10.38 Show that under the unitary Lorentz transformation (10.273), the action density (10.258) is Lorentz covariant (10.259).

10.39 Show that under the unitary Lorentz transformations (10.257 & 10.273), the Majorana mass terms (10.266 & 10.279) are Lorentz covariant.

**Solution:** Transposition flips the sign of $\sigma_2$ but leaves $\sigma_1$ and $\sigma_3$ invariant. And because $\sigma_2$ anti commutes with both $\sigma_1$ and $\sigma_3$, it follows that

$$\sigma_2 \boldsymbol{z} \cdot \boldsymbol{\sigma}^{\mathsf{T}} \sigma_2 = - \boldsymbol{z} \cdot \boldsymbol{\sigma}. \tag{10.311}$$

Since $\sigma_2^2 = 1$, we have for any $2 \times 2$ matrix $A$, the relation $(\sigma_2 A \sigma_2)^2 = \sigma_2 A^2 \sigma_2$ as well as $(\sigma_2 A \sigma_2)^n = \sigma_2 A^n \sigma_2$. Thus

$$\sigma_2 e^A \sigma_2 = e^{\sigma_2 A \sigma_2}. \tag{10.312}$$

Thus under a Lorentz transformation

$$\begin{aligned} U(L)\xi^{\mathsf{T}}(x)\sigma_2\xi(x) &= \left(e^{\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}\xi(Lx)\right)^{\mathsf{T}}\sigma_2 e^{\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}\xi(Lx) \\ &= \xi(Lx)^{\mathsf{T}}\left(e^{\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}\right)^{\mathsf{T}}\sigma_2 e^{\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}\xi(Lx) \\ &= \xi(Lx)^{\mathsf{T}}\exp\left(\boldsymbol{z}\cdot\boldsymbol{\sigma}^{\mathsf{T}}/2\right)\sigma_2 e^{\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}\xi(Lx) \\ &= \xi(Lx)^{\mathsf{T}}\sigma_2\sigma_2\exp\left(\boldsymbol{z}\cdot\boldsymbol{\sigma}^{\mathsf{T}}/2\right)\sigma_2\sigma_2\sigma_2 e^{\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}\xi(Lx) \\ &= \xi(Lx)^{\mathsf{T}}\sigma_2\exp\left(\sigma_2(\boldsymbol{z}\cdot\boldsymbol{\sigma}^{\mathsf{T}}/2)\sigma_2\right)\sigma_2\sigma_2 e^{\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}\xi(Lx) \\ &= \xi(Lx)^{\mathsf{T}}\sigma_2\exp\left(-\boldsymbol{z}\cdot\boldsymbol{\sigma}/2\right)e^{\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}\xi(Lx) \\ &= \xi(Lx)^{\mathsf{T}}\sigma_2 e^{-\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}e^{\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}\xi(Lx) \\ &= \xi(Lx)^{\mathsf{T}}\sigma_2\xi(Lx). \end{aligned} \tag{10.313}$$

10.40 Show that the definitions of the gamma matrices (10.281) and of the generators (10.283) imply that the gamma matrices transform as a 4-vector under Lorentz transformations (10.284).

10.41 Show that (10.283) and (10.284) imply that the generators $J^{ab}$ satisfy the commutation relations of the Lorentz group.

10.42 Show that the spinor $\zeta = \sigma_2 \xi^*$ defined by (10.295) is right handed (10.273) if $\xi$ is left handed (10.257.

10.43 Use (10.303) to get (10.304 & 10.305).

10.44 Derive (10.306) from (10.285, 10.299, & 10.305).