17

The Renormalization Group

17.1 The Renormalization Group in Quantum Field Theory

Most quantum field theories are non-linear with infinitely many degrees of freedom, and because they describe point particles, they are rife with infinities. But short-distance effects, probably the finite sizes of the fundamental constituents of matter, mitigate these infinities so that we can cope with them consistently without knowing what happens at very short distances and very high energies. This procedure is called **renormalization**.

For instance, in the theory described by the Lagrange density

$$\mathcal{L} = -\frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{24}\phi^4 \tag{17.1}$$

we can cut off divergent integrals at some high energy Λ. The amplitude for the elastic scattering of two bosons of initial four-momenta p_1 and p_2 into two of final momenta p_1' and p_2' to one-loop order (Weinberg, 1996, chap. 18) then is proportional to (Zee, 2010, chaps. III & VI)

$$A = g - \frac{g^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^6}{stu}\right) - i\pi + 3\right] \tag{17.2}$$

as long as the absolute values of the Mandelstam variables $s = -(p_1+p_2)^2$, $t = -(p_1-p_1')^2$, and $u = -(p_1-p_2')^2$, which satisfy $stu > 0$ and $s+t+u = 4m^2$, are all much larger than m^2 (Stanley Mandelstam, 1928–). We define the **physical coupling constant** g_μ, as opposed to the **bare** one g that comes with $\mathcal{L}$, to be the real part of the amplitude A at $s = -t = -u = \mu^2$

$$g_\mu = g - \frac{3g^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) + 1\right]. \tag{17.3}$$

Thus the bare coupling constant is $g = g_\mu + 3g^2\left[\ln(\Lambda^2/\mu^2) + 1\right]$, and using this formula, we can write our expression (17.2) for the amplitude A in a form in which the cutoff Λ no longer appears

$$A = g_\mu - \frac{g^2}{32\pi^2}\left[\ln\left(\frac{\mu^6}{stu}\right) - i\pi\right]. \tag{17.4}$$

This is the magic of renormalization.

The physical coupling "constant" g_μ is the right coupling at energy μ because when all the Mandelstam variables are near the renormalization point $stu = \mu^6$, the one-loop correction is tiny, and $A \approx g_\mu$.

How does the physical coupling g_μ depend upon the energy μ? The amplitude A must be independent of the renormalization energy μ, and so

$$\frac{dA}{d\mu} = \frac{dg_\mu}{d\mu} - \frac{g^2}{32\pi^2}\frac{6}{\mu} = 0 \tag{17.5}$$

which is a version of the **Callan-Symanzik equation**.

We assume that when the cutoff Λ is big but finite, the bare and **running** coupling constants g and g_μ are so tiny that they differ by terms of order g^2 or g_μ^2. Then to lowest order in g and g_μ, we can replace g^2 by g_μ^2 in (17.5) and arrive at the simple differential equation

$$\mu\,\frac{dg_\mu}{d\mu} \equiv \beta(g_\mu) = \frac{3\,g_\mu^2}{16\pi^2} \tag{17.6}$$

which we can integrate

$$\ln\frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_E} \frac{dg_\mu}{\beta(g_\mu)} = \frac{16\pi^2}{3}\int_{g_M}^{g_E}\frac{dg_\mu}{g_\mu^2} = \frac{16\pi^2}{3}\left(\frac{1}{g_M} - \frac{1}{g_E}\right) \tag{17.7}$$

to find the running physical coupling constant g_μ at energy $\mu = E$

$$g_E = \frac{g_M}{1 - 3\,g_M\,\ln(E/M)/16\pi^2}. \tag{17.8}$$

As the energy $E = \sqrt{s}$ rises above M, while staying below the singular value $E = M\exp(16\pi^2/3g_M)$, the running coupling g_E slowly increases. And so does the scattering amplitude, $A \approx g_E$.

Example 17.1 (Quantum Electrodynamics) Vacuum polarization makes the amplitude for the scattering of two electrons proportional to (Weinberg, 1995, chap. 11)

$$A(q^2) = e^2\,\left[1 + \pi(q^2)\right] \tag{17.9}$$

rather than to e^2. Here e is the renormalized charge, $q = p_1' - p_1$ is the four-momentum transferred to the first electron, and

$$\pi(q^2) = \frac{e^2}{2\pi^2} \int_0^1 x(1-x) \ln\left[1 + \frac{q^2 x(1-x)}{m^2}\right] dx \tag{17.10}$$

represents the polarization of the vacuum. We define the square of the running coupling constant e_μ^2 to be the amplitude (17.9) at $q^2 = \mu^2$

$$e_\mu^2 = A(\mu^2) = e^2 \left[1 + \pi(\mu^2)\right]. \tag{17.11}$$

For $\mu^2 \gg m^2$, the vacuum polarization term $\pi(\mu^2)$ is (exercise 17.1)

$$\pi(\mu^2) \approx \frac{e^2}{6\pi^2}\left[\ln\frac{\mu}{m} - \frac{5}{6}\right]. \tag{17.12}$$

The amplitude (17.9) then is

$$A(q^2) = e_\mu^2 \frac{1+\pi(q^2)}{1+\pi(\mu^2)} \tag{17.13}$$

and since it must be independent of μ, we have

$$0 = \frac{d}{d\mu}\frac{A(q^2)}{1+\pi(q^2)} = \frac{d}{d\mu}\frac{e_\mu^2}{1+\pi(\mu^2)} \approx \frac{d}{d\mu}\left\{e_\mu^2 \left[1 - \pi(\mu^2)\right]\right\}. \tag{17.14}$$

So we find

$$0 = 2e_\mu\left(\frac{de_\mu}{d\mu}\right)\left[1-\pi(\mu^2)\right] - e_\mu^2 \frac{d\pi(\mu^2)}{d\mu} = 2e_\mu\left(\frac{de_\mu}{d\mu}\right)\left[1-\pi(\mu^2)\right] - e_\mu^2 \frac{e^2}{6\pi^2\mu}. \tag{17.15}$$

Thus since by (17.10 & 17.11) $\pi(\mu^2) = \mathcal{O}(e^2)$ and $e_\mu^2 = e^2 + \mathcal{O}(e^4)$, we find to lowest order in e_μ

$$\mu\frac{de_\mu}{d\mu} \equiv \beta(e_\mu) = \frac{e_\mu^3}{12\pi^2}. \tag{17.16}$$

We can integrate this differential equation

$$\ln\frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{e_M}^{e_E} \frac{de_\mu}{\beta(e_\mu)} = 12\pi^2 \int_{e_M}^{e_E} \frac{de_\mu}{e_\mu^3} = 6\pi^2\left(\frac{1}{e_M^2} - \frac{1}{e_E^2}\right) \tag{17.17}$$

and so get for the running coupling constant the formula

$$e_E^2 = \frac{e_M^2}{1 - e_M^2 \ln(E/M)/6\pi^2} \tag{17.18}$$

which shows that it slowly increases with the energy E. Thus, the fine-structure constant $e_\mu^2/4\pi$ rises from $\alpha = 1/137.036$ at m_e to

$$\frac{e^2(45.5\text{GeV})}{4\pi} = \frac{\alpha}{1 - 2\alpha\ln(45.5/0.00051)/3\pi} = \frac{1}{134.6} \tag{17.19}$$

at $\sqrt{s} = 91$ GeV. When all light charged particles are included, one finds that the fine-structure constant rises to $\alpha = 1/128.87$ at $E = 91$ GeV. □

Example 17.2 (Quantum Chromodynamics) Because of the cubic interaction of the gauge fields of a nonabelian gauge theory, the running coupling constant g_μ can slowly decrease with rising energy. If the gauge group is $SU(3)$, then due to this cubic interaction and that of the ghost fields (16.255), the running coupling constant g_μ is to order g_M^3

$$g_\mu = g_M \left[1 - \frac{11 g_M^2}{16\pi^2} \ln\left(\frac{\mu}{M}\right) \right]. \tag{17.20}$$

It differs from g_M only by terms of order g_M^3 and so satisfies the differential equation

$$\mu \frac{dg_\mu}{d\mu} \equiv \beta(g_\mu) = -\frac{11 g_M^3}{16\pi^2} \approx -\frac{11 g_\mu^3}{16\pi^2} \tag{17.21}$$

in which the beta-function is **negative**. Integrating

$$\ln \frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_G} \frac{dg_\mu}{\beta(g_\mu)} = -\frac{16\pi^2}{11} \int_{g_M}^{g_E} \frac{dg_\mu}{g_\mu^3} = \frac{8\pi^2}{11} \left(\frac{1}{g_M^2} - \frac{1}{g_E^2} \right) \tag{17.22}$$

we find

$$g_E^2 = g_M^2 \left[1 + \frac{11 g_M^2}{8\pi^2} \ln \frac{E}{M} \right]^{-1} \tag{17.23}$$

which shows that as the energy E of a scattering process increases, the running coupling slowly **decreases**, going to zero at infinite energy, an effect called **asymptotic freedom**.

If the gauge group is $SU(N)$, and the theory has n_f flavors of quarks with masses below μ, then the beta function is

$$\beta(g_\mu) = -\frac{g_\mu^3}{4\pi^2} \left(\frac{11N}{12} - \frac{n_f}{6} \right) \tag{17.24}$$

which remains negative as long as $n_f < 11N/2$. Using this beta-function with $N = 3$ and again integrating, we get instead of (17.23)

$$g_E^2 = g_M^2 \left[1 + \frac{(11 - 2n_f/3) g_M^2}{16\pi^2} \ln \frac{E^2}{M^2} \right]^{-1}. \tag{17.25}$$

So with

$$\Lambda^2 \equiv M^2 \exp\left(-\frac{16\pi^2}{(11 - 2n_f/3) g_M^2} \right) \tag{17.26}$$

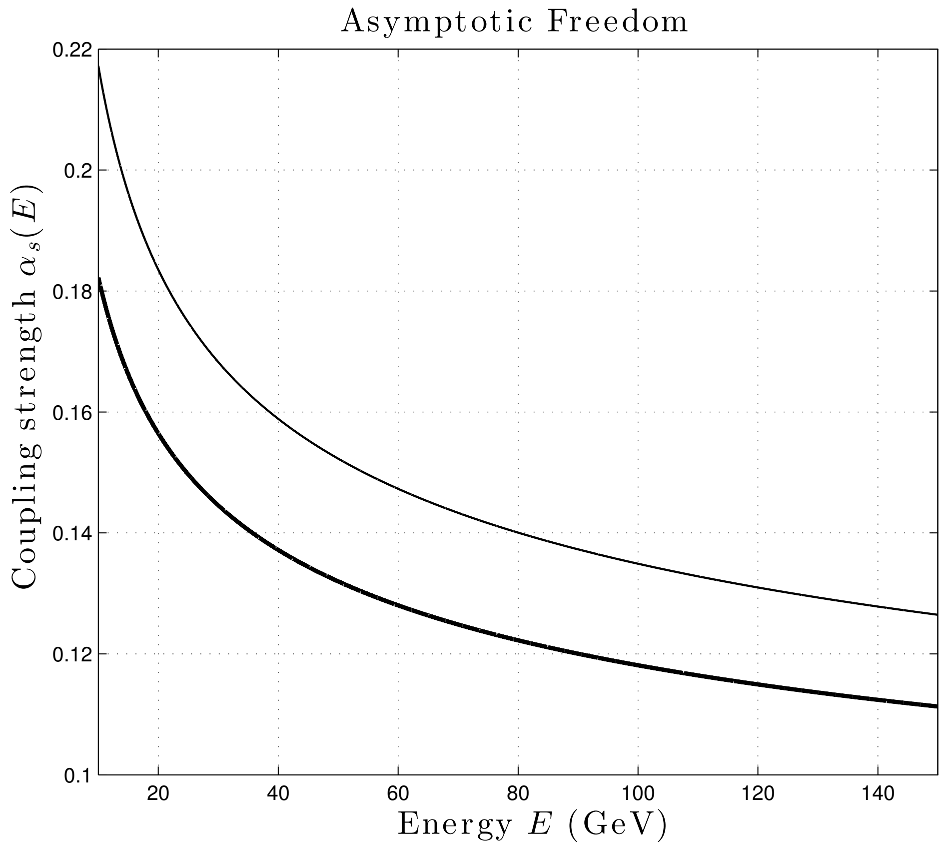

Figure 17.1 The strong-structure constant $\alpha_s(E)$ as given by the one-loop formula (17.27) (thin curve) and by a three-loop formula (thick curve) with $\Lambda = 230$ MeV and $n_f = 5$ is plotted for $m_b \ll E \ll m_t$.

we find (exercise 17.2)

$$\alpha_s(E) \equiv \frac{g^2(E)}{4\pi} = \frac{12\pi}{(33 - 2n_f)\ln(E^2/\Lambda^2)}. \tag{17.27}$$

This formula expresses the dimensionless strong-structure constant $\alpha_s(E)$ appropriate to energy E in terms of a parameter Λ that has the dimension of energy. Some call this **dimensional transmutation**. For $\Lambda = 230$ MeV and $n_f = 5$, Fig. 17.1 displays $\alpha_s(E)$ in the range $4.19 = m_b \ll E \ll m_t = 172$ GeV as given by the one-loop formula (17.27) (thin curve) and a three-loop formula (Weinberg, 1996, p. 156) (thick curve). □

17.2 The Renormalization Group in Lattice Field Theory

Let us consider a quantum field theory on a lattice (Gattringer and Lang, 2010, chap. 3) in which the strength of the nonlinear interactions depends upon a single dimensionless coupling constant g. The spacing a of the lattice regulates the infinities, which return as $a \to 0$. The value of an observable P computed on this lattice will depend upon the lattice spacing a and on the coupling constant g, and so will be a function $P(a, g)$ of these two parameters. The *right* value of the coupling constant is the value that makes the result of the computation be as close as possible to the physical value P. So the correct coupling constant is not a constant at all, but rather a function $g(a)$ that varies with the lattice spacing or cutoff a. Thus as we vary the lattice spacing and go to the continuum limit $a \to 0$, we must adjust the coupling function $g(a)$ so that what we compute, $P(a, g(a))$, is equal to the physical value P. That is, $g(a)$ must vary with a so as to keep $P(a, g(a)) = P$. But then $P(a, g(a))$ must remain constant as a varies, so

$$\frac{dP(a, g(a))}{da} = 0. \tag{17.28}$$

Writing this condition as a dimensionless derivative

$$a\,\frac{dP(a, g(a))}{da} = \frac{da}{d\ln a}\,\frac{dP(a, g(a))}{da} = \frac{dP(a, g(a))}{d\ln a} = 0 \tag{17.29}$$

we arrive at the **Callan-Symanzik equation**

$$0 = \frac{dP(a, g(a))}{d\ln a} = \left(\frac{\partial}{\partial \ln a} + \frac{dg}{d\ln a}\,\frac{\partial}{\partial g}\right) P(a, g(a)). \tag{17.30}$$

The coefficient of the second partial derivative with a minus sign

$$\beta_L(g) \equiv -\frac{dg}{d\ln a} \tag{17.31}$$

is the lattice β-function. Since the lattice spacing a and the energy scale μ are inversely related, the lattice β-function differs from the continuum beta-function by a minus sign.

In $SU(N)$ gauge theory, the first two terms of the lattice β-function for small g are

$$\beta_L(g) = -\beta_0\, g^3 - \beta_1\, g^5 \tag{17.32}$$

where for n_f flavors of light quarks

$$\beta_0 = \frac{1}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}n_f\right)$$
$$\beta_1 = \frac{1}{(4\pi)^4}\left(\frac{34}{3}N^2 - \frac{10}{3}Nn_f - \frac{N^2-1}{N}n_f\right). \quad (17.33)$$

In quantum chromodynamics, $N = 3$.

Combining the definition (17.31) of the β-function with its expansion (17.32) for small g, one arrives at the differential equation

$$\frac{dg}{d\ln a} = \beta_0\, g^3 + \beta_1\, g^5 \quad (17.34)$$

which one may integrate

$$\int d\ln a = \ln a - \ln c = \int \frac{dg}{\beta_0 g^3 + \beta_1 g^5} = -\frac{1}{2\beta_0 g^2} + \frac{\beta_1}{2\beta_0^2}\ln\left(\frac{\beta_0 + \beta_1 g^2}{g^2}\right) \quad (17.35)$$

to find

$$a(g) = c\left(\frac{\beta_0 + \beta_1 g^2}{g^2}\right)^{\beta_1/2\beta_0^2} e^{-1/2\beta_0 g^2} \quad (17.36)$$

in which c is a constant of integration. The term $\beta_1 g^2$ is of higher order in g, and if one drops it and absorbs a factor of β_0^2 into a new constant of integration Λ, then one gets

$$a(g) = \frac{1}{\Lambda}\left(\beta_0 g^2\right)^{-\beta_1/2\beta_0^2} e^{-1/2\beta_0 g^2}. \quad (17.37)$$

As $g \to 0$, the lattice spacing $a(g)$ goes to zero *very fast* (as long as $n_f < 17$ for $N = 3$). The inverse of this relation (17.37) is

$$g(a) \approx \left[\beta_0 \ln(a^{-2}\Lambda^{-2}) + (\beta_1/\beta_0)\,\ln\left(\ln(a^{-2}\Lambda^{-2})\right)\right]^{-1/2}. \quad (17.38)$$

It shows that the coupling constant slowly goes to zero with a, which is a lattice version of **asymptotic freedom**. □

17.3 The Renormalization Group in Condensed-Matter Physics

The study of condensed matter is concerned mainly with properties that emerge in the bulk, such as the melting point, the boiling point, or the conductivity. So we want to see what happens to the physics when we increase the distance scale many orders of magnitude beyond the size a of an individual molecule or the distance between nearest neighbors.

As a simple example, let's consider a euclidean action in d dimensions

$$S = \int d^d x \left(\frac{1}{2} (\partial\phi)^2 + \sum_n g_n \phi^n \right) \tag{17.39}$$

in which $g_2\phi^2 \equiv m^2\phi^2/2$ is a mass term and $g_4\phi^4 \equiv \lambda\phi^4/24$ is a quartic self-interaction. In terms of an ultraviolet cutoff $\Lambda = 1/a$, we may define a partition function

$$Z(\Lambda) = \int_\Lambda e^{-S} D\phi \tag{17.40}$$

to be one in which the field

$$\phi(x) = \int_\Lambda e^{ikx} \phi(k) \, \frac{d^d k}{(2\pi)^d} \tag{17.41}$$

only has Fourier coefficients $\phi(k)$ with $k^2 < \Lambda^2$. Corresponding to each such field $\phi(x)$, we introduce a "stretched" field

$$\phi_L(x) = A(L) \, \phi(x/L) \quad \text{for} \quad L \ge 1 \tag{17.42}$$

in which $A(L)$ is a scale factor that we will use to keep the kinetic part of the action invariant. Since

$$\phi_L(x) = A(L) \, \phi(x/L) = A(L) \int_\Lambda \exp\left(i \frac{kx}{L} \right) \phi(k) \, \frac{d^d k}{(2\pi)^d} \tag{17.43}$$

the momenta of the stretched field are reduced by the factor $1/L$.

We may define a new partition function in which we integrate over the stretched fields $\phi_L(x)$

$$Z(\Lambda/L) = \int_{\Lambda/L} e^{-S} D\phi \equiv \int_\Lambda e^{-S} D\phi_L. \tag{17.44}$$

The kinetic action of a stretched field is

$$S_k = \int d^d x \, \frac{A^2(L)}{2} \left(\frac{\partial\phi(x/L)}{\partial x} \right)^2 = \int d^d(x/L) \, L^d \, \frac{A^2(L)}{2} \left(\frac{\partial\phi(x/L)}{L\partial x/L} \right)^2 \tag{17.45}$$

and so if we choose

$$A(L) = L^{-(d-2)/2} \tag{17.46}$$

then letting $x' = x/L$, we find that the kinetic action S_k is invariant

$$S_k = \int d^d x' \, \frac{1}{2} \left(\frac{\partial\phi(x')}{\partial x'} \right)^2 . \tag{17.47}$$

The full action of a stretched field is

$$S(\phi_L) = \int d^d x \left(\frac{1}{2}(\partial\phi)^2 + \sum_n g_{d,n}(L)\phi^n \right) \tag{17.48}$$

in which

$$g_{d,n}(L) = L^d A^n(L)\, g_n = L^{d-n(d-2)/2}\, g_{d,n}. \tag{17.49}$$

The beta-function

$$\beta(g_{d,n}) \equiv \frac{L}{g_{d,n}(L)} \frac{dg_{d,n}(L)}{dL} = d - n(d-2)/2 \tag{17.50}$$

is just the exponent of the coupling "constant" $g_{d,n}(L)$. If it is positive, then the coupling constant $g_{d,n}(L)$ gets stronger as $L \to \infty$; such couplings are called **relevant**. Couplings with vanishing exponents are insensitive to changes in L and are **marginal**. Those with negative exponents shrink with increasing L; they are **irrelevant**.

The coupling constant $g_{d,n,p}$ of a term with p derivatives and n powers of the field ϕ in a space of d dimensions varies as

$$g_{d,n,p}(L) = L^d\, A^n(L)\, L^{-p}\, g_{n,p} = L^{d-n(d-2)/2-p}\, g_{d,n,p}. \tag{17.51}$$

Example 17.3 (QCD) In quantum chromodynamics, there is a cubic term $g\, f_{abc}\, A_0^a A_i^b \partial_0 A_i^c$ which in effect looks like $g\, f_{abc}\, \phi_a \phi_b \dot{\phi}_c$. Is it relevant? Well, if we stretch space but not time, then the time derivative has no effect, and $d = 3$. So the cubic, $n = 3$, grows as $L^{3/2}$

$$g_{3,3,0}(L) = L^{d-n(d-2)/2}\, g_{3,3,0} = L^{3/2}\, g_{3,3,0}. \tag{17.52}$$

Since this cubic term drives asymptotic freedom, its strengthening as space is stretched by the dimensionless factor L may point to a qualitative explanation of confinement. For if $g_{3,3,0}(L)$ grows with distance as $L^{3/2}$, then $\alpha_s(L) = g_{3,3,0}^2(L)/4\pi$ grows as L^3, and so the strength $\alpha_s(Lr)/(Lr)^2$ of the force between two quarks separated by a distance Lr grows linearly with L

$$F(Lr) = \frac{\alpha_s(Lr)}{(Lr)^2} = \frac{L^3\, \alpha_s(r)}{(Lr)^2} = L\, \frac{\alpha_s(r)}{r^2} \tag{17.53}$$

which may be enough for quark confinement.

On the other hand, if we stretch both space and time, then the cubic $g_{4,3,1}(L)$ and quartic $g_{4,4,0}(L)$ couplings are marginal. □

Exercises

17.1 Show that for $\mu^2 \gg m^2$, the vacuum polarization term (17.10) reduces to (17.12). Hint: Use $\ln a\, b = \ln a + \ln b$ when integrating.

17.2 Show that by choosing the energy scale Λ according to (17.26), one can derive (17.27) from (17.25).

17.3 Show that if we stretch both space and time, then in the notation of (17.51), the cubic $g_{4,3,1}(L)$ and quartic $g_{4,4,0}(L)$ couplings are marginal, that is, are independent of L.