The integral over q^0 here yields a delta function in time, so this is equivalent to a correction to the interaction Hamiltonian $V(t)$, of the form

$$-\frac{1}{2}\int d^3x \int d^3y \,\frac{j^0(\mathbf{x},t)\, j^0(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}\,.$$

This is just right to cancel the Coulomb interaction (8.4.25). Our result is that the photon propagator can be taken effectively as the covariant quantity

$$\Delta^{\rm eff}_{\mu\nu}(x-y) = (2\pi)^{-4}\int d^4q \,\frac{\eta_{\mu\nu}}{q^2-i\epsilon}e^{iq\cdot(x-y)} \tag{8.5.8}$$

with the Coulomb interaction dropped from now on. We see that the apparent violation of Lorentz invariance in the instantaneous Coulomb interaction is cancelled by another apparent violation of Lorentz invariance, that as noted in Section 5.9 the fields $a^\mu(x)$ are not four-vectors, and therefore have a non-covariant propagator. From a practical point of view, the important point is that in the momentum space Feynman rules, the contribution of an internal photon line is simply given by

$$\frac{-i}{(2\pi)^4}\frac{\eta_{\mu\nu}}{q^2-i\epsilon} \tag{8.5.9}$$

and the Coulomb interaction is dropped.

8.6 Feynman Rules for Spinor Electrodynamics

We are now in a position to state the Feynman rules for calculating the S-matrix in quantum electrodynamics. For definiteness, we will consider the electrodynamics of a single species of spin $\frac{1}{2}$ particles of charge $q=-e$ and mass m. We will call these fermions electrons, but the same formalism applies to muons and other such particles. The simplest gauge- and Lorentz-invariant Lagrangian for this theory is*

$$\mathscr{L} = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \bar{\Psi}\left(\gamma^\mu[\partial_\mu + i\,e\,A_\mu] + m\right)\Psi\,. \tag{8.6.1}$$

The electric current four-vector is then simply

$$J^\mu = \frac{\partial\mathscr{L}}{\partial A_\mu} = -i\,e\,\bar{\Psi}\gamma^\mu\Psi\,. \tag{8.6.2}$$

* In Chapter 12 we will discuss reasons why more complicated terms are excluded from the Lagrangian density.

The interaction (8.4.23) in the interaction picture is here

$$V(t) = +ie \int d^3x \; (\bar{\psi}(\mathbf{x},t)\gamma^\mu\psi(\mathbf{x},t)) \; a_\mu(\mathbf{x},t) + V_{\text{Coul}}(t)\,. \tag{8.6.3}$$

(There is no V_{matter} here.) As we have seen, the Coulomb term $V_{\text{Coul}}(t)$ just serves to cancel a part of the photon propagator that is non-covariant and local in time.

Following the general results of Section 6.3, we can state the momentum space Feynman rules for the connected part of the S-matrix in this theory as follows:

(i) Draw all Feynman diagrams with up to some given number of vertices. The diagrams consist of electron lines carrying arrows and photon lines without arrows, with the lines joined at vertices, at each of which there is one incoming and one outgoing electron line and one photon line. There is one external line coming into the diagram from below or going upwards out of the diagram for each particle in the initial or final states, respectively; electrons are represented by external lines carrying arrows pointing upwards into or out of the diagram, while positrons are represented by lines carrying arrows pointing downwards into or out of the diagram. There are also as many internal lines as are needed to give each vertex the required number of attached lines. Each internal line is labelled with an off-mass-shell four-momentum flowing in a definite direction along the line (taken conventionally to flow along the direction of the arrow for electron lines.) Each external line is labelled with the momentum and spin z-component or helicity of the electron or photon in the initial and final states.

(ii) Associate factors with the components of the diagram as follows:

Vertices

Label each vertex with a four-component Dirac index α at the electron line with its arrow coming into the vertex, a Dirac index β at the electron line with its arrow going out of the vertex, and a spacetime index μ at the photon line. For each such vertex, include a factor

$$(2\pi)^4\, e\,(\gamma^\mu)_{\beta\alpha}\delta^4(k - k' + q)\,, \tag{8.6.4}$$

where k and k' are the electron four-momenta entering and leaving the vertex, and q is the photon four-momentum entering the vertex (or minus the photon momentum leaving the vertex).

External lines:

Label each external line with the three-momentum **p** and spin z-component or helicity σ of the particle in the initial or final state. For each line for an electron in the final state running out of a vertex carrying a Dirac label β on this line, include a factor*

$$\frac{\bar{u}_\beta(\mathbf{p},\sigma)}{(2\pi)^{3/2}} . \tag{8.6.5}$$

For each line for a positron in the final state running into a vertex carrying a Dirac label α on this line, include a factor

$$\frac{v_\alpha(\mathbf{p},\sigma)}{(2\pi)^{3/2}} . \tag{8.6.6}$$

For each line for an electron in the initial state running into a vertex carrying a Dirac label α on this line, include a factor

$$\frac{u_\alpha(\mathbf{p},\sigma)}{(2\pi)^{3/2}} . \tag{8.6.7}$$

For each line for a positron in the initial state running out of a vertex carrying a Dirac label β on this line, include a factor

$$\frac{\bar{v}_\beta(\mathbf{p},\sigma)}{(2\pi)^{3/2}} . \tag{8.6.8}$$

The us and vs are the four-component spinors discussed in Section 5.5. For each line for a photon in the final state connected to a vertex carrying a spacetime label μ on this line, include a factor

$$\frac{e^*_\mu(\mathbf{p},\sigma)}{(2\pi)^{3/2}\sqrt{2p^0}} . \tag{8.6.9}$$

For each line for a photon in the initial state connected to a vertex carrying a spacetime label μ on this line, include a factor

$$\frac{e_\mu(\mathbf{p},\sigma)}{(2\pi)^{3/2}\sqrt{2p^0}} . \tag{8.6.10}$$

The e_μ are the photon polarization four-vectors described in the previous section.

Internal lines:

For each internal electron line carrying a four-momentum **k** and running from a vertex carrying a Dirac label β to another vertex carrying a Dirac

* A matrix β has been extracted from the interaction in (8.6.4), so that $\bar{u}$ and $\bar{v}$ appear instead of $u^\dagger$ and $v^\dagger$.

label α, include a factor

$$\frac{-i}{(2\pi)^4}\,\frac{[-i\,\not{k}+m]_{\alpha\beta}}{k^2+m^2-i\epsilon}\,. \tag{8.6.11}$$

(We are here using the very convenient 'Dirac slash' notation; for any four-vector v^μ, $\not{v}$ denotes $\gamma_\mu v^\mu$.) For each internal photon line carrying a four-momentum q that runs between two vertices carrying spacetime labels μ and ν include a factor

$$\frac{-i}{(2\pi)^4}\,\frac{\eta_{\mu\nu}}{q^2-i\epsilon}\,. \tag{8.6.12}$$

(iii) Integrate the product of all these factors over the four-momenta carried by the internal lines, and sum over all Dirac and spacetime indices.

(iv) Add up the results obtained in this way from each Feynman diagram.

Additional combinatoric factors and fermionic signs may need to be included, as described in parts (v) and (vi) of Section 6.1.

The difficulty of evaluating Feynman diagrams increases rapidly with the number of internal lines and vertices, so it is important to have some idea of what numerical factors tend to suppress the contributions of the more complicated diagrams. We shall estimate these numerical factors including not only the factors of the electronic charge e associated with vertices, but also the factors of 2 and π from vertices, propagators, and momentum space integrals.

Consider a connected Feynman diagram with V vertices, I internal lines, E external lines, and L loops. These quantities are not independent, but are subject to relations already used in Section 6.3:

$$L = I - V + 1\,, \qquad 2I + E = 3V\,.$$

There is a factor $e(2\pi)^4$ from each vertex, a factor $(2\pi)^{-4}$ from each internal line, and a four-dimensional momentum space integral for each loop. The volume element in four-dimensional Euclidean space in terms of a radius parameter κ is $\pi^2\kappa^2 d\kappa^2$, so each loop contributes a factor π^2. Thus the diagram will contain a factor

$$(2\pi)^{4V}e^V(2\pi)^{-4I}\pi^{2L} = (2\pi)^4 e^{E-2}\left(\frac{e^2}{16\pi^2}\right)^L\,.$$

The number E of external lines is fixed for a given process, so we see that the expansion parameter that governs the suppression of Feynman graphs for each additional loop is

$$\frac{e^2}{16\pi^2} = \frac{\alpha}{4\pi} = 5.81\times 10^{-4}\,.$$

Fortunately this is small enough that good accuracy can usually be obtained from Feynman diagrams with at most a few loops.

* * *

We must say a little more about the spin states of photons and electrons in realistic experiments, where not every particle in the initial and final states has a definite known helicity or spin z-component. This consideration is especially important for photons, which in practice are often characterized by a state of transverse or elliptical polarization rather than helicity. As we saw in the previous section, for photons of helicity ± 1, the polarization vectors are

$$e(\mathbf{p}, \pm 1) = R(\hat{\mathbf{p}}) \begin{bmatrix} 1/\sqrt{2} \\ \pm i/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \,,$$

where $R(\hat{\mathbf{p}})$ is the standard rotation that takes the z-axis to the $\mathbf{p}$ direction. These are not the only possible photon states; in general, a photon state can be a linear combination of helicity states $\Psi_{\mathbf{p},\pm 1}$

$$\alpha_+ \Psi_{\mathbf{p},+1} + \alpha_- \Psi_{\mathbf{p},-1} \tag{8.6.13}$$

which is properly normalized if

$$|\alpha_+|^2 + |\alpha_-|^2 = 1 \,. \tag{8.6.14}$$

To calculate the S-matrix element for absorbing or emitting such a photon, we simply replace $e_\mu(\mathbf{p}, \pm 1)$ in the Feynman rules with

$$e_\mu(\mathbf{p}) = \alpha_+ \, e_\mu(\mathbf{p}, +1) + \alpha_- \, e_\mu(\mathbf{p}, -1) \,. \tag{8.6.15}$$

The polarization vectors for definite helicity satisfy the normalization condition

$$e^*_\mu(\mathbf{p}, \lambda') \, e^\mu(\mathbf{p}, \lambda) = \delta_{\lambda'\lambda} \tag{8.6.16}$$

and therefore in general

$$e^*_\mu(\mathbf{p}) \, e^\mu(\mathbf{p}) = 1 \,. \tag{8.6.17}$$

The two extreme cases are *circular polarization*, for which $\alpha_- = 0$ or $\alpha_+ = 0$, and *linear polarization*, for which $|\alpha_+| = |\alpha_-| = 1/\sqrt{2}$. For linear polarization, by an adjustment of the overall phase of the state (8.6.13), we can make α_+ and α_- complex conjugates, so that they can be expressed as

$$\alpha_\pm = \exp(\mp i\phi)\Big/\sqrt{2} \,. \tag{8.6.18}$$

Then in the Feynman rules we should use a polarization vector

$$e_\mu(\mathbf{p}) = R(\hat{\mathbf{p}}) \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \\ 0 \end{bmatrix} . \tag{8.6.19}$$

That is, ϕ is the azimuthal angle of the photon polarization in the plane perpendicular to $\mathbf{p}$. Note that the photon polarization vector here is *real*, which is only possible for linear polarization. In between the extremes of circular and linear polarization are the states of *elliptic* polarization, for which $|\alpha_+|$ and $|\alpha_-|$ are non-zero and unequal.

More generally, an initial photon may be prepared in a statistical mixture of spin states. In the most general case, an initial photon may have any number of possible polarization vectors $e_\mu^{(r)}(\mathbf{p})$, each with probability P_r. The rate for absorbing such a photon in a given process will then be of the form

$$\Gamma = \sum_r P_r |e_\mu^{(r)}(\mathbf{p})\, M^\mu|^2 = M^{\mu *} M^\nu \rho_{\nu\mu} , \tag{8.6.20}$$

where ρ is the *density matrix*

$$\rho_{\nu\mu} \equiv \sum_r P_r\, e_\nu^{(r)}(\mathbf{p})\, e_\mu^{(r)*}(\mathbf{p}) . \tag{8.6.21}$$

Since ρ is obviously a Hermitian positive matrix of unit trace (because $\sum_r P_r = 1$) with $\rho_{\nu 0} = \rho_{0\mu} = 0$ and $\rho_{\nu\mu} p^\nu = \rho_{\nu\mu} p^\mu = 0$, it may be written as

$$\rho_{\nu\mu} = \sum_{s=1,2} \lambda_s\, e_\nu(\mathbf{p};s)\, e_\mu^*(\mathbf{p};s) , \tag{8.6.22}$$

where $e_\mu(\mathbf{p};s)$ are the two orthonormal eigenvectors of ρ with

$$e_0(\mathbf{p};s) = e_\mu(\mathbf{p};s) p^\mu = 0 \tag{8.6.23}$$

and λ_s are the corresponding eigenvalues, with

$$\lambda_s \geq 0 , \qquad \sum_{s=1,2} \lambda_s = 1 .$$

We may then write the rate for the photon absorption process as

$$\Gamma = \sum_{s=1,2} \lambda_s |e_\nu(\mathbf{p};s) M^\nu|^2 . \tag{8.6.24}$$

Thus any statistical mixture of initial photon states is always equivalent to having just two orthonormal polarizations $e_\nu(\mathbf{p};s)$ with probabilities λ_s.

In particular, if we know nothing whatever about the initial photon polarization, then the two probabilities λ_s for the polarization vectors

$e_\nu(\mathbf{p};s)$ are equal, so that $\lambda_1 = \lambda_2 = \frac{1}{2}$, and the density matrix (and hence the absorption rate) is an average over initial polarizations

$$\rho_{ij} = \frac{1}{2} \sum_{s=1,2} e_i(\mathbf{p};s) e_j^*(\mathbf{p};s) = \frac{1}{2} \left(\delta_{ij} - \hat{p}_i \hat{p}_j \right) . \tag{8.6.25}$$

Fortunately, this result does not depend on the particular pair of polarization vectors $e_i(\mathbf{p};s)$ over which we average; for unpolarized photons we can average the absorption rate over any pair of orthonormal polarization vectors. Similarly, if we make no attempt to measure the polarization of a photon in the final state, then the rate may be calculated by summing over any pair of orthonormal final photon polarization vectors.

The same remarks apply to electrons and positrons; if (as is usually the case) we make no attempt to prepare an electron or positron so that some spin states are more likely than others, then the rate is to be calculated by *averaging* over any two orthonormal initial spin states, such as those with spin z-component $\sigma = \pm\frac{1}{2}$; if we make no attempt to measure a final electron's or positron's spin state, then we must *sum* the rate over any two orthonormal initial spin states, such as those with spin z-component $\sigma = \pm\frac{1}{2}$. Such sums may be performed using the relations (5.5.37) and (5.5.38):

$$\sum_\sigma u_\alpha(\mathbf{p},\sigma)\bar{u}_\beta(\mathbf{p},\sigma) = \left(\frac{-i \not{p} + m}{2p^0} \right)_{\alpha\beta} , \tag{8.6.26}$$

$$\sum_\sigma v_\alpha(\mathbf{p},\sigma)\bar{v}_\beta(\mathbf{p},\sigma) = \left(\frac{-i \not{p} - m}{2p^0} \right)_{\alpha\beta} , \tag{8.6.27}$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$. For instance, if the initial state contains an electron with momentum $\mathbf{p}$ and spin z-component σ, and a positron with momentum $\mathbf{p}'$ and spin z-component σ', then the S-matrix element for the process will be of the form $(\bar{v}_\alpha(\mathbf{p}',\sigma')\,\mathscr{M}_{\alpha\beta}\,u_\beta(\mathbf{p},\sigma))$. Hence if neither electron nor positron spins are observed, the rate will be proportional to

$$\begin{aligned} &\frac{1}{4} \sum_{\sigma',\sigma} \left| (\bar{v}_\alpha(\mathbf{p}',\sigma')\,\mathscr{M}_{\alpha\beta}\,u_\beta(\mathbf{p},\sigma)) \right|^2 \\ &\quad = \frac{1}{4}\mathrm{Tr}\left\{ \beta\,\mathscr{M}^\dagger\,\beta\, \left(\frac{-i \not{p}' - m}{2p^{0\prime}} \right) \mathscr{M} \left(\frac{-i \not{p} + m}{2p^0} \right) \right\} . \end{aligned}$$

Techniques for the calculation of such traces are described in the Appendix to this chapter.

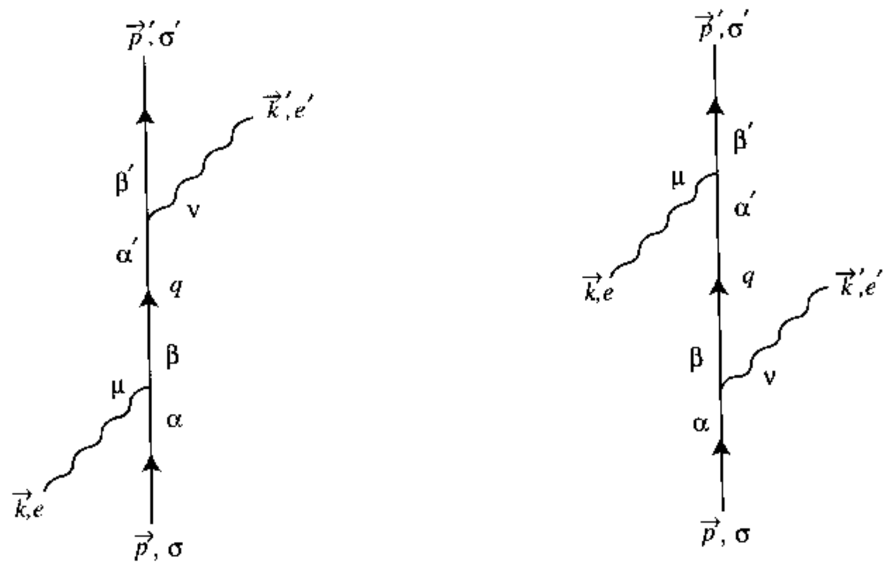

Figure 8.1. The two lowest-order Feynman diagrams for Compton scattering. Straight lines are electrons; wavy lines are photons.

8.7 Compton Scattering

As an example of the methods described in this chapter, we shall consider here the scattering of a photon by an electron (or other particle of spin $\frac{1}{2}$ and charge $-e$), to lowest order in e. We label the initial and final photon momenta and polarization vectors by k^μ, e^μ and k'^μ, e'^μ, where $k^0 = |\mathbf{k}|$ and $k^{0\prime} = |\mathbf{k}'|$. Also, the initial and final electron momenta and spin z-components are labelled p^μ, σ and p'^μ, σ', where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$ and $p'^0 = \sqrt{\mathbf{p}'^2 + m^2}$, with m the electron mass. The lowest order Feynman diagrams for this process are shown in Figure 8.1. Using the rules outlined in the previous section, the corresponding S-matrix element is

$$
\begin{aligned}
&S(\mathbf{p},\sigma \;+\; \mathbf{k},e \;\to\; \mathbf{p}',\sigma' \;+\; \mathbf{k}',e') = \\
&\quad \frac{\bar{u}(\mathbf{p}',\sigma')_{\beta'}}{(2\pi)^{3/2}} \; \frac{e^{*\prime}_\nu}{(2\pi)^{3/2}\sqrt{2k^{0\prime}}} \; \frac{u(\mathbf{p},\sigma)_\alpha}{(2\pi)^{3/2}} \; \frac{e_\mu}{(2\pi)^{3/2}\sqrt{2k^0}} \\
&\quad \times \int d^4q \; \left[\frac{-i}{(2\pi)^4}\right] \; \left[\frac{-i\; \not{q} + m}{q^2 + m^2 - i\epsilon}\right]_{\alpha'\beta} \\
&\quad \times \Big\{ \; \left[e(2\pi)^4 \gamma^\nu_{\beta'\alpha'} \delta^4(q - p' - k')\right] \; \left[e(2\pi)^4 \gamma^\mu_{\beta\alpha} \delta^4(q - p - k)\right] \\
&\quad + \left[e(2\pi)^4 \gamma^\mu_{\beta'\alpha'} \delta^4(q + k - p')\right] \; \left[e(2\pi)^4 \gamma^\nu_{\beta\alpha} \delta^4(q + k' - p)\right] \Big\} \,. \qquad (8.7.1)
\end{aligned}
$$

Performing the (trivial) q-integral, collecting factors of i and 2π, and rewriting the result in matrix notation, we have more simply

$$S = \frac{-ie^2\delta^4(p'+k'-p-k)}{(2\pi)^2\sqrt{2k^{0\prime}\cdot 2k^0}}\bar{u}(\mathbf{p}',\sigma')\left[\not{e}^{*\prime}\left(\frac{-i(\not{p}+\not{k})+m}{(p+k)^2+m^2}\right)\not{e}\right.$$
$$\left. + \not{e}\left(\frac{-i(\not{p}-\not{k}')+m}{(p-k')^2+m^2}\right)\not{e}^{*\prime}\right]u(\mathbf{p},\sigma)\,. \tag{8.7.2}$$

(Here $\not{e}^*$ means $e^*_\mu\gamma^\mu$, not $(\not{e})^*$. Also, we drop the $-i\epsilon$, because the denominators here do not vanish.) Because $p^2 = -m^2$ and $k^2 = k'^2 = 0$, the denominators can be simplified

$$(p+k)^2+m^2 = 2p\cdot k\,, \tag{8.7.3}$$
$$(p-k')^2+m^2 = -2p'\cdot k\,. \tag{8.7.4}$$

Also, the 'Feynman amplitude' M is defined in general by Eq. (3.3.2), which (because some scattering is assumed to take place) here reads

$$S = -2\pi i\delta^4(p'+k'-p-k)M\,, \tag{8.7.5}$$

so

$$M = \frac{e^2}{4(2\pi)^3\sqrt{k^0k^{0\prime}}}\bar{u}(\mathbf{p}'\sigma')\Big\{\not{e}^{*\prime}\left[-i(\not{p}+\not{k})+m\right]\not{e}\big/p\cdot k$$
$$-\not{e}\left[-i(\not{p}-\not{k}')+m\right]\not{e}^{*\prime}\big/p\cdot k'\Big\}u(\mathbf{p},\sigma)\,. \tag{8.7.6}$$

The differential cross-section is given in terms of M by Eq. (3.4.15), which here reads

$$d\sigma = (2\pi)^4u^{-1}|M|^2\delta^4(p'+k'-p-k)d^3p'd^3k'\,. \tag{8.7.7}$$

Since one of the particles here is massless, Eq. (3.4.17) for the initial velocity gives

$$u = |p\cdot k|\big/p^0k^0\,. \tag{8.7.8}$$

To go further, it will be convenient to adopt a specific coordinate frame. Since electrons in atoms move non-relativistically, the laboratory frame for high-energy (X ray or gamma ray) photon–electron scattering experiments is usually (though not always) one in which the initial electron can be taken to be at rest. We will adopt this frame here, so that

$$\mathbf{p} = 0\,, \qquad p^0 = m\,. \tag{8.7.9}$$

The velocity (8.7.8) is then simply

$$u = 1\,. \tag{8.7.10}$$

To save writing, we denote the photon energies by

$$\omega = k^0 = |\mathbf{k}| = -p \cdot k \Big/ m\,, \tag{8.7.11}$$

$$\omega' = k^{0'} = |\mathbf{k}'| = -p \cdot k' \Big/ m\,. \tag{8.7.12}$$

The three-momentum delta function in Eq. (8.7.7) just serves to eliminate the differential d^3p', setting $\mathbf{p}' = \mathbf{k} - \mathbf{k}'$. This leaves the remaining energy delta function

$$\delta(p^{'0} + k^{'0} - p^0 - k^0) = \delta\left(\sqrt{(\mathbf{k}-\mathbf{k}')^2 + m^2} + \omega' - m - \omega\right)\,. \tag{8.7.13}$$

This fixes ω' to satisfy

$$\sqrt{\omega^2 - 2\omega\omega'\cos\theta + \omega'^2 + m^2} = \omega + m - \omega'\,,$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. Squaring both sides and cancelling $\omega^{'2}$ terms gives*

$$\omega' = \omega\,\frac{m}{m + \omega(1-\cos\theta)} \equiv \omega_c(\theta)\,. \tag{8.7.14}$$

The energy delta function (8.7.13) can be written

$$\begin{aligned}\delta(p^{'0} + k^{'0} - p^0 - k^0) &= \frac{\delta(\omega' - \omega_c(\theta))}{|\partial[\sqrt{\omega^2 - 2\omega\omega'\cos\theta + \omega'^2 + m^2} + \omega']/\partial\omega'|} \\ &= \frac{\delta(\omega' - \omega_c(\theta))}{|(\omega' - \omega cos\theta)/p'^0 + 1|} \\ &= \frac{p'^0\omega'}{m\omega}\delta(\omega' - \omega_c(\theta))\,.\end{aligned} \tag{8.7.15}$$

Also, the differential d^3k' can be written

$$d^3k' = \omega^{'2}d\omega' d\Omega\,, \tag{8.7.16}$$

where $d\Omega$ is the solid angle into which the final photon is scattered. The final delta function in Eq. (8.7.15) just serves to eliminate the differential $d\omega'$ in Eq. (8.7.16), leaving us with a differential cross-section

$$d\sigma = (2\pi)^4 |M|^2\,\frac{p^{'0}\omega^{'3}}{m\omega}\,d\Omega \tag{8.7.17}$$

with $p^{'0} = m + \omega - \omega'$, and ω' given by Eq. (8.7.14).

* Equivalently, there is an increase in wavelength

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1-\cos\theta}{m}\,.$$

The verification of this formula in the scattering of X rays by electrons by A.H. Compton in 1922–3 played a key role in confirming Einstein's 1905 proposal of a quantum of light, which soon after Compton's experiments came to be known as the photon.

Usually we do not measure the spin z-component of the initial or final electron. In such cases, we must sum over σ' and average over σ, or in other words take half the sum over σ and σ':

$$d\bar{\sigma}(\mathbf{p} + \mathbf{k}, e \to \mathbf{p}' + \mathbf{k}', e') \equiv \tfrac{1}{2}\sum_{\sigma',\sigma} d\sigma(\mathbf{p},\sigma + \mathbf{k}, e \to \mathbf{p}',\sigma' + \mathbf{k}', e') \,. \tag{8.7.18}$$

To calculate this, we use the standard formula

$$\sum_\sigma u_\alpha(\mathbf{p},\sigma)\bar{u}_\beta(\mathbf{p},\sigma) = \frac{(-i\not{p}+m)_{\alpha\beta}}{2p^0} \tag{8.7.19}$$

and likewise for the sum over σ'. It follows that for an arbitrary 4×4 matrix A

$$\begin{aligned}\sum_{\sigma,\sigma'} \left|\bar{u}(\mathbf{p}',\sigma')Au(\mathbf{p},\sigma)\right|^2 &= \sum_{\sigma,\sigma'}(\bar{u}(\mathbf{p}',\sigma')Au(\mathbf{p},\sigma))(\bar{u}(\mathbf{p},\sigma)\beta A^\dagger\beta u(\mathbf{p}',\sigma')) \\ &= \sum_{\sigma,\sigma'} A_{\beta\alpha}u_\alpha(\mathbf{p},\sigma)\bar{u}_\gamma(\mathbf{p},\sigma)(\beta A^\dagger\beta)_{\gamma\delta}u_\delta(\mathbf{p}',\sigma')\bar{u}_\beta(\mathbf{p}',\sigma') \\ &= \mathrm{Tr}\left\{A\left(\frac{-i\not{p}+m}{2p^0}\right)\beta A^\dagger\beta\left(\frac{-i\not{p}'+m}{2p'^0}\right)\right\}. \end{aligned}\tag{8.7.20}$$

Recalling that $\beta\gamma_\mu^\dagger\beta = -\gamma_\mu$, Eq. (8.7.6) gives now

$$\begin{aligned}\sum_{\sigma,\sigma'}|M|^2 &= \frac{e^4}{64(2\pi)^6\omega\omega' p^0 p'^0} \\ &\times \mathrm{Tr}\left[\left\{\not{e}'^*\frac{[-i(\not{p}+\not{k})+m]}{p\cdot k}\not{e} - \not{e}\frac{[-i(\not{p}-\not{k}')+m]}{p\cdot k'}\not{e}'^*\right\}(-i\not{p}+m)\right. \\ &\times \left.\left\{\not{e}^*\frac{[-i(\not{p}+\not{k})+m]}{p\cdot k}\not{e}' - \not{e}'\frac{[-i(\not{p}-\not{k}')+m]}{p\cdot k'}\not{e}^*\right\}(-i\not{p}'+m)\right].\end{aligned}\tag{8.7.21}$$

(Recall again that $\not{e}^*$ means $e^*_\mu\gamma^\mu$, not $(e_\mu\gamma^\mu)^*$, and likewise for $\not{e}'^*$.) We work in a 'gauge' in which

$$e\cdot p = e^*\cdot p = e'\cdot p = e'^*\cdot p = 0 \tag{8.7.22}$$

such as for instance Coulomb gauge in the laboratory frame, where $e^0 = e'^0 = 0$ and $\mathbf{p} = 0$. This implies that

$$\begin{aligned}[-i\not{p}+m]\,\not{e}[-i\not{p}+m] &= \not{e}[i\not{p}+m]\,[-i\not{p}+m] \\ &= \not{e}(\not{p}^2+m^2) = \not{e}(p_\mu p^\mu + m^2) = 0\end{aligned}$$

and likewise for $\not{e}'^*$, $\not{e}'$, and $\not{e}^*$. Eq. (8.7.21) can therefore be written in the greatly simplified form

$$\sum_{\sigma,\sigma'}|M|^2 = \frac{-e^4}{64(2\pi)^6\omega\omega' p^0 p'^0}\mathrm{Tr}\left[\left\{\frac{\not{e}'^*\,\not{k}\,\not{e}}{p\cdot k}+\frac{\not{e}\,\not{k}'\,\not{e}'^*}{p\cdot k'}\right\}(-i\,\not{p}+m)\right.$$
$$\left.\times\ \left\{\frac{\not{e}^*\,\not{k}\,\not{e}'}{p\cdot k}+\frac{\not{e}'\,\not{k}'\,\not{e}^*}{p\cdot k'}\right\}(-i\,\not{p}'+m)\right]\ . \tag{8.7.23}$$

The trace of any product of an odd number of gamma matrices vanishes, so this breaks up into terms of zeroth and second order in m:

$$\sum_{\sigma,\sigma'}|M|^2 = \frac{e^4}{64(2\pi)^6\omega\omega' p^0 p'^0}\left(\frac{T_1}{(p\cdot k)^2}+\frac{T_2}{(p\cdot k)(p\cdot k')}+\frac{T_3}{(p\cdot k)(p\cdot k')}\right.$$
$$\left.+\frac{T_4}{(p\cdot k')^2}-\frac{m^2t_1}{(p\cdot k)^2}-\frac{m^2t_2}{(p\cdot k)(p\cdot k')}-\frac{m^2t_3}{(p\cdot k)(p\cdot k')}-\frac{m^2t_4}{(p\cdot k')^2}\right) \tag{8.7.24}$$

where

$$T_1 = \mathrm{Tr}\left\{\not{e}'^*\,\not{k}\,\not{e}\,\not{p}\,\not{e}^*\,\not{k}\,\not{e}'\,\not{p}'\right\}\ , \tag{8.7.25}$$

$$T_2 = \mathrm{Tr}\left\{\not{e}'^*\,\not{k}\,\not{e}\,\not{p}\,\not{e}'\,\not{k}'\,\not{e}^*\,\not{p}'\right\}\ , \tag{8.7.26}$$

$$T_3 = \mathrm{Tr}\left\{\not{e}\,\not{k}'\,\not{e}'^*\,\not{p}\,\not{e}^*\,\not{k}\,\not{e}'\,\not{p}'\right\}\ , \tag{8.7.27}$$

$$T_4 = \mathrm{Tr}\left\{\not{e}\,\not{k}'\,\not{e}'^*\,\not{p}\,\not{e}'\,\not{k}'\,\not{e}^*\,\not{p}'\right\}\ , \tag{8.7.28}$$

$$t_1 = \mathrm{Tr}\left\{\not{e}'^*\,\not{k}\,\not{e}\,\not{e}^*\,\not{k}\,\not{e}'\right\}\ , \tag{8.7.29}$$

$$t_2 = \mathrm{Tr}\left\{\not{e}'^*\,\not{k}\,\not{e}\,\not{e}'\,\not{k}'\,\not{e}^*\right\}\ , \tag{8.7.30}$$

$$t_3 = \mathrm{Tr}\left\{\not{e}\,\not{k}'\,\not{e}'^*\,\not{e}^*\,\not{k}\,\not{e}'\right\}\ , \tag{8.7.31}$$

$$t_4 = \mathrm{Tr}\left\{\not{e}\,\not{k}'\,\not{e}'^*\,\not{e}'\,\not{k}'\,\not{e}^*\right\}\ . \tag{8.7.32}$$

The Appendix to this chapter shows how to calculate any trace $\mathrm{Tr}\{\not{a}\,\not{b}\,\not{c}\,\not{d}\ldots\}$ as a sum of products of scalar products of the four-vectors $a, b, c, d, \ldots$. In general, traces of products of 6 or 8 gamma matrices like the t_k or T_k would be given by a sum of 15 or 105 terms, respectively, but fortunately here most scalar products vanish; in addition to Eq. (8.7.22), we also have $k\cdot k = k'\cdot k' = 0$. (Furthermore, $e\cdot e^* = e'\cdot e'^* = 1$.) To simplify the calculation further, let us specialize to the case of *linear* polarization, where e^μ and e'^μ are real. Dropping the asterisks in Eqs. (8.7.25)–(8.7.32), we have then

$$T_1 = \mathrm{Tr}\left\{\not{e}'\,\not{k}\,\not{e}\,\not{p}\,\not{e}\,\not{k}\,\not{e}'\,\not{p}'\right\}\ .$$

Since $e^\mu p_\mu = 0$ and $e^\mu e_\mu = 1$, we have

$$\not{e}\,\not{p}\,\not{e} = -\not{p}\,\not{e}\,\not{e} = -\not{p}$$

so

$$T_1 = -\mathrm{Tr}\left\{ \not{e}'\,\not{k}\,\not{p}\,\not{k}\,\not{e}'\,\not{p}' \right\}.$$

Also, $k^\mu k_\mu = 0$, so

$$\not{k}\,\not{p}\,\not{k} = -\not{k}\,\not{k}\,\not{p} + 2\not{k}p\cdot k = 2\not{k}p\cdot k$$

and hence

$$T_1 = -2p\cdot k\ \mathrm{Tr}\left\{ \not{e}'\,\not{k}\,\not{e}'\,\not{p}' \right\}.$$

Using Eq. (8.A.6), this is

$$T_1 = -8p\cdot k\ [2e'\cdot k\ e'\cdot p' - k\cdot p'] .$$

It is convenient to make the substitutions

$$\begin{aligned} e'\cdot p' &= e'\cdot[p+k-k'] = e'\cdot k \\ k\cdot p' &= -\tfrac{1}{2}(p'-k)^2 - \tfrac{1}{2}m^2 = -\tfrac{1}{2}(p-k')^2 - \tfrac{1}{2}m^2 = p\cdot k' \end{aligned}$$

so

$$T_1 = -16\ p\cdot k(e'\cdot k)^2 + 8\ p\cdot k\ p\cdot k' . \tag{8.7.33}$$

A similar (though more lengthy) calculation gives

$$\begin{aligned} T_2 = T_3 = &-8(e\cdot k')^2(p\cdot k) + 16\ (e\cdot e')^2 p\cdot k' p\cdot k + 8(e\cdot e')^2 k\cdot k' m^2 \\ &-8(e\cdot e')m^2(k\cdot e')(k'\cdot e) + 8(e'\cdot k)^2 p\cdot k' \\ &-4(k\cdot p)^2 + 4(k\cdot k')(p\cdot p') - 4(k\cdot p')(p\cdot k') , \end{aligned} \tag{8.7.34}$$

$$T_4 = -16\ p\cdot k'(e\cdot k')^2 + 8(p\cdot k)(p\cdot k') , \tag{8.7.35}$$

$$t_1 = t_4 = 0 , \tag{8.7.36}$$

$$t_2 = t_3 = -8\ e\cdot e'\ k\cdot e'\ k'\cdot e + 8(p\cdot k')(e\cdot e')^2 - 4(k\cdot k') . \tag{8.7.37}$$

Combining all these terms in Eq. (8.7.24) gives

$$\sum_{\sigma,\sigma'} |M|^2 = \frac{e^4}{64(2\pi)^6 \omega\omega' p^0 p'^0} \left[\frac{8(k\cdot k')^2}{(k\cdot p)(k'\cdot p)} + 32(e\cdot e')^2 \right] . \tag{8.7.38}$$

All this applies in any Lorentz frame. In the *laboratory* frame, we have the special results

$$k\cdot k' = \omega\omega'(\cos\theta - 1) = m\omega\omega'\left(\frac{1}{\omega} - \frac{1}{\omega'}\right) ,$$

$$p\cdot k = -m\omega \qquad p\cdot k' = -m\omega' .$$

Combining Eq. (8.7.38) with Eq. (8.7.17), the laboratory frame cross-section is

$$\tfrac{1}{2}\sum_{\sigma,\sigma'} d\sigma(\mathbf{p},\sigma\ +\ \mathbf{k},e\ \to\ \mathbf{p}',\sigma'\ +\ \mathbf{k}',e') = \frac{e^4\omega'^2 d\Omega}{64\pi^2 m^2\omega^2}$$
$$\times\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}\ -2\ +4(e\cdot e')^2\right]. \tag{8.7.39}$$

This is the celebrated formula derived (using old-fashioned perturbation theory) by O. Klein and Y. Nishina[4] in 1929.

As discussed in Section 8.6, if the incoming photon is (as usual) not prepared in a state with any particular polarization, then we must average over two orthonormal values of **e**. This average gives

$$\tfrac{1}{2}\sum_e e_i e_j = \tfrac{1}{2}(\delta_{ij}-\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j)$$

and the differential cross-section is then

$$\tfrac{1}{4}\sum_{e,\sigma,\sigma'} d\sigma(\mathbf{p},\sigma+\mathbf{k},e\ \to\ \mathbf{p}',\sigma'+\mathbf{k},e') = \frac{e^4\omega'^2 d\Omega}{64\pi^2 m^2\omega^2}\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-2(\hat{\mathbf{k}}\cdot e')^2\right]. \tag{8.7.40}$$

We see that the scattered photon is preferentially polarized in a direction perpendicular to the incident as well as the final photon direction, i.e., perpendicular to the plane in which the scattering takes place. This is a well-known result, responsible among other things for the polarization of light from eclipsing binary stars.**

To calculate the cross-section for experiments in which the final photon polarization is not measured, we must sum Eq. (8.7.40) over e', using

$$\sum_{e'} e'_i e'_j = \delta_{ij}-\hat{\mathbf{k}}'_i\hat{\mathbf{k}}'_j\,.$$

This gives

$$\tfrac{1}{4}\sum_{e,e',\sigma,\sigma'} d\sigma(\mathbf{p},\sigma\ +\ \mathbf{k},e\ \to\ \mathbf{p}',\sigma'\ +\ \mathbf{k}',e')$$
$$= \frac{e^4\omega'^2 d\Omega}{32\pi^2 m^2\omega^2}\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-1+\cos^2\theta\right], \tag{8.7.41}$$

where θ is the angle between $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$. In the non-relativistic case, $\omega << m$,

** The light from one of the stars is polarized when it is scattered by free electrons in the outer atmosphere of the other, cooler, star when both are along the same line of sight. This polarization is normally undetectable because it cancels when the astronomer adds up light from all parts of the star's disk. The polarization has been observed in eclipsing binary stars at times when the cooler star blocks the light from just one side of the hotter star.

Eq. (8.7.41) gives

$$\tfrac{1}{4}\sum_{e,e',\sigma,\sigma'} d\sigma = \frac{e^4\, d\Omega}{32\pi^2 m^2}(1+\cos^2\theta)\,. \tag{8.7.42}$$

The solid angle integral is

$$\int [1+\cos^2\theta]d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} [1+\cos^2\theta]\ \sin\theta\ d\theta = \frac{16\pi}{3}\,,$$

giving a total cross-section for $\omega \ll m$:

$$\sigma_{\rm T} = \frac{e^4}{6\pi^2 m^2}\,. \tag{8.7.43}$$

This is often written $\sigma_{\rm T} = 8\pi r_0^2/3$, where $r_0 = e^2/4\pi m = 2.818 \times 10^{-13}$ cm is known as the *classical electron radius.* Expression (8.7.43) is called the *Thomson cross-section,* after J. J. Thomson, the discoverer of the electron. Eqs. (8.7.42) and (8.7.43) were originally derived using classical mechanics and electrodynamics, by calculating the reradiation of light by a non-relativistic point charge in a plane wave electromagnetic field.

8.8 Generalization : *p*-form Gauge Fields*

The antisymmetric field strength tensor $F_{\mu\nu}$ of electromagnetism is a special case of a general class of tensors of special importance in physics and mathematics. A *p-form* is an antisymmetric covariant tensor of rank *p*. From a *p*-form $t_{\mu_1,\mu_2,\cdots\mu_p}$ one may construct a $(p+1)$-form called the *exterior derivative*** dt by taking the derivative and then antisymmetrizing with respect to all indices:

$$\begin{aligned}(dt)_{\mu_1\mu_2\cdots\mu_{p+1}} &\equiv \partial_{[\mu_1} t_{\mu_2\mu_3\cdots\mu_{p+1}]} \\ &\equiv \partial_{\mu_1} t_{\mu_2\mu_3\cdots\mu_{p+1}} - \partial_{\mu_2} t_{\mu_1\mu_3\cdots\mu_{p+1}} + \cdots + (-1)^p \partial_{\mu_{p+1}} t_{\mu_1\mu_2\cdots\mu_p}\end{aligned} \tag{8.8.1}$$

with square brackets indicating antisymmetrization with respect to the indices within the brackets. Because derivatives commute, repeated exterior derivatives vanish

$$d(dt) = 0\,. \tag{8.8.2}$$

A *p*-form whose exterior derivative vanishes is called *closed,* while a *p*-form that is itself an exterior derivative is called *exact*. From Eq. (8.8.2)

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* This section lies somewhat out of the book's main line of development, and may be omitted in a first reading.

** Exterior derivatives and *p*-forms play a special role in general relativity, in part because the exterior derivative of a tensor transforms like a tensor even though it is calculated using ordinary rather than covariant derivatives.[5]