbol{H}$ vanishes (like that of any curl), and so the Maxwell-Ampère law and Gauss's law imply that free charge is conserved

$$0 = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) = \boldsymbol{\nabla} \cdot \boldsymbol{j}_f + \boldsymbol{\nabla} \cdot \dot{\boldsymbol{D}} = \boldsymbol{\nabla} \cdot \boldsymbol{j}_f + \dot{\rho}_f. \tag{10}$$

If we use this **continuity equation** to replace $\boldsymbol{\nabla} \cdot \boldsymbol{j}_f$ with $-\dot{\rho}_f$ in this same equation $0 = \boldsymbol{\nabla} \cdot \boldsymbol{j}_f + \boldsymbol{\nabla} \cdot \dot{\boldsymbol{D}}$, then we see that the Maxwell-Ampère law preserves Gauss's law (a constraint) in time

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{j}_f + \boldsymbol{\nabla} \cdot \dot{\boldsymbol{D}} = \frac{\partial}{\partial t} \left(-\rho_f + \boldsymbol{\nabla} \cdot \boldsymbol{D} \right). \tag{11}$$

Similarly, Faraday's law preserves the constraint $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$

$$0 = -\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{12}$$

In a **linear, isotropic** medium, the **electric displacement** $\boldsymbol{D}$ is related to the electric field $\boldsymbol{E}$ by the **permittivity** $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and the **magnetic** or **magnetizing** field $\boldsymbol{H}$ differs from the magnetic induction $\boldsymbol{B}$ by the **permeability** $\boldsymbol{H} = \boldsymbol{B}/\mu$.

On a sub-nanometer scale, the microscopic form of Maxwell's equations applies. On this scale, the homogeneous equations (8) are unchanged, but the inhomogeneous ones are

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \, \boldsymbol{j} + \epsilon_0 \, \mu_0 \, \dot{\boldsymbol{E}} = \mu_0 \, \boldsymbol{j} + \frac{\dot{\boldsymbol{E}}}{c^2} \tag{13}$$

in which ρ and $\boldsymbol{j}$ are the total charge and current densities, and $\epsilon_0 = 8.854 \times 10^{-12}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 are the **electric** and **magnetic constants**, whose product is the inverse of the square of the speed of light, $\epsilon_0 \mu_0 = 1/c^2$. Gauss's law and the Maxwell-Ampère law (13) imply (exercise **??**) that the microscopic (total) current 4-vector $j = (c\rho, \boldsymbol{j})$ obeys the continuity equation $\dot{\rho} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$. Electric charge is conserved.

In vacuum, $\rho = \boldsymbol{j} = 0$, $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$, and $\boldsymbol{H} = \boldsymbol{B}/\mu_0$, and Maxwell's equations become

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad &\text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \quad &\text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{c^2}\dot{\boldsymbol{E}}. \end{aligned} \tag{14}$$

Two of these equations $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$ are constraints. Taking the curl of the other two equations, we find

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\frac{1}{c^2}\ddot{\boldsymbol{E}} \quad \text{and} \quad \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = -\frac{1}{c^2}\ddot{\boldsymbol{B}}. \tag{15}$$

One may use the Levi-Civita identity (**??**) to show (exercise **??**) that

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E}) - \triangle\boldsymbol{E} \text{ and } \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \triangle\boldsymbol{B} \tag{16}$$

in which $\triangle \equiv \boldsymbol{\nabla}^2$. Since in vacuum the divergence of $\boldsymbol{E}$ vanishes, and since that of $\boldsymbol{B}$ always vanishes, these identities and the curl-curl equations (15) tell us that waves of $\boldsymbol{E}$ and $\boldsymbol{B}$ move at the speed of light

$$\frac{1}{c^2}\ddot{\boldsymbol{E}} - \triangle\boldsymbol{E} = 0 \quad \text{and} \quad \frac{1}{c^2}\ddot{\boldsymbol{B}} - \triangle\boldsymbol{B} = 0. \tag{17}$$

We may write the two homogeneous Maxwell equations (8) as

$$\begin{aligned} \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} &= \partial_i \left(\partial_j A_k - \partial_k A_j\right) + \partial_k \left(\partial_i A_j - \partial_j A_i\right) \\ &+ \partial_j \left(\partial_k A_i - \partial_i A_k\right) = 0 \end{aligned} \tag{18}$$

(exercise **??**). This relation, known as the **Bianchi identity**, actually is a generally covariant tensor equation

$$\epsilon^{\ell ijk} \partial_i F_{jk} = 0 \tag{19}$$

in which $\epsilon^{\ell ijk}$ is totally antisymmetric, as explained in Sec. **??**. There are four versions of this identity (corresponding to the four ways of choosing three different indices i, j, k from among four and leaving out one, ℓ). The $\ell = 0$ case gives the scalar equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, and the three that have $\ell \neq 0$ give the vector equation $\boldsymbol{\nabla} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0$.

In tensor notation, the microscopic form of the two inhomogeneous equations (13)—the laws of Gauss and Ampère—are

$$\partial_i F^{ki} = \mu_0 \, j^k \tag{20}$$

in which j^k is the current 4-vector

$$j^k = (c\rho, \boldsymbol{j}) \,. \tag{21}$$

The **Lorentz force law** for a particle of charge q is

$$m \frac{d^2 x^i}{d\tau^2} = m \frac{du^i}{d\tau} = \frac{dp^i}{d\tau} = f^i = q\, F^{ij} \frac{dx_j}{d\tau} = q\, F^{ij}\, u_j. \tag{22}$$

We may cancel a factor of $dt/d\tau$ from both sides and find for $i = 1, 2, 3$

$$\frac{dp^i}{dt} = q\left(-F^{i0} + \epsilon_{ijk} B_k v_j\right) \quad \text{or} \quad \frac{d\boldsymbol{p}}{dt} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right). \tag{23}$$

This equation for $i = 0$ says that the time derivative of the energy E is

$$\frac{dE}{dt} = q\, \boldsymbol{E} \cdot \boldsymbol{v} \tag{24}$$

which shows that only the electric field does work. The only special-relativistic correction needed in Maxwell's electrodynamics is a factor of $1/\sqrt{1 - v^2/c^2}$ in these equations. That is, we use $\boldsymbol{p} = m\boldsymbol{u} = m\boldsymbol{v}/\sqrt{1 - v^2/c^2}$ not $\boldsymbol{p} = m\boldsymbol{v}$ in (23), and we use the total energy E not the kinetic energy in (24). The reason why so little of classical electrodynamics was changed by special relativity is that electric and magnetic effects were accessible to measurement during the 1800's. Classical electrodynamics was almost perfect.

Keeping track of factors of the speed of light is a lot of trouble and a distraction; in what follows, we'll often use units with $c = 1$.

In natural units, the action is

$$S[A] = \int -\tfrac{1}{4} F_{ab} F^{ab} + j_b A^b + \mathcal{L}_m \, d^4x \tag{25}$$

in which

$$F_{ab} = \partial_a A_b - \partial_b A_a \tag{26}$$

is Faraday's tensor and $\mathcal{L}_m$ is the action density of the matter fields. So the action is

$$\begin{aligned} S[A] &= \int -\tfrac{1}{4} \left(\partial_a A_b - \partial_b A_a\right) \left(\partial^a A^b - \partial^b A^a\right) + j_b A^b + \mathcal{L}_m \, d^4x \\ &= \int \tfrac{1}{2} \left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right) + \boldsymbol{j} \cdot \boldsymbol{A} - j^0 A^0 + \mathcal{L}_m \, d^4x. \end{aligned} \tag{27}$$

Electrodynamics is invariant under **local gauge** transformations

$$\begin{aligned} \psi'(x) &= e^{i\lambda(x)}\psi(x) \\ A'_b(x) &= A_b(x) + i\left(\partial_b e^{i\lambda(x)}\right)e^{-i\lambda(x)} = A_b(x) - \partial_b\lambda(x) \end{aligned} \tag{28}$$

in which the function $\lambda(x)$ depends upon the space-time coordinates x.

Gauss's law

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho/\epsilon_0 \tag{29}$$

is a constraint. It says that the field A_b must satisfy

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \boldsymbol{\nabla}\cdot\left(\boldsymbol{c}\boldsymbol{\nabla}\boldsymbol{A_0} - \dot{\boldsymbol{A}}\right) = \rho/\epsilon_0. \tag{30}$$

We transform to the Coulomb (or radiation) gauge by choosing $\lambda(x)$ so that

$$\boldsymbol{\nabla}\cdot\boldsymbol{A'} = \boldsymbol{\nabla}\cdot\left(\boldsymbol{A(x)} - \boldsymbol{\nabla\lambda(x)}\right) = 0. \tag{31}$$

That is, we set

$$\triangle\lambda(x) = \boldsymbol{\nabla}\cdot\boldsymbol{A(x)}. \tag{32}$$

In this gauge, Gauss's law (30) says that

$$\triangle A_0(x) = -\triangle A^0(x) = \frac{\rho(x)}{c\epsilon_0} = j^0. \tag{33}$$

That is, the field $A_0(x)$ is determined by the charge density $j^0 = \rho/c\epsilon_0$

$$A^0(\boldsymbol{x}, t) = \int \frac{j^0(\boldsymbol{y}, t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\, d^3y. \tag{34}$$

The field A^0 is a **dependent** variable.

The momentum canonically conjugate to the transverse field $\boldsymbol{A}$ is for $i = 1$, 2, and 3

$$\pi^i = \frac{\partial\mathcal{L}}{\partial\dot{A}_i} = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_i)} \quad \text{while} \quad \pi^0 = 0. \tag{35}$$

Using our formula (27) for the action, we find

$$\pi^i = -\partial^0 A^i = \partial_0 A^i = \dot{A}^i. \tag{36}$$

So the hamiltonian density is

$$
\begin{aligned}
\mathcal{H} &= \pi^i \dot{A}^i - \mathcal{L} = \pi^{i2} - \tfrac{1}{2}\boldsymbol{E}^2 + \tfrac{1}{2}\boldsymbol{B}^2 - \boldsymbol{j}\cdot\boldsymbol{A} + j^0 A^0 - \mathcal{L}_m \\
&= \pi^{i2} - \tfrac{1}{2}\left(\boldsymbol{\pi} + \boldsymbol{\nabla}\boldsymbol{A^0}\right)^2 + \tfrac{1}{2}\left(\boldsymbol{\nabla}\times\boldsymbol{A}\right)^2 - \boldsymbol{j}\cdot\boldsymbol{A} + j^0 A^0 - \mathcal{L}_m \\
&= \tfrac{1}{2}\pi^{i2} + \tfrac{1}{2}\left(\boldsymbol{\nabla}\times\boldsymbol{A}\right)^2 - \boldsymbol{\pi}\cdot\boldsymbol{\nabla}\boldsymbol{A^0} - \tfrac{1}{2}\left(\boldsymbol{\nabla}A^0\right)^2 - \boldsymbol{j}\cdot\boldsymbol{A} + j^0 A^0 - \mathcal{L}_m .
\end{aligned}
\tag{37}
$$

Integrating by parts and dropping the surface term, we see that the term $-\boldsymbol{\pi}\cdot\boldsymbol{\nabla}\boldsymbol{A^0} \to \boldsymbol{A^0}\boldsymbol{\nabla}\cdot\boldsymbol{\pi}$ vanishes because $\boldsymbol{\pi}$ is transverse. Integrating by parts further (dropping the surface term) and using Gauss's law (33), we get

$$
\int -\tfrac{1}{2}\left(\boldsymbol{\nabla}A^0\right)^2 \, d^3x = \int \tfrac{1}{2}A^0\boldsymbol{\nabla}^2 A^0 \, d^3x = \int -\tfrac{1}{2}A^0 j^0 \, d^3x \tag{38}
$$

The hamiltonian then is

$$
H = H_m + \int \left[\tfrac{1}{2}\boldsymbol{\pi^2} + \tfrac{1}{2}(\boldsymbol{\nabla}\times\boldsymbol{A})^2 - \boldsymbol{A}\cdot\boldsymbol{j}\right] \, d^3x + V_C \tag{39}
$$

in natural units where

$$
V_C = \int \tfrac{1}{2}A^0 j^0 \, d^3x = \tfrac{1}{2}\int \frac{j^0(\boldsymbol{x},t)j^0(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \, d^3x d^3y. \tag{40}
$$

Clearly like charges raise the energy V_C, while opposite charges lower it.

We quantize the transverse field $\boldsymbol{A}$ and its transverse time derivative $\boldsymbol{\pi} = \dot{\boldsymbol{A}}$. Their commutation relations are

$$
\begin{aligned}
&[A_i(t,\boldsymbol{x}), \pi_j(t,\boldsymbol{y})] = i\left[\delta_{ij}\delta^3(\boldsymbol{x}-\boldsymbol{y}) + \frac{\partial^2}{\partial x^i \partial x^j}\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\right] \\
&[A_i(t,\boldsymbol{x}), A_j(t,\boldsymbol{y})] = 0 = [\pi_i(t,\boldsymbol{x}), \pi_j(t,\boldsymbol{y})].
\end{aligned}
\tag{41}
$$

We may represent these commutation relations by introducing the photon annihilation $a_\lambda(\boldsymbol{k})$ and creation $a_\lambda^\dagger(\boldsymbol{k})$ operators which satisfy the commutation relations

$$
\begin{aligned}
&[a_\lambda(\boldsymbol{k}), a_{\lambda'}^\dagger(\boldsymbol{k}')] = \delta_{\lambda,\lambda'}\,\delta^3(\boldsymbol{k}-\boldsymbol{k}') \\
&[a_\lambda(\boldsymbol{k}), a_{\lambda'}(\boldsymbol{k}')] = 0 = [a_\lambda^\dagger(\boldsymbol{k}), a_{\lambda'}^\dagger(\boldsymbol{k}')]
\end{aligned}
\tag{42}
$$

for the two values of the polarization index λ. With each index λ, we associate the polarization vectors

$$
\boldsymbol{\epsilon}_\pm(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\,R(\boldsymbol{k})\begin{pmatrix}1\\ \pm i\\ 0\end{pmatrix} \tag{43}
$$

in which $R(\boldsymbol{k})$ is a standard 3×3 matrix that rotates the vector $k\hat{z}$ into $\boldsymbol{k}$. In terms of these operators and vectors, the field A_i is

$$\boldsymbol{A}(x) = \sum_{\lambda=\pm} \int \left[\boldsymbol{\epsilon}_\lambda(\boldsymbol{k}) a_\lambda(\boldsymbol{k}) e^{ikx} + \boldsymbol{\epsilon}^*_\lambda(\boldsymbol{k}) a^\dagger_\lambda(\boldsymbol{k}) e^{-ikx} \right] \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}. \tag{44}$$