in which t_m is the time at the end of the matter era, and ρ_v is the density of dark energy, which while vastly less than the energy density ρ_i that drove inflation, currently amounts to 76% of the total energy density.

11.50 Yang-Mills Theory

The gauge transformation of an **abelian** gauge theory like electrodynamics multiplies a *single* charged field by a space-time-dependent *phase factor* $\phi'(x) = \exp(iq\theta(x))\,\phi(x)$. Yang and Mills generalized this gauge transformation to one that multiplies a *vector* ϕ of matter fields by a space-time dependent *unitary matrix* $U(x)$

$$\phi'_a(x) = \sum_{b=1}^{n} U_{ab}(x)\,\phi_b(x) \quad \text{or} \quad \phi'(x) = U(x)\,\phi(x) \tag{11.468}$$

and showed how to make the action of the theory invariant under such **non-abelian** gauge transformations. (The fields ϕ are scalars for simplicity.)

Since the matrix U is unitary, inner products like $\phi^\dagger(x)\,\phi(x)$ are automatically invariant

$$\Big(\phi^\dagger(x)\,\phi(x)\Big)' = \phi^\dagger(x)U^\dagger(x)U(x)\phi(x) = \phi^\dagger(x)\phi(x). \tag{11.469}$$

But inner products of derivatives $\partial^i\phi^\dagger\,\partial_i\phi$ are not invariant because the derivative acts on the matrix $U(x)$ as well as on the field $\phi(x)$.

Yang and Mills made derivatives $D_i\phi$ that transform like vectors

$$(D_i\phi)' = U\,D_i\phi. \tag{11.470}$$

To do so, they introduced **gauge-field matrices** A_i that play the role of the connections Γ_i in general relativity and set

$$D_i = \partial_i + A_i \tag{11.471}$$

in which A_i like ∂_i is anti-hermitian. They required that under the gauge transformation (11.468), the gauge-field matrix A_i transform to A'_i in such a way as to make the derivatives transform as in (11.470)

$$(D_i\phi)' = \left(\partial_i + A'_i\right)\phi' = \left(\partial_i + A'_i\right)U\phi = U\,D_i\phi = U\,(\partial_i + A_i)\,\phi. \tag{11.472}$$

So they set

$$\left(\partial_i + A'_i\right)U\phi = U\,(\partial_i + A_i)\,\phi \quad \text{or} \quad (\partial_i U)\,\phi + A'_i\,U\phi = U A_i\,\phi. \tag{11.473}$$

and made the gauge-field matrix A_i transform as

$$A'_i = U A_i U^{-1} - (\partial_i U)\,U^{-1}. \tag{11.474}$$

Then under the gauge transformation (11.468), the derivative $D_i\phi$ transforms as in (11.470), like the vector ϕ in (11.468), and the inner product of covariant derivatives

$$\left[\left(D^i\phi\right)^\dagger D_i\phi\right]' = \left(D^i\phi\right)^\dagger U^\dagger U D_i\phi = \left(D^i\phi\right)^\dagger D_i\phi \tag{11.475}$$

remains invariant.

To make an invariant action density for the gauge-field matrices A_i, they used the transformation law (11.472) which implies that $D_i' U\phi = U D_i\,\phi$ or $D_i' = U D_i\, U^{-1}$. So they defined their generalized Faraday tensor as

$$F_{ik} = [D_i, D_k] = \partial_i A_k - \partial_k A_i + [A_i, A_k] \tag{11.476}$$

which transforms covariantly

$$F_{ik}' = U F_{ik} U^{-1}. \tag{11.477}$$

They then generalized the action density $F_{ik}F^{ik}$ of electrodynamics to the trace $\mathrm{Tr}\left(F_{ik}F^{ik}\right)$ of the square of the Faraday matrices which is invariant under gauge transformations since

$$\mathrm{Tr}\left(U F_{ik} U^{-1} U F^{ik} U^{-1}\right) = \mathrm{Tr}\left(U F_{ik} F^{ik} U^{-1}\right) = \mathrm{Tr}\left(F_{ik}F^{ik}\right). \tag{11.478}$$

As an action density for fermionic matter fields, they replaced the ordinary derivative in Dirac's formula $\overline{\psi}(\gamma^i\partial_i + m)\psi$ by the covariant derivative (11.471) to get $\overline{\psi}(\gamma^i D_i + m)\psi$ (Chen-Ning Yang 1922–, Robert L. Mills 1927–1999).

In an abelian gauge theory, the square of the 1-form $A = A_i\, dx^i$ vanishes $A^2 = A_i\, A_k\, dx^i \wedge dx^k = 0$, but in a non-abelian gauge theory the gauge fields are matrices, and $A^2 \neq 0$. The sum $dA + A^2$ is the Faraday 2-form

$$\begin{aligned} F = dA + A^2 &= \left(\partial_i\, A_k + A_i\, A_k\right)\, dx^i \wedge dx^k \\ &= \tfrac{1}{2}\left(\partial_i\, A_k - \partial_k\, A_i + [A_i\, A_k]\right)\, dx^i \wedge dx^k = \tfrac{1}{2} F_{ik}\, dx^i \wedge dx^k. \end{aligned} \tag{11.479}$$

The scalar matter fields ϕ may have self-interactions described by a potential $V(\phi)$ such as $V(\phi) = \lambda(\phi^\dagger\phi - m^2/\lambda)^2$ which is positive unless $\phi^\dagger\phi = m^2/\lambda$. The kinetic action of the matter fields is $(D^i\phi)^\dagger D_i\phi$. At low temperatures, the matter fields assume a mean value $\langle 0|\phi|0\rangle = \phi_0$ in the vacuum with $\phi_0^\dagger\phi_0 = m^2/\lambda$ so as to minimize their potential energy density $V(\phi)$. Their kinetic action $(D^i\phi)^\dagger D_i\phi = (\partial^i\phi + A^i\phi)^\dagger(\partial_i\phi + A_i\phi)$ then is in effect $\phi_0^\dagger\, A^i\, A_i\phi_0$. The gauge-field matrix $A^i_{ab} = t^\alpha_{ab} A^i_\alpha$ is a linear combination of the generators t^α of the gauge group. So the action of the matter fields contains the term $\phi_0^\dagger\, A^i\, A_i\, \phi_0 = M^2_{\alpha\beta}\, A^i_\alpha\, A_{i\beta}$ in which the mass-squared matrix for the gauge fields is $M^2_{\alpha\beta} = \phi_0^{*a}\, t^\alpha_{ab}\, t^\beta_{bc}\, \phi_0^c$. This **Higgs mechanism** gives masses to those linear combinations $b_\beta A_{i\beta}$ of the gauge fields for which $M^2_{\alpha\beta} b_\beta \neq 0$.

The Higgs mechanism also gives masses to the fermions. The mass term m in the Yang-Mills-Dirac action is replaced by something like $c\,\phi$ in which c is a constant different for each fermion. In the vacuum and at low temperatures, each fermion in effect acquires as its mass $c\,\phi_0$ (Peter Higgs 1929–).

11.51 Gauge Theory and Vectors

We can formulate Yang-Mills theory in terms of vectors as we did relativity. To accomodate noncompact groups, we will generalize the unitary matrices $U(x)$ of the Yang-Mills gauge group to non-singular matrices $V(x)$ that act on n matter fields $\psi^a(x)$ as

$$\psi'^a(x) = \sum_{a=1}^{n} V^a_{\ b}(x)\,\psi^a(x). \tag{11.480}$$

The field

$$\Psi(x) = \sum_{a=1}^{n} e_a(x)\,\psi^a(x) \tag{11.481}$$

will be gauge invariant $\Psi'(x) = \Psi(x)$ if the vectors $e_a(x)$ transform as

$$e'_a(x) = \sum_{b=1}^{n} e_b(x)\,V^{-1b}{}_a(x). \tag{11.482}$$

In what follows, we will sum over repeated indices from 1 to n and often will suppress explicit mention of the space-time coordinates. In this compressed notation, the field Ψ is gauge invariant because

$$\Psi' = e'_a\,\psi'^a = e_b\,V^{-1b}{}_a\,V^a{}_c\,\psi^c = e_b\,\delta^b{}_c\,\psi^c = e_b\,\psi^b = \Psi \tag{11.483}$$

which is $e'^{\mathsf{T}}\psi' = e^{\mathsf{T}}V^{-1}V\psi = e^{\mathsf{T}}\psi$ in matrix notation.

The inner product of two basis vectors is an internal "metric tensor"

$$e^*_a \cdot e_b = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} e^{\alpha *}_a\,\eta_{\alpha\beta} e^{\alpha}_b = \sum_{\alpha=1}^{N} e^{\alpha *}_a\,e^{\alpha}_b = g_{ab} \tag{11.484}$$

in which I set $\eta = I$ for simplicity. As in relativity, we'll assume the matrix g_{ab} to be non-singular. We then can use its inverse to construct dual vectors $e^a = g^{ab} e_b$ that satisfy $e^{a\dagger} \cdot e_b = \delta^a_b$.