16.8 Show that for the Hamiltonian (16.60) of the simple harmonic oscillator the action $S[q_c]$ of the classical path is (16.67).

16.9 Show that the harmonic-oscillator action of the loop (16.68) is (16.69).

16.10 Show that the harmonic-oscillator amplitude (16.72) for $q_0 = 0$ and $q_0' = 0$ reduces as $t \to 0$ to the one-dimensional version of the free-particle amplitude (16.54).

16.11 Work out the path-integral formula for the amplitude for a mass $m$ to fall to the ground from height $h$ in a gravitational field of local acceleration $g$ to lowest order and then including loops. Hint: use the technique of section 16.7.

16.12 Show that the action (16.74) of the stationary solution (16.77) is (16.79).


16.14 Derive identity (16.136). Split the time integral at $t = 0$ into two halves, use

$$e^{\pm \epsilon t} = \pm \frac{d}{dt} e^{\pm \epsilon t}$$

(16.257)

and then integrate each half by parts.

**Solution:** We assume that the limits $\lim_{t \to \pm \infty} f(t) = f(\pm \infty)$ exist. We split the integral and integrate by parts using the hint (16.257):

$$\epsilon \int_{-\infty}^{\infty} f(t) e^{-\epsilon |t|} dt = \int_{-\infty}^{0} f(t) \epsilon e^{\epsilon t} dt + \int_{0}^{\infty} f(t) \epsilon e^{-\epsilon t} dt$$

$$= \int_{-\infty}^{0} f(t) \frac{de^{\epsilon t}}{dt} dt - \int_{0}^{\infty} f(t) \frac{de^{-\epsilon t}}{dt} dt$$

$$= \int_{-\infty}^{0} \left[ \frac{d(f(t)e^{\epsilon t})}{dt} - e^{\epsilon t} \frac{df(t)}{dt} \right] dt$$

$$- \int_{0}^{\infty} \left[ \frac{d(f(t)e^{-\epsilon t})}{dt} - e^{-\epsilon t} \frac{df(t)}{dt} \right] dt$$

$$= f(0) - \int_{-\infty}^{0} e^{\epsilon t} \dot{f}(t) dt + f(0) + \int_{0}^{\infty} e^{-\epsilon t} \dot{f}(t) dt$$

(16.258)

in which the dot means time derivative. We now let $\epsilon \to 0$ and get

$$\epsilon \int_{-\infty}^{\infty} f(t) e^{-t|t|} dt = 2f(0) - \int_{-\infty}^{0} \dot{f}(t) dt + \int_{0}^{\infty} \dot{f}(t) dt$$

$$= 2f(0) - f(0) + f(-\infty) + f(\infty) - f(0)$$

(16.259)
which is the desired identity (16.136)
\[
\epsilon \int_{-\infty}^{\infty} f(t) e^{-\epsilon |t|} dt = f(\infty) + f(-\infty). \tag{16.260}
\]

16.15 Derive the third term in equation (16.138) from its second term.
16.16 Derive equation (16.147) from equations (16.144, 16.147, & 16.146).
16.17 Derive the formula (16.148) for \( Z_0[j] \) from the expression (16.147) for \( S_0[\phi, \epsilon, j] \).
16.19 Derive equation (16.154) from the formula (16.149) for \( Z_0[j] \).
16.20 Show that the time integral of the Coulomb term (16.159) is the negative of the term that is quadratic in \( j^0 \) in the number \( F \) defined by (16.164).
16.21 By following steps analogous to those the led to (16.150), derive the formula (16.177) for the photon propagator in Feynman’s gauge.
16.22 Derive expression (16.192) for the inner product \( \langle \zeta | \theta \rangle \).
16.23 Derive the representation (16.195) of the identity operator \( I \) for a single fermionic degree of freedom from the rules (16.182 & 16.185) for Grassmann integration and the anti-commutation relations (16.178 & 16.184).
16.24 Derive the eigenvalue equation (16.200) from the definition (16.198 & 16.199) of the eigenstate \( |\theta\rangle \) and the anti-commutation relations (16.196 & 16.197).
16.25 Derive the eigenvalue relation (16.213) for the Fermi field \( \psi_m(x, t) \) from the anti-commutation relations (16.209 & 16.210) and the definitions (16.211 & 16.212).
16.26 Derive the formula (16.214) for the inner product from the definition (16.212) of the ket \( |\chi\rangle \).