Coherent-state path integrals

A coherent state of argument α

$$\begin{aligned}|\alpha\rangle &= e^{-|\alpha|^2/2} e^{\alpha a^\dagger}|0\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!}|0\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle\end{aligned} \tag{1}$$

is an eigenstate of the annihilation operator a with eigenvalue α

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad \text{which implies that} \quad \langle\alpha|a^\dagger = \langle\alpha|\alpha^*. \tag{2}$$

The coherent states are complete (indeed supercomplete) and provide for the identity operator the expression

$$I = \int |\alpha\rangle\langle\alpha| \, \frac{d^2\alpha}{\pi} \tag{3}$$

in which $d^2\alpha = d\mathrm{Re}\,\alpha \, d\mathrm{Im}\,\alpha$. The inner product of two coherent states is

$$\langle\beta|\alpha\rangle = e^{\beta^*\alpha - (|\alpha|^2 + |\beta|^2)/2} \tag{4}$$

so that

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\beta-\alpha|^2} \tag{5}$$

which shows that they are normalized $\langle\alpha|\alpha\rangle = 1$.

Suppose that $H(a^\dagger, a)$ is any normally ordered hamiltonian, that is, any hermitian expression that has its a's standing to the right of its $a^\dagger$'s. Then to first order in ϵ

$$\langle\beta|e^{-i\epsilon H(a^\dagger,a)}|\alpha\rangle = \langle\beta|\alpha\rangle e^{-i\epsilon H(\beta^*,\alpha)} = e^{\beta^*\alpha - (|\alpha|^2+|\beta|^2)/2 - i\epsilon H(\beta^*,\alpha)}. \tag{6}$$

We replace β and α by α_2 and α_1 so that

$$\langle\alpha_2|e^{-i\epsilon H(a^\dagger,a)}|\alpha_1\rangle = \exp\left[\alpha_2^*\alpha_1 - (|\alpha_2|^2 + |\alpha_1|^2)/2 - i\epsilon H(\alpha_2^*, \alpha_1)\right]. \tag{7}$$

Now we let

$$\alpha_2 = \alpha_1 + \epsilon\dot{\alpha}_1 \equiv \alpha + \epsilon\dot{\alpha} \tag{8}$$

and get again to first order in ϵ

$$\begin{aligned}\langle\alpha + \epsilon\dot{\alpha}|e^{-i\epsilon H(a^\dagger,a)}|\alpha\rangle &= \exp\left[(\alpha+\epsilon\dot{\alpha})^*\alpha - \tfrac{1}{2}(\alpha+\epsilon\dot{\alpha})^*(\alpha+\epsilon\dot{\alpha}) - \tfrac{1}{2}\alpha^*\alpha - i\epsilon H(\alpha^*,\alpha)\right] \\ &= \exp\left[\epsilon\dot{\alpha}^*\alpha - \tfrac{1}{2}\epsilon\dot{\alpha}^*\alpha - \tfrac{1}{2}\epsilon\alpha^*\dot{\alpha} - i\epsilon H(\alpha^*,\alpha)\right] \\ &= \exp\left[\frac{\epsilon}{2}\left(\dot{\alpha}^*\alpha - \alpha^*\dot{\alpha}\right) - i\epsilon H(\alpha^*,\alpha)\right].\end{aligned} \tag{9}$$

If we set $\alpha = (q+ip)/\sqrt{2}$, then we find

$$\begin{aligned}\langle\alpha + \epsilon\dot{\alpha}|e^{-i\epsilon H(a^\dagger,a)}|\alpha\rangle &= \exp\left\{\frac{\epsilon}{4}\left[(\dot{q}-i\dot{p})\left(q+ip\right) - (q-ip)\left(\dot{q}+i\dot{p}\right)\right] - i\epsilon H(p,q)\right\} \\ &= \exp\left\{i\epsilon\left[\tfrac{1}{2}\left(\dot{q}p - q\dot{p}\right) - H(p,q)\right]\right\} \\ &= \exp\left\{i\epsilon\left[\tfrac{1}{2}\dot{q}p - \tfrac{1}{2}\frac{d(qp)}{dt} + \tfrac{1}{2}\dot{q}p - H(p,q)\right]\right\} \\ &= \exp\left\{i\epsilon\left[\dot{q}p - H(p,q) - \tfrac{1}{2}\frac{d(qp)}{dt}\right]\right\}.\end{aligned} \tag{10}$$

Now the hamiltonian H is related to the lagrangian L by $H = \dot{q}p - L$, so $\dot{q}p - H(p,q) = L$, and we have

$$\langle\alpha + \epsilon\dot{\alpha}|e^{-i\epsilon H(a^\dagger,a)}|\alpha\rangle = \exp\left\{i\epsilon\left[L - \tfrac{1}{2}\frac{d(qp)}{dt}\right]\right\}. \tag{11}$$

If we put together $N = T/\epsilon$ matrix elements $\langle\alpha+\epsilon\dot{\alpha}|\exp\left[-i\epsilon H(a^\dagger,a)\right]|\alpha\rangle$, then we get the path integral

$$\begin{aligned}\langle f|e^{-iTH}|i\rangle &= \int \langle f|\alpha(T)\rangle e^{-i\frac{1}{2}q(T)p(T)} \exp\left[i\int_0^T L(\alpha)dt\right] e^{i\frac{1}{2}q(0)p(0)}\langle\alpha(0)|i\rangle D\alpha \\ &= \int \langle f|\alpha(T)\rangle e^{(\bar{\alpha}^2(T)-\alpha^2(T))/4} \exp\left[i\int_0^T L(\alpha)dt\right] e^{(\alpha^2(0)-\bar{\alpha}^2(0))/4}\langle\alpha(0)|i\rangle D\alpha\end{aligned} \tag{12}$$

in which $|f\rangle$ and $|i\rangle$ are the final and initial states, and $D\alpha$ is a product of $d^2\alpha/\pi$, one for each time slice. The factors $e^{(\bar\alpha^2(T)-\alpha^2(T))/4}$ and $e^{(\alpha^2(0)-\bar\alpha^2(0))/4}$ are phases.

If the initial and final states are both the ground state $|0\rangle$, then the amplitude is

$$\langle 0|e^{-iTH}|0\rangle = \int e^{(\bar\alpha^2(T)-\alpha^2(T))/4-|\alpha(T)|^2/2} \exp\left[i\int_0^T L(\alpha)dt\right] e^{(\alpha^2(0)-\bar\alpha^2(0))/4-|\alpha(0)|^2/2} D\alpha. \tag{13}$$

Since the position eigenstates $|q\rangle$ and the coherent states $\alpha\rangle$ are so important, it makes sense to compute their inner product, which by definition (1) is

$$\langle q|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle q|n\rangle. \tag{14}$$

The wave function $q|n\rangle$ of the nth excited state of the harmonic oscillator is

$$\langle q|n\rangle = \frac{\sqrt{s}\,e^{-(sq)^2/2}}{\sqrt{2^n n!\sqrt{\pi}}}\,H_n(sq) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{e^{-m\omega q^2/2\hbar}}{\sqrt{2^n n!}}\,H_n\left(\left(\frac{m\omega}{\hbar}\right)^{1/2} q\right) \tag{15}$$

where $s = \sqrt{m\omega/\hbar}$. The generating function for the Hermite polynomials $H_n(x)$ is

$$e^{2xy-y^2} = \sum_{n=0}^{\infty} H_n(x)\,\frac{y^n}{n!}. \tag{16}$$

So using (14, 15, & 16), we get

$$
\begin{aligned}
\langle q|\alpha\rangle &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle q|n\rangle \\
&= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\sqrt{s}\, e^{-(sq)^2/2}}{\sqrt{2^n n! \sqrt{\pi}}} \, H_n(sq) \\
&= \sqrt{\frac{s}{\sqrt{\pi}}}\, e^{-|\alpha|^2/2-(sq)^2/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha}{\sqrt{2}}\right)^n \, H_n(sq) \\
&= \sqrt{\frac{s}{\sqrt{\pi}}}\, e^{-|\alpha|^2/2-(sq)^2/2} e^{\sqrt{2} sq\alpha - \alpha^2/2} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(- \frac{|\alpha|^2}{2} - \frac{\alpha^2}{2} - \frac{m\omega}{2\hbar} q^2 + \sqrt{\frac{2m\omega}{\hbar}}\, q\, \alpha \right).
\end{aligned}
\tag{17}
$$

We can extend this path integral to one for a systems of several degrees of freedom or of infinitely many degrees of freedom by using multimode coherent states $|\{\alpha\}\rangle$ which are eigenstates of the annihilation operators of the various modes

$$a_\lambda(\boldsymbol{k})|\{\alpha\}\rangle = \alpha_\lambda(\boldsymbol{k})|\{\alpha\}\rangle. \tag{18}$$

The multimode identity operator is

$$I = \int |\{\alpha\}\rangle\langle\{\alpha\}| \prod_{\lambda,\boldsymbol{k}} \frac{d^2\alpha_\lambda(\boldsymbol{k})}{\pi}. \tag{19}$$

Our single-mode path integral (12) generalizes to the multimode path integral

$$\langle f|e^{-iTH}|i\rangle = \int \langle f|\{\alpha(T)\}\rangle e^{i\phi(T)} \exp\left[i \int_0^T L(\{\alpha\}) d^4x \right] e^{-i\phi(0)} \langle\{\alpha(0)\}|i\rangle D\alpha \tag{20}$$

in which now

$$D\alpha = \prod_{\lambda,k} \frac{d^2\alpha_\lambda(k)}{\pi}. \tag{21}$$

has an integration for every 4-vector k and the phase $\phi(t)$ for $t = T$ and $t = 0$ is a sum over the modes

$$\phi(t) = \sum_{\lambda,\boldsymbol{k}} \frac{1}{4} \left(\bar{\alpha}_\lambda^2(t, \boldsymbol{k}) - \alpha_\lambda^2(t, \boldsymbol{k}) \right) . \tag{22}$$

Consider a scalar field for a particle of mass m like

$$\phi(x) = \int \left[a(\boldsymbol{k}) e^{ikx} + a^\dagger(\boldsymbol{k}) e^{-ikx} \right] \frac{d^3 k}{\sqrt{(2\pi)^3 2\omega_k}} \tag{23}$$

in which $kx = \boldsymbol{k} \cdot \boldsymbol{x} - k^0 t = \boldsymbol{k} \cdot \boldsymbol{x} - \omega t$. The annihilation $a(\boldsymbol{k})$ and creation $a^\dagger(\boldsymbol{k})$ operators satisfy the commutation relations

$$\begin{aligned} [a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] &= \delta^3(\boldsymbol{k} - \boldsymbol{k}') \\ [a(\boldsymbol{k}), a(\boldsymbol{k}')] &= 0 = [a^\dagger(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')]. \end{aligned} \tag{24}$$

The field ϕ and its conjugate momentum $\pi = \dot{\phi}$ satisfy the equal-time commutation relations

$$[\phi(\boldsymbol{x}, t), \pi(\boldsymbol{y}, t)] = i\delta^3(\boldsymbol{x} - \boldsymbol{y}). \tag{25}$$

and

$$[\phi(\boldsymbol{x}, t), \phi(\boldsymbol{y}, t)] = 0 = [\pi(\boldsymbol{x}, t), \pi(\boldsymbol{y}, t)] \, . \tag{26}$$

In class, we derived for the mean value in the vacuum $|0\rangle$ of the time-ordered exponential of a space-time integral of a classical (c-number, external) current $j(x)$ the path-integral formula

$$\begin{aligned} Z_0[j] &\equiv \langle 0 | \mathcal{T} \left\{ \exp \left[i \int j(x) \, \phi(x) \, d^4 x \right] \right\} |0\rangle \\ &= \frac{\int \exp \left[i \int j(x) \, \phi(x) \, d^4 x \right] e^{iS_0[\phi, \epsilon]} \, D\phi}{\int e^{iS_0[\phi, \epsilon]} \, D\phi} \end{aligned} \tag{27}$$

in which $S_0[\phi, \epsilon]$ is the action of a free scalar field that describes a particle of mass m. The action $S_0[\phi, \epsilon]$ is quadratic in the fields

$$\begin{aligned} S_0[\phi, \epsilon] &= \int \tfrac{1}{2} \left[\dot{\phi}^2(x) - (\nabla\phi(x))^2 - (m^2 - i\epsilon)\phi^2(x) \right] d^4x \\ &= \int \tfrac{1}{2} \left[-\partial_a\phi(x)\partial^a\phi(x) - (m^2 - i\epsilon)\phi^2(x) \right] d^4x \end{aligned} \tag{28}$$

in which I simplified the $i\epsilon$-term. The field $\phi(x)$ obeys the Klein-Gordon equation

$$(m^2 - \Box)\phi(x) = 0. \tag{29}$$

We used path integrals to show that

$$Z_0[j] = \exp\left[\frac{i}{2} \int j(x)\, \Delta(x - x')\, j(x')\, d^4x\, d^4x' \right] \tag{30}$$

in which $\Delta(x - x')$ is Feynman's **propagator**

$$\Delta(x - x') = \int \frac{e^{ip(x-x')}}{p^2 + m^2 - i\epsilon} \frac{d^4p}{(2\pi)^4}. \tag{31}$$

The functional derivative of $Z_0[j]$ is

$$\frac{1}{i} \frac{\delta Z_0[j]}{\delta j(x)} = \langle 0 | \mathcal{T} \left[\phi(x) \exp\left(i \int j(x)\phi(x) d^4x \right) \right] | 0 \rangle \tag{32}$$

while that of equation (86) is

$$\frac{1}{i} \frac{\delta Z_0[j]}{\delta j(x)} = Z_0[j] \int \Delta(x - x')\, j(x')\, d^4x'. \tag{33}$$

So the second functional derivative of $Z_0[j]$ evaluated at $j = 0$ gives

$$\langle 0 | \mathcal{T} \left[\phi(x)\phi(x') \right] | 0 \rangle = \frac{1}{i^2} \frac{\delta^2 Z_0[j]}{\delta j(x) \delta j(x')} \bigg|_{j=0} = -i\, \Delta(x - x') \equiv -i\, \Delta_F(x - x'). \tag{34}$$

We can derive the same formula more directly from the operator expression (23) for the field at the space-time point x. The commutator of the positive-frequency part

$$\phi^{+}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \, \exp[i(\boldsymbol{p}\cdot\boldsymbol{x} - p^0 x^0)] \, a(\boldsymbol{p}) \tag{35}$$

of the scalar field $\phi = \phi^+ + \phi^-$ with its negative-frequency part

$$\phi^{-}(y) = \int \frac{d^3q}{\sqrt{(2\pi)^3 2q^0}} \, \exp[-i(\boldsymbol{q}\cdot\boldsymbol{y} - q^0 y^0)] \, a^\dagger(\boldsymbol{q}) \tag{36}$$

is the Lorentz-invariant function $\Delta_+(x-y)$

$$\begin{aligned} [\phi^+(x), \phi^-(y)] &= \int \frac{d^3p \, d^3q}{(2\pi)^3 2\sqrt{q^0 p^0}} \, e^{ipx - iqy} \, [a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})] \\ &= \int \frac{d^3p}{(2\pi)^3 2p^0} \, e^{ip(x-y)} = \Delta_+(x-y) \end{aligned} \tag{37}$$

in which $p(x-y) = \boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y}) - p^0(x^0 - y^0)$.

Incidentally, at points x that are space-like, that is, for which $x^2 = \boldsymbol{x}^2 - (x^0)^2 \equiv r^2 > 0$, the Lorentz-invariant function $\Delta_+(x)$ depends only upon $r = +\sqrt{x^2}$ and has the value (Weinberg, QFTI, p. 202)

$$\Delta_+(x) = \frac{m}{4\pi^2 r} \, K_1(mr) \tag{38}$$

in which the Hankel function K_1 is

$$K_1(z) = -\frac{\pi}{2} \, [J_1(iz) + iN_1(iz)] = \frac{1}{z} + \frac{z}{2j+2} \left[\ln\left(\frac{z}{2}\right) + \gamma - \frac{1}{2j+2} \right] + \dots \tag{39}$$

where J_1 is the first Bessel function, N_1 is the first Neumann function, and $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

The Feynman propagator $\Delta_F(x)$ is a Green's function for the Klein-Gordon differential operator

$$(m^2 - \Box)\Delta_F(x) = \delta^4(x) \tag{40}$$

in which $x = (x^0, \boldsymbol{x})$ and

$$\Box = \triangle - \frac{\partial^2}{\partial t^2} = \triangle - \frac{\partial^2}{\partial (x^0)^2} \tag{41}$$

is the four-dimensional version of the laplacian $\triangle \equiv \nabla \cdot \nabla$. Here $\delta^4(x)$ is the four-dimensional Dirac delta function

$$\delta^4(x) = \int \frac{d^4q}{(2\pi)^4} \exp[i(\boldsymbol{q} \cdot \boldsymbol{x} - q^0 x^0)] = \int \frac{d^4q}{(2\pi)^4} e^{iqx} \tag{42}$$

in which $qx = \boldsymbol{q} \cdot \boldsymbol{x} - q^0 x^0$ is the Lorentz-invariant inner product of the 4-vectors q and x. There are many Green's functions that satisfy Eq.(40). Feynman's propagator $\Delta_F(x)$ is the one that satisfies boundary conditions that will become evident when we analyze the effect of its $i\epsilon$

$$\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4} \frac{\exp(iqx)}{q^2 + m^2 - i\epsilon} = \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dq^0}{2\pi} \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x} - iq^0x^0}}{q^2 + m^2 - i\epsilon}. \tag{43}$$

The quantity $q^0 = E_{\boldsymbol{q}} = \sqrt{\boldsymbol{q}^2 + m^2}$ is the energy of a particle of mass m and momentum $\boldsymbol{q}$ in natural units with the speed of light $c = 1$. Using this abbreviation and setting $\epsilon' = \epsilon/2E_q$, we may write the denominator as

$$q^2 + m^2 - i\epsilon = \boldsymbol{q} \cdot \boldsymbol{q} - \left(q^0\right)^2 + m^2 - i\epsilon = \left(E_q - i\epsilon' - q^0\right)\left(E_q - i\epsilon' + q^0\right) + \epsilon'^2 \tag{44}$$

in which ϵ'^2 is negligible. Dropping the prime on ϵ, we do the q^0 integral

$$I(\boldsymbol{q}) = -\int_{-\infty}^{\infty} \frac{dq^0}{2\pi} e^{-iq^0x^0} \frac{1}{[q^0 - (E_{\boldsymbol{q}} - i\epsilon)]\,[q^0 - (-E_{\boldsymbol{q}} + i\epsilon)]}. \tag{45}$$

As shown in Fig. 1, the integrand

$$e^{-iq^0x^0} \frac{1}{[q^0 - (E_{\boldsymbol{q}} - i\epsilon)]\,[q^0 - (-E_{\boldsymbol{q}} + i\epsilon)]} \tag{46}$$

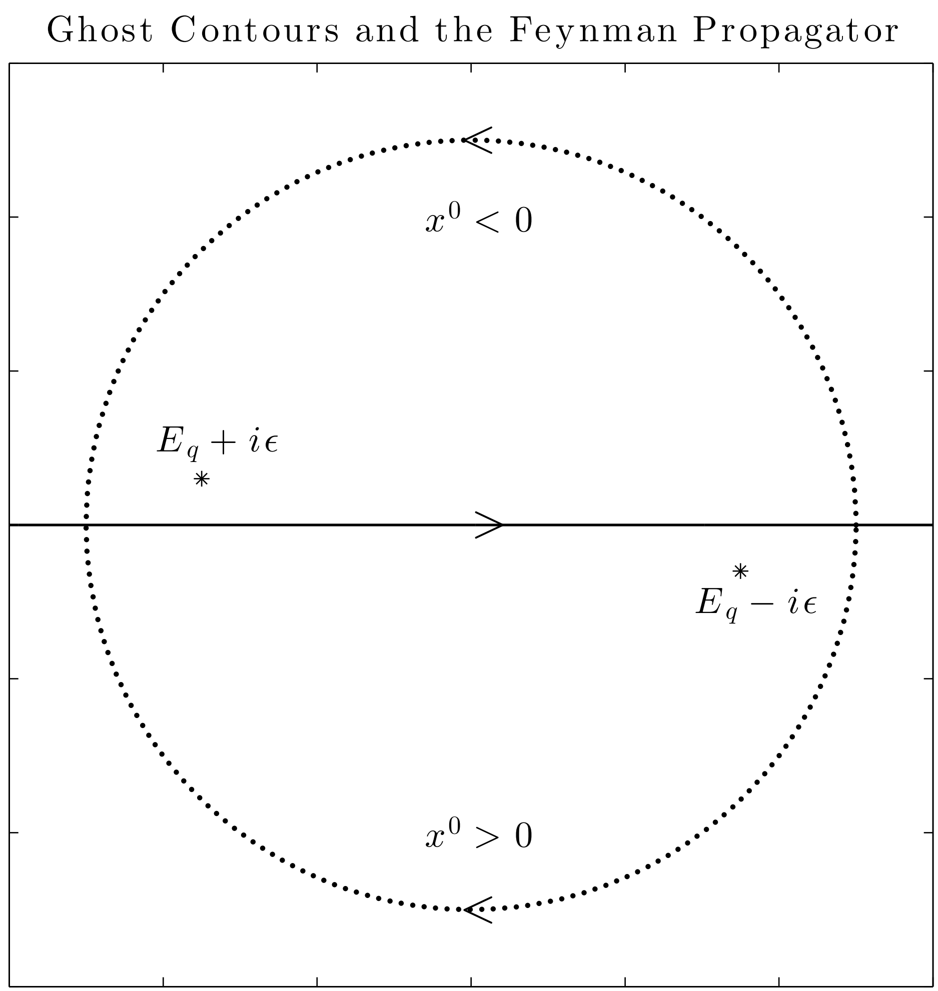

Figure 1: In equation (46), the function $f(q^0)$ has poles at $\pm(E_{\boldsymbol{q}} - i\epsilon)$, and the function $\exp(-iq^0x^0)$ is exponentially suppressed in the lower half plane if $x^0 > 0$ and in the upper half plane if $x^0 < 0$. So we can add a ghost contour (dots) in the LHP if $x^0 > 0$ and in the UHP if $x^0 < 0$.

has poles at $E_{\boldsymbol{q}} - i\epsilon$ and at $-E_{\boldsymbol{q}} + i\epsilon$. When $x^0 > 0$, we can add a ghost contour that goes clockwise around the lower half-plane and get

$$I(\boldsymbol{q}) = i e^{-iE_{\boldsymbol{q}} x^0} \, \frac{1}{2E_{\boldsymbol{q}}} \qquad x^0 > 0. \tag{47}$$

When $x^0 < 0$, our ghost contour goes counterclockwise around the upper half-plane, and we get

$$I(\boldsymbol{q}) = i e^{iE_{\boldsymbol{q}} x^0} \, \frac{1}{2E_{\boldsymbol{q}}} \qquad x^0 < 0. \tag{48}$$

Using the step function $\theta(x) = (x + |x|)/2$, we combine (47) and (48)

$$-iI(\boldsymbol{q}) = \frac{1}{2E_{\boldsymbol{q}}} \left[\theta(x^0)\, e^{-iE_{\boldsymbol{q}} x^0} + \theta(-x^0)\, e^{iE_{\boldsymbol{q}} x^0} \right]. \tag{49}$$

In terms of the Lorentz-invariant function

$$\Delta_+(x) = \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_{\boldsymbol{q}}} \, \exp[i(\boldsymbol{q}\cdot\boldsymbol{x} - E_{\boldsymbol{q}} x^0)] \tag{50}$$

and with a factor of $-i$, Feynman's propagator (43) is

$$-i\Delta_F(x) = \theta(x^0)\, \Delta_+(x) + \theta(-x^0)\, \Delta_+(\boldsymbol{x}, -x^0). \tag{51}$$

The integral (50) defining $\Delta_+(x)$ is insensitive to the sign of $\boldsymbol{q}$, and so

$$\begin{aligned} \Delta_+(-x) &= \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_{\boldsymbol{q}}} \, \exp[i(-\boldsymbol{q}\cdot\boldsymbol{x} + E_{\boldsymbol{q}} x^0)] \\ &= \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_{\boldsymbol{q}}} \, \exp[i(\boldsymbol{q}\cdot\boldsymbol{x} + E_{\boldsymbol{q}} x^0)] = \Delta_+(\boldsymbol{x}, -x^0). \end{aligned} \tag{52}$$

Thus we arrive at the standard form of the Feynman propagator

$$-i\Delta_F(x) = \theta(x^0)\, \Delta_+(x) + \theta(-x^0)\, \Delta_+(-x). \tag{53}$$

We now can see why the Feynman propagator is the mean value in the vacuum of the **time-ordered product** of the fields $\phi(x)$ and $\phi(y)$

$$\mathcal{T}\{\phi(x)\phi(y)\} \equiv \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x). \tag{54}$$

The operators $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ respectively annihilate the vacuum ket $a(\boldsymbol{p})|0\rangle = 0$ and bra $\langle 0|a^\dagger(\boldsymbol{p}) = 0$, and so by (35 & 36) do the positive- and negative-frequency parts of the field $\phi^+(z)|0\rangle = 0$ and $\langle 0|\phi^-(z) = 0$. Thus the mean value in the vacuum of the time-ordered product is

$$\begin{aligned}\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle &= \langle 0|\theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)|0\rangle \\ &= \langle 0|\theta(x^0 - y^0)\phi^+(x)\phi^-(y) + \theta(y^0 - x^0)\phi^+(y)\phi^-(x)|0\rangle \\ &= \langle 0|\theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] + \theta(y^0 - x^0)[\phi^+(y), \phi^-(x)]|0\rangle. \end{aligned} \tag{55}$$

But by (37), these commutators are $\Delta_+(x-y)$ and $\Delta_+(y-x)$. Thus the mean value in the vacuum of the time-ordered product

$$\begin{aligned}\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle &= \theta(x^0 - y^0)\Delta_+(x-y) + \theta(y^0 - x^0)\Delta_+(y-x) \\ &= -i\Delta_F(x-y)\end{aligned} \tag{56}$$

is the Feynman propagator (51) multiplied by $-i$ in accord with our path-integral result (90).

We can perform the same manipulations on the electromagnetic field. With each index $\lambda = \pm$, we associate the polarization vectors

$$\boldsymbol{\epsilon}_\pm(\boldsymbol{k}) = \frac{1}{\sqrt{2}} R(\boldsymbol{k}) \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \tag{57}$$

in which $R(\boldsymbol{k})$ is a standard 3×3 matrix that rotates the vector $k\hat{z}$ into $\boldsymbol{k}$. In terms of these vectors and the photon annihilation $a_\lambda(\boldsymbol{k})$ and creation $a^\dagger_\lambda(\boldsymbol{k})$ operators which satisfy the commutation relations

$$\begin{aligned}[a_\lambda(\boldsymbol{k}), a^\dagger_{\lambda'}(\boldsymbol{k}')] &= \delta_{\lambda,\lambda'}\,\delta^3(\boldsymbol{k} - \boldsymbol{k}') \\ [a_\lambda(\boldsymbol{k}), a_{\lambda'}(\boldsymbol{k}')] &= 0 = [a^\dagger_\lambda(\boldsymbol{k}), a^\dagger_{\lambda'}(\boldsymbol{k}')]\end{aligned} \tag{58}$$

the field A_i is

$$\boldsymbol{A}(x) = \sum_{\lambda=\pm} \int \left[\boldsymbol{\epsilon}_\lambda(\boldsymbol{k}) a_\lambda(\boldsymbol{k}) e^{ikx} + \boldsymbol{\epsilon}^*_\lambda(\boldsymbol{k}) a^\dagger_\lambda(\boldsymbol{k}) e^{-ikx} \right] \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \tag{59}$$

in which $kx = \boldsymbol{k} \cdot \boldsymbol{x} - k^0 t$. You showed in your homework that this field is transverse, that is, that it satisfies the Coulomb-gauge condition $\boldsymbol{\nabla} \cdot \boldsymbol{A}(x) = 0$ because $\boldsymbol{k} \cdot \boldsymbol{\epsilon}_\lambda(\boldsymbol{k}) = 0$.

The sum over polarizations is

$$M^{ij}(\boldsymbol{k}) \equiv \sum_{\lambda=\pm} \epsilon^i_\lambda(\boldsymbol{k}) \epsilon^{*j}_\lambda(\boldsymbol{k}) = \delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2} \tag{60}$$

because M is a 3×3 hermitian matrix of trace 2 that satisfies

$$\sum_{i=1}^3 k^i M^{ij}(\boldsymbol{k}) = 0 = \sum_{j=1}^3 M^{ij}(\boldsymbol{k}) k^j. \tag{61}$$

More directly, we have in matrix notation

$$\begin{aligned} M(\boldsymbol{k}) &\equiv \sum_{\lambda=\pm} \epsilon_\lambda(\boldsymbol{k}) \epsilon^*_\lambda(\boldsymbol{k}) = \frac{1}{2} \sum_{\lambda=\pm} R(\boldsymbol{k}) \begin{pmatrix} 1 \\ \lambda i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -\lambda i & 0 \end{pmatrix} R^{-1}(\boldsymbol{k}) = \frac{1}{2} R(\boldsymbol{k}) \sum_{\lambda=\pm} \begin{pmatrix} 1 & -\lambda i & 0 \\ \lambda i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} R^{-1}(\boldsymbol{k}) \\ &= R(\boldsymbol{k}) \left(I - \hat{z} \hat{z}^{\mathsf{T}} \right) R^{-1}(\boldsymbol{k}) = \left(I - \hat{\boldsymbol{k}} \hat{\boldsymbol{k}}^{\mathsf{T}} \right) \end{aligned} \tag{62}$$

which is (60).

It follows from this relation that

$$[A^i(t,\boldsymbol{x}), A^j(t,\boldsymbol{y})] = 0 = [\dot{A}^i(t,\boldsymbol{x}), \dot{A}^j(t,\boldsymbol{y})] \tag{63}$$

which I leave to you as a homework problem, and that

$$[A^i(t,\boldsymbol{x}), \Pi^j(t,\boldsymbol{y})] = [A^i(t,\boldsymbol{x}), \dot{A}^j(t,\boldsymbol{y})] = i\delta^{ij} \delta(\boldsymbol{x} - \boldsymbol{y}) + i \frac{\partial^2}{\partial x^i \partial x^j} \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \tag{64}$$

as I showed in class.

I also showed in class that the mean value in the vacuum of the time-ordered product of the (free) field $A_i(x)$ given by the integral (59) over d^3k is the **photon propagator**

$$\langle 0|\mathcal{T}\left[A_i(x)A_j(y)\right]|0\rangle = -i\Delta_{ij}(x-y) = \int \frac{d^3k}{(2\pi)^3 2k^0}\left(\delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2}\right)\left[e^{ik(x-y)}\theta(x-y) + e^{ik(y-x)}\theta(y-x)\right]. \quad (65)$$

In the Coulomb gauge, $A_0(x)$ is a dependent variable, and does not appear in the theory. So we can extend this last formula to the tensor relation

$$\langle 0|\mathcal{T}\left[A_a(x)A_b(y)\right]|0\rangle = -i\Delta_{ab}(x-y) = \int \frac{d^3k}{(2\pi)^3 2k^0} P_{ab}(k)\left[e^{ik(x-y)}\theta(x-y) + e^{ik(y-x)}\theta(y-x)\right] \quad (66)$$

in which

$$P_{0b} = P_{a0} = P_{00} = 0 \quad \text{while} \quad P_{ij}(k) = \delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2}. \quad (67)$$

By reverse-engineering our discussion of the Feynman propagator, we can write the photon propagator as

$$\langle 0|\mathcal{T}\left[A_a(x)A_b(y)\right]|0\rangle = -i\Delta_{ab}(x-y) = -i\int \frac{P_{ab}(k)}{k^2 - i\epsilon}\, e^{ik(x-y)}\, \frac{d^4k}{(2\pi)^4}. \quad (68)$$

One can rewrite P_{ab} as

$$P_{ab}(\boldsymbol{k}) = \eta_{ab} + \frac{k^0 k_a n_b + k^0 k_b n_a - k_a k_b + k^2 n_a n_b}{|\boldsymbol{k}|^2} \quad (69)$$

in which $n^a = (1,0,0,0)$ is a fixed time-like vector and $k^2 = \boldsymbol{k}^2 - (k^0)^2$, but k^0 is *entirely arbitrary*. In perturbation theory, the photon propagator will occur between conserved currents $\partial_a j^a(x) = 0$. In momentum space, current conservation is $k^a j^a(k) = 0$. So in effect

$$j^a(k)P_{ab}(\boldsymbol{k})j^b(k) = j^a(k)\left\{\eta_{ab} + \frac{k^0 k_a n_b + k^0 k_b n_a - k_a k_b + k^2 n_a n_b}{|\boldsymbol{k}|^2}\right\} j^b(k) = j^a(k)\eta_{ab}j^b(k) + j^0(k)\frac{k^2}{|\boldsymbol{k}|^2}j^0(k). \quad (70)$$

In the second term k^2 cancels the k^2 in the denominator $1/(k^2 - i\epsilon)$ and gives us just the Coulomb energy

$$-\frac{1}{2}\int d^3x d^3y \frac{j^0(t,\boldsymbol{x})j^0(t,\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}. \tag{71}$$

So we are left with in effect

$$\langle 0|\mathcal{T}\left[A_a(x)A_b(y)\right]|0\rangle^{eff} = -i\Delta^{eff}_{ab}(x-y) = -i\int \frac{\eta_{ab}}{k^2 - i\epsilon}\, e^{ik(x-y)}\, \frac{d^4k}{(2\pi)^4} \tag{72}$$

which is the Feynman-gauge propagator (16.177). So we get to the same nice photon propagator both from path integrals and from the operator formalism.

Let's see if we can get this same expression from the gauge-fixed action

$$S_\alpha = \int -\frac{1}{4}F_{ab}\,F^{ab} - \frac{1}{2}(\partial_b A^b)^2 + A^b j_b + \mathcal{L}_m \; d^4x \tag{73}$$

in which I set $\alpha = 1$ and

$$F_{ab} = \frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b} = -F_{ba} \tag{74}$$

is the Faraday tensor. The part of the action density that is quadratic in the gauge field is

$$\begin{aligned}
S_q &= -\int \frac{1}{4}F_{ab}\,F^{ab} + \frac{1}{2}(\partial_b A^b)^2\, d^4x = -\int \frac{1}{4}\left(\frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b}\right)\left(\frac{\partial A^b}{\partial x_a} - \frac{\partial A^a}{\partial x_b}\right) + \frac{1}{2}(\partial_b A^b)^2\, d^4x \\
&= -\int \frac{1}{4}\left(\partial_a A_b - \partial_b A_a\right)\left(\partial^a A^b - \partial^b A^a\right) + \frac{1}{2}(\partial_b A^b)^2\, d^4x \\
&= -\int \frac{1}{4}\left[\partial_a A_b \partial^a A^b + \partial_b A_a \partial^b A^a - \partial_b A_a \partial^a A^b - \partial_a A_b \partial^b A^a + \frac{1}{2}(\partial_b A^b)^2\right]\, d^4x \\
&= -\frac{1}{2}\int \left[\partial_a A_b \partial^a A^b - \partial_a A_b \partial^b A^a + (\partial_b A^b)^2\right]\, d^4x \\
&= \frac{1}{2}\int \left[A_b \partial_a \partial^a A^b - A_b \partial_a \partial^b A^a + A^a \partial_a \partial_b A^b\right]\, d^4x = \frac{1}{2}\int A_b \partial_a \partial^a A^b\, d^4x
\end{aligned} \tag{75}$$

in which I integrated by parts.

The part of the action involving A is

$$S[A,j] = \int \left[\tfrac{1}{2} A_b \partial_a \partial^a A^b + A^b j_b\right] \, d^4x. \tag{76}$$

We write the electromagnetic field in terms of its Fourier transform

$$A_b(x) = \int e^{ipx} \tilde{A}_b(p) \, \frac{d^4p}{(2\pi)^4} = \int e^{-ipx} \tilde{A}^*_b(p) \, \frac{d^4p}{(2\pi)^4} \tag{77}$$

and also write the conserved current in terms of its Fourier transform

$$j_b(x) = \int e^{ipx} \tilde{j}_b(p) \, \frac{d^4p}{(2\pi)^4}. \tag{78}$$

So the action $S[A,j]$ is

$$\begin{aligned}
S[A,j] &= \tfrac{1}{2}\int e^{-ipx}\tilde{A}_b^*(p)\,\frac{d^4p}{(2\pi)^4}\,\partial_a\partial^a e^{ip'x}\tilde{A}^b(p')\,\frac{d^4p'}{(2\pi)^4}d^4x + \tfrac{1}{2}\int e^{-ipx}\tilde{A}^{b*}(p)\,\frac{d^4p}{(2\pi)^4}e^{ip'x}\tilde{j}_b(p')\,\frac{d^4p'}{(2\pi)^4}\,d^4x \\
&+ \tfrac{1}{2}\int e^{ipx}\tilde{A}^b(p)\,\frac{d^4p}{(2\pi)^4}e^{-ip'x}\tilde{j}_b^*(p')\,\frac{d^4p'}{(2\pi)^4}\,d^4x \\
&= \tfrac{1}{2}\int e^{-ipx}\tilde{A}_b^*(p)\,\frac{d^4p}{(2\pi)^4}\,(-p'^2)e^{ip'x}\tilde{A}^b(p')\,\frac{d^4p'}{(2\pi)^4}d^4x + \tfrac{1}{2}\int e^{-ipx}\tilde{A}^{b*}(p)\,\frac{d^4p}{(2\pi)^4}e^{ip'x}\tilde{j}_b(p')\,\frac{d^4p'}{(2\pi)^4}\,d^4x \\
&+ \tfrac{1}{2}\int e^{ipx}\tilde{A}^b(p)\,\frac{d^4p}{(2\pi)^4}e^{-ip'x}\tilde{j}_b^*(p')\,\frac{d^4p'}{(2\pi)^4}\,d^4x \\
&= \tfrac{1}{2}\int \tilde{A}_b^*(p)\,\frac{d^4p}{(2\pi)^4}\,(-p'^2)\tilde{A}^b(p')\,d^4p'\delta(p'-p) + \tfrac{1}{2}\int \tilde{A}^{b*}(p)\,\frac{d^4p}{(2\pi)^4}\tilde{j}_b(p')\,d^4p'\delta(p'-p) \\
&+ \tfrac{1}{2}\int \tilde{A}^b(p)\,\frac{d^4p}{(2\pi)^4}\tilde{j}_b^*(p')\,d^4p'\delta(p'-p) \\
&= \tfrac{1}{2}\int \left[\tilde{A}_b^*(p)(-p^2)\tilde{A}^b(p) + \tilde{A}^{b*}(p)\tilde{j}_b(p) + \tilde{A}^b(p)\tilde{j}_b^*(p)\right]\,\frac{d^4p}{(2\pi)^4} \\
&= \tfrac{1}{2}\int \left[\tilde{A}^{a*}(p)(-\eta_{ab}p^2)\tilde{A}^b(p) + \tilde{A}^{b*}(p)\tilde{j}_b(p) + \tilde{A}^b(p)\tilde{j}_b^*(p)\right]\,\frac{d^4p}{(2\pi)^4}.
\end{aligned} \tag{79}$$

So we let

$$\tilde{A}^b = \hat{A}^b + \frac{\eta^{bc}}{p^2}\tilde{j}_c \tag{80}$$

and substitute into (79)

$$\begin{aligned}
S[A,j] &= \tfrac{1}{2}\int \left\{\left[\hat{A}^{a*} + \frac{\eta^{ad}}{p^2}\tilde{j}_d^*\right](-\eta_{ab}\,p^2)\left[\hat{A}^b + \frac{\eta^{bc}}{p^2}\tilde{j}_c\right] + \left[\hat{A}^{b*} + \frac{\eta^{bc}}{p^2}\tilde{j}_c^*\right]\tilde{j}_b + \left[\hat{A}^b + \frac{\eta^{bc}}{p^2}\tilde{j}_c\right]\tilde{j}_b^*\right\}\frac{d^4p}{(2\pi)^4} \\
&= \tfrac{1}{2}\int \left[-p^2\,\hat{A}_a^*(p)\hat{A}^a(p) + \frac{\tilde{j}_a^*(p)\,j^a(p)}{p^2}\right]\frac{d^4p}{(2\pi)^4}.
\end{aligned} \tag{81}$$

As in our treatment of path integrals for scalar fields, we now add in the $i\epsilon$ terms and get

$$S[A, j, \epsilon] = \frac{1}{2} \int \left[-(p^2 - i\epsilon)\, \hat{A}^*_a(p) \hat{A}^a(p) + \frac{\tilde{j}^*_a(p)\, j^a(p)}{(p^2 - i\epsilon)} \right] \frac{d^4p}{(2\pi)^4} = S[\hat{A}, 0, \epsilon] + \frac{1}{2} \int \frac{\tilde{j}^*_a(p)\, j^a(p)}{(p^2 - i\epsilon)} \frac{d^4p}{(2\pi)^4}. \tag{82}$$

The mean value of a time-ordered product of operators in the ground state $|0\rangle$ of the free theory is

$$\langle 0| \mathcal{T} \left[\mathcal{O}_1 \dots \mathcal{O}_n \right] |0\rangle = \frac{\displaystyle\int \mathcal{O}_1 \dots \mathcal{O}_n \, e^{iS[A,\psi,\epsilon]} \, DA \, D\psi}{\displaystyle\int e^{iS[A,\psi,\epsilon]} \, DA \, D\psi} \tag{83}$$

in which ψ represents the charged fields.

We can use these formulas (82 & 83) to express the mean value in the vacuum $|0\rangle$ of the time-ordered exponential of a space-time integral of $j_b(x) A^b(x)$, in which j is a classical (c-number, external) current, as the ratio

$$\begin{aligned} Z_0[j] \equiv \langle 0| \mathcal{T} \left\{ \exp\left[i \int j_b(x) \; A^b(x) \, d^4x \right] \right\} |0\rangle &= \frac{\displaystyle\int \exp\left[i \int j_b(x) \; A^b(x) \, d^4x \right] e^{iS[A,\epsilon]} \, DA}{\displaystyle\int e^{iS[A,\epsilon]} \, DA} \\ = \frac{\displaystyle\int e^{iS[A,j,\epsilon]} \, DA}{\displaystyle\int e^{iS[A,\epsilon]} \, DA} = \frac{\displaystyle\int e^{iS[\hat{A},0,\epsilon]} \, D\hat{A}}{\displaystyle\int e^{iS[A,\epsilon]} \, DA} \exp\left[\frac{i}{2} \int \frac{\tilde{j}^*_a(p) \, \tilde{j}^a(p)}{p^2 - i\epsilon} \frac{d^4p}{(2\pi)^4} \right]. \end{aligned} \tag{84}$$

Since the transformation (80) between A or $\tilde{A}$ and $\hat{A}$ is linear, their differentials are the same, $DA = D\hat{A}$, and so the ratio of path integrals is unity:

$$Z_0[j] = \exp\left[\frac{i}{2} \int \frac{\tilde{j}^*_a(p) \, \tilde{j}^a(p)}{p^2 - i\epsilon} \frac{d^4p}{(2\pi)^4} \right]. \tag{85}$$

Going back to position space, one finds (homework problem)

$$Z_0[j] = \langle 0|\,\mathcal{T}\left\{\exp\left[i\int j^b(x)\,A_b(x)\,d^4x\right]\right\}|0\rangle = \exp\left[\frac{i}{2}\int j(x)^a\,\Delta_{ab}(x-x')\,j^b(x')\,d^4x\,d^4x'\right] \tag{86}$$

in which $\Delta_{ab}(x-x')$ is Feynman's **propagator** (43) for the case of a massless vector field

$$\Delta_{ab}(x-x') = \int e^{ip(x-x')}\frac{\eta_{ab}}{p^2-i\epsilon}\,\frac{d^4p}{(2\pi)^4}. \tag{87}$$

The functional derivative of $Z_0[j]$, defined by (84), is

$$\frac{1}{i}\frac{\delta Z_0[j]}{\delta j^a(x)} = \langle 0|\,\mathcal{T}\left[A_a(x)\exp\left(i\int j^b(x')A_b(x')d^4x'\right)\right]|0\rangle \tag{88}$$

while that of equation (86) is

$$\frac{1}{i}\frac{\delta Z_0[j]}{\delta j^a(x)} = Z_0[j]\int \Delta_{ab}(x-x')\,A^b(x')\,d^4x'. \tag{89}$$

Thus the second functional derivative of $Z_0[j]$ evaluated at $j=0$ gives

$$\langle 0|\,\mathcal{T}\left[A_a(x)A_b(x')\right]|0\rangle = \frac{1}{i^2}\frac{\delta^2 Z_0[j]}{\delta j^a(x)\delta j^b(x')}\bigg|_{j=0} = -i\,\Delta_{ab}(x-x') = -i\int e^{ip(x-x')}\frac{\eta_{ab}}{p^2-i\epsilon}\,\frac{d^4p}{(2\pi)^4}. \tag{90}$$

which is what we got from SW's physical argument which led to the same expression (72).