Dirac Spinors

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A 4-component spin-one-half Majorana of Dirac field ψ with action density

$$\mathcal{L} = -\overline{\psi}\,(\gamma^a \partial_a + m)\,\psi \equiv -\overline{\psi}\,(\not{\partial} + m)\,\psi \tag{1}$$

(where $\bar{\psi} \equiv \psi^\dagger i\gamma^0$) obeys the Dirac equation

$$(\gamma^a \partial_a + m)\,\psi = (\not{\partial} + m)\,\psi = 0. \tag{2}$$

Two Majorana fields of the same mass ψ_1 and ψ_2 have the Fourier expansions

$$\psi_\ell^{(i)}(x) = \sum_{s=-}^{+} \int \left[u_\ell(\vec{p}, s) a(p, s, i) e^{ipx} + v_\ell(\vec{p}, s) a^\dagger(p, s, i) e^{-ipx} \right] \frac{d^3p}{(2\pi)^{3/2}} \tag{3}$$

while a Dirac field made of them has the Fourier expansion

$$\begin{aligned} \psi_\ell(x) &= \frac{1}{\sqrt{2}} \left(\psi_\ell^{(1)}(x) + i\psi_\ell^{(2)}(x) \right) \\ &= \sum_{s=-}^{+} \int \left[u_\ell(\vec{p}, s) b(p, s) e^{ipx} + v_\ell(\vec{p}, s) c^\dagger(p, s) e^{-ipx} \right] \frac{d^3p}{(2\pi)^{3/2}} \end{aligned} \tag{4}$$

in which the Dirac index ℓ runs from 1 to 4, $px = \vec{p} \cdot \vec{x} - p^0 t$, $p^0 = \sqrt{\vec{p}^2 + m^2}$, and the annihilation a_i and creation $a_j^\dagger$ operators satisfy the anticommutation relations

$$\begin{aligned} \{a(p,s,i), a(p',s',j)\} &\equiv a(p,s,i)\, a(p',s',j) + a(p',s',j)\, a(p,s,i) = 0 \\ \{a(p,s,i), a^\dagger(p',s',j)\} &= \delta_{i,j}\, \delta_{s,s'}\, \delta^{(3)}(\vec{p} - \vec{p}'). \end{aligned} \tag{5}$$

The state $|p,s,i\rangle = a^\dagger(p,s,i)|0\rangle$ represents a particle of type i, 4-momentum p, and spin $s\frac{1}{2}$ in the z-direction. The particle annihilation $b(p,s)$ and antiparticle creation $c^\dagger(p,s)$ operators are

$$b(p,s) = \tfrac{1}{\sqrt{2}}\,(a(p,s,1) + ia(p,s,2)) \quad \text{and} \quad c^\dagger(p,s) = \tfrac{1}{\sqrt{2}}\left(a^\dagger(p,s,1) + ia^\dagger(p,s,2)\right). \tag{6}$$

They satisfy the anticommutation relations

$$\begin{aligned} \{b(p,s), b(p',s')\} &= 0 = \{c(p,s)\, c(p',s')\} = \{b(p,s), c^\dagger(p',s')\} \\ \{b(p,s), b^\dagger(p',s')\} &= \delta_{s,s'}\, \delta^{(3)}(\vec{p} - \vec{p}') = \{c(p,s), c^\dagger(p',s')\}. \end{aligned} \tag{7}$$

The state $|p,s,b\rangle = b^\dagger(p,s)|0\rangle$ represents a particle of 4-momentum p and spin $s\frac{1}{2}$ in the z-direction, while the state $|p,s,c\rangle = c^\dagger(p,s)|0\rangle$ represents an antiparticle of 4-momentum p and spin $s\frac{1}{2}$ in the z-direction.

The Majorana field will satisfy the Dirac equation (2) if the spinors obey the rules

$$(i\gamma^a p_a + m)\, u(\vec{p},s) = 0 = (-i\gamma^a p_a + m)\, v(\vec{p},s) \tag{8}$$

which the spinors will obey if their momentum-dependence is

$$u(\vec{p},s) = (m - i\gamma^a p_a)\, u(\vec{0},s) \quad \text{and} \quad v(\vec{p},s) = (m + i\gamma^a p_a)\, v(\vec{0},s). \tag{9}$$

For we then have

$$\begin{aligned} (i\gamma^a p_a + m)\, u(\vec{p},s) &= -\,(i\gamma^a p_a + m)\left(i\gamma^b p_b - m\right) u(\vec{0},s) \\ &= \left(\gamma^a p_a \gamma^b p_b + m^2\right) u(\vec{0},s) = \left[\left(\tfrac{1}{2}\{\gamma^a,\gamma^b\} + \tfrac{1}{2}[\gamma^a,\gamma^b]\right) p_a p_b + m^2\right] u(\vec{0},s) \\ &= \left(\eta^{ab} p_a p_b + m^2\right) u(\vec{0},s) = \left(-(p^0)^2 + \vec{p}^2 + m^2\right) u(\vec{0},s) = 0 \end{aligned} \tag{10}$$

as well as

$$\begin{aligned}(-i\gamma^a p_a + m)\, v(\vec{p}, s) =& \; -(-i\gamma^a p_a + m)\left(-i\gamma^b p_b - m\right) v(\vec{0}, s) \\ =& \; \left(\gamma^a p_a \gamma^b p_b + m^2\right) v(\vec{0}, s) = \left[\left(\tfrac{1}{2}\{\gamma^a, \gamma^b\} + \tfrac{1}{2}[\gamma^a, \gamma^b]\right) p_a p_b + m^2\right] v(\vec{0}, s) \\ =& \; \left(\eta^{ab} p_a p_b + m^2\right) v(\vec{0}, s) = \left(-(p^0)^2 + \vec{p}^2 + m^2\right) v(\vec{0}, s) = 0.\end{aligned} \tag{11}$$

The zero-momentum spinors obey the rules (8) at $\vec{p} = 0$

$$\begin{aligned}\left(i\gamma^0 k_0 + m\right) u(\vec{0}, s) =& \; \left(-i\gamma^0 m + m\right) u(\vec{0}, s) = 0 \\ \left(-i\gamma^0 k_0 + m\right) v(\vec{0}, s) =& \; \left(i\gamma^0 m + m\right) v(\vec{0}, s) = 0.\end{aligned} \tag{12}$$

That is,

$$i\gamma^0 u(\vec{0}, s) = u(\vec{0}, s) \quad \text{and} \quad i\gamma^0 v(\vec{0}, s) = -v(\vec{0}, s) \tag{13}$$

or since

$$\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and incidentally} \quad \vec{\gamma} = -i \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{14}$$

the zero-momentum spinors satisfy

$$\begin{aligned}i\gamma^0 u(\vec{0}, s) =& \; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u(\vec{0}, s) = u(\vec{0}, s) \\ i\gamma^0 v(\vec{0}, s) =& \; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v(\vec{0}, s) = -v(\vec{0}, s).\end{aligned} \tag{15}$$

The natural choices are

$$u(\vec{0}, +) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u(\vec{0}, -) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v(\vec{0}, +) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v(\vec{0}, -) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{16}$$

They are suitably orthonormal

$$\begin{aligned} u^\dagger(\vec{0}, s) u(\vec{0}, s') &= \delta_{s,s'} \\ v^\dagger(\vec{0}, s) v(\vec{0}, s') &= \delta_{s,s'} \\ u^\dagger(\vec{0}, s) v(\vec{0}, s') &= 0 \\ v^\dagger(\vec{0}, s) u(\vec{0}, s') &= 0. \end{aligned} \tag{17}$$

Because the zero-momentum spinors are eigenvectors of γ^0, we can rewrite the momentum dependence (9) of the spinors as

$$\begin{aligned} u(\vec{p}, s) = &\ (-i\gamma^a p_a + m)\, u(\vec{0}, s) = \left(-i\gamma^a p_a i\gamma^0 + m\right) u(\vec{0}, s) = \left(\gamma^a p_a \gamma^0 + m\right) u(\vec{0}, s) \\ v(\vec{p}, s) = &\ (+i\gamma^a p_a + m)\, v(\vec{0}, s) = \left(i\gamma^a p_a(-i\gamma^0) + m\right) v(\vec{0}, s) = \left(\gamma^a p_a \gamma^0 + m\right) v(\vec{0}, s). \end{aligned} \tag{18}$$

We may use the orthonormality (17) of the zero-momentum spinors, their dependence (18) upon their momentum, and the relations $\vec{\gamma}^\dagger = \vec{\gamma}$ and $\gamma^{0\dagger} = -\gamma^0$ to show that the spinors $u(\vec{p}, s)$ are orthonormal

$$\begin{aligned} u^\dagger(\vec{p}, s) u(\vec{p}, s') &= u^\dagger(\vec{0}, s)\, (m - ip_a\gamma^a)^\dagger \left(m - ip_b\gamma^b\right) u(\vec{0}, s')/[2p^0(p^0 + m)] \\ &= u^\dagger(\vec{0}, s) \left(i\vec{\gamma}\cdot\vec{p} + ip^0\gamma^0 + m\right)\left(-i\vec{\gamma}\cdot\vec{p} + ip^0\gamma^0 + m\right) u(\vec{0}, s')/[2p^0(p^0 + m)] \\ &= u^\dagger(\vec{0}, s) \left[(m + ip^0\gamma^0)^2 + (\vec{\gamma}\cdot\vec{p})^2\right] u(\vec{0}, s')/[2p^0(p^0 + m)] \\ &= u^\dagger(\vec{0}, s) \left[m^2 + p^{02} + 2mip^0\gamma^0 + (\vec{p})^2\right] u(\vec{0}, s')/[2p^0(p^0 + m)] \\ &= u^\dagger(\vec{0}, s) \left[m^2 + p^{02} + 2mp^0 + (\vec{p})^2\right] u(\vec{0}, s')/[2p^0(p^0 + m)] \\ &= u^\dagger(\vec{0}, s) u(\vec{0}, s') = \delta_{s,s'}. \end{aligned} \tag{19}$$

More generally, one has

$$\begin{aligned} u^\dagger(\vec{p}, s) u(\vec{p}, s') &= \delta_{s,s'} \\ v^\dagger(\vec{p}, s) v(\vec{p}, s') &= \delta_{s,s'} \\ u^\dagger(\vec{p}, s) v(-\vec{p}, s') &= 0 \\ v^\dagger(\vec{p}, s) u(-\vec{p}, s') &= 0. \end{aligned} \tag{20}$$

We have seen in the notes on group theory that the Dirac representation of the Lorentz group is the direct sum of the $D^{(1/2,0)}$ and $D^{(0,1/2)}$ representations. The standard boost that takes $k \equiv (m,0,0,0)$ into $p = (p^0, \vec{p}) = Lk$ is

$$L(p) = R(p)B(p)R^{-1}(p) \tag{21}$$

in which $B(p)$ is the boost in the z-direction that takes the 4-vector $(m,0,0,0)$ to $(p^0,0,0,p)$ and the rotation $R(p)$ rotates the z-axis into the direction $\hat{\boldsymbol{p}}$. In the group-theory notes, we saw that

$$D^{(1/2,0)}(L) = \frac{p^0 + m - \vec{p}\cdot\vec{\sigma}}{\sqrt{2m(p^0+m)}} \quad \text{and} \quad D^{(0,1/2)}(L) = \frac{p^0 + m + \vec{p}\cdot\vec{\sigma}}{\sqrt{2m(p^0+m)}}. \tag{22}$$

Thus the 4×4 matrix that represents the standard boost L is $D(L) = D^{(1/2,0)}(L) \oplus D^{(0,1/2)}(L)$

$$D(L) = \begin{pmatrix} D^{(1/2,0)}(L) & 0 \\ 0 & D^{(0,1/2)}(L) \end{pmatrix} = \frac{1}{\sqrt{2m(p^0+m)}} \begin{pmatrix} p^0 + m - \boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \\ 0 & p^0 + m + \boldsymbol{p}\cdot\boldsymbol{\sigma} \end{pmatrix}. \tag{23}$$

As a first homework problem, show that

$$D(L) = \frac{\gamma^a p_a \gamma^0 + m}{\sqrt{2m(p^0+m)}}. \tag{24}$$

It is useful to further scale the spinors $u(\vec{p},s)$ and $v(\vec{p},s)$ so that

$$u(\vec{p},s) = \sqrt{\frac{m}{p^0}}\, D(L)\, u(\vec{0},s) \quad \text{and} \quad v(\vec{p},s) = \sqrt{\frac{m}{p^0}}\, D(L)\, v(\vec{0},s). \tag{25}$$

As a second homework problem, show that the spin sums of the zero-momentum spinors are

$$\begin{aligned} \sum_{s=-}^{+} u_\ell(\vec{0},s) u^*_{\ell'}(\vec{0},s) &= \frac{1}{2}\left(I + i\gamma^0\right)_{\ell\ell'} = \frac{1}{2}\left(\delta_{\ell\ell'} + i\gamma^0_{\ell\ell'}\right) \\ \sum_{s=-}^{+} v_\ell(\vec{0},s) v^*_{\ell'}(\vec{0},s) &= \frac{1}{2}\left(I - i\gamma^0\right)_{\ell\ell'} = \frac{1}{2}\left(\delta_{\ell\ell'} - i\gamma^0_{\ell\ell'}\right). \end{aligned} \tag{26}$$

As a third homework problem, show that the spin sums of the spinors are

$$\begin{aligned}\sum_{s=-}^{+} u_\ell(\vec{p},s) u^*_{\ell'}(\vec{p},s) &= \frac{1}{2p^0}\left[\left(-i\gamma^a p_a + m\right) i\gamma^0\right]_{\ell\ell'} \\ \sum_{s=-}^{+} v_\ell(\vec{p},s) v^*_{\ell'}(\vec{p},s) &= \frac{1}{2p^0}\left[\left(-i\gamma^a p_a - m\right) i\gamma^0\right]_{\ell\ell'} .\end{aligned} \tag{27}$$

The effect of a unitary transformation $U(\Lambda)$ that implements a Lorentz transformation Λ on a state $|p,s,b\rangle$ is

$$U(\Lambda)|p,s,b\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'=-j}^{j} D^{(j)}_{s's}(W(\Lambda,p))|\Lambda p, s', b\rangle \tag{28}$$

in which the $(2j+1)\times(2j+1)$ unitary matrix $D^{(j)}_{s's}(W(\Lambda,p))$ represents the Wigner rotation

$$W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p) \tag{29}$$

where $L(p)$ is the standard boost (21) that takes the 4-vector $(m,0,0,0)$ to p.

Let's compute the Feynman propagator for a Dirac field. It is defined as the mean value in the vacuum (of the theory without interactions (the free theory)) of the time-ordered product defined as

$$\begin{aligned}\langle 0|\mathcal{T}[\psi_\ell(x)\psi^\dagger_m(y)]|0\rangle &\equiv \langle 0|\theta(x^0-y^0)\psi_\ell(x)\psi^\dagger_m(y) - \theta(y^0-x^0)\psi^\dagger_m(y)\psi_\ell(x)|0\rangle \\ &= \langle 0|\theta(x^0-y^0)\psi^+_\ell(x)\psi^{\dagger-}_m(y) - \theta(y^0-x^0)\psi^{\dagger+}_m(y)\psi^-_\ell(x)|0\rangle \\ &= \langle 0|\theta(x^0-y^0)\{\psi^+_\ell(x),\psi^{\dagger-}_m(y)\} - \theta(y^0-x^0)\{\psi^{\dagger+}_m(y)\psi^-_\ell(x)\}|0\rangle \\ &= \theta(x^0-y^0)\{\psi^+_\ell(x),\psi^{\dagger-}_m(y)\} - \theta(y^0-x^0)\{\psi^{\dagger+}_m(y),\psi^-_\ell(x)\}.\end{aligned} \tag{30}$$

We can use the Fourier expansion of the Dirac field (4) and of its adjoint

$$\begin{aligned}\psi^\dagger_\ell(x) &= \frac{1}{\sqrt{2}}\left(\psi^{(1)\dagger}_\ell(x) - i\psi^{(2)\dagger}_\ell(x)\right) \\ &= \sum_{s=-}^{+}\int \left[u^*_\ell(\vec{p},s)b^\dagger(p,s)e^{-ipx} + v^*_\ell(\vec{p},s)c(p,s)e^{+ipx}\right]\frac{d^3p}{(2\pi)^{3/2}}\end{aligned} \tag{31}$$

as well as the spin sums (27) to compute these anticommutators:

$$\begin{aligned}
\{\psi_\ell^+(x), \psi_m^{\dagger -}(y)\} &= \sum_{s,t=-}^{+} \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3q}{(2\pi)^{3/2}} u_\ell(\vec{p}, s) e^{ipx} u_m^*(\vec{q}, t) e^{-iqy} \{b(p,s), b^\dagger(q,t)\} \\
&= \sum_{s,t=-}^{+} \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3q}{(2\pi)^{3/2}} u_\ell(\vec{p}, s) e^{ipx} u_m^*(\vec{q}, t) e^{-iqy} \delta_{s,t} \delta^{(3)}(\vec{p} - \vec{q}) \\
&= \sum_{s=-}^{+} \int \frac{d^3p}{(2\pi)^3} u_\ell(\vec{p}, s) u_m^*(\vec{p}, s) e^{ip(x-y)} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left[(-i\gamma^a p_a + m)\, i\gamma^0\right]_{\ell m} e^{ip(x-y)} \\
&= \left[(-\gamma^a \partial_a + m)\, i\gamma^0\right]_{\ell m} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{ip(x-y)} = \left[(-\gamma^a \partial_a + m)\, i\gamma^0\right]_{\ell m} \Delta_+(x-y)
\end{aligned} \tag{32}$$

and since $\{\psi_m^{\dagger +}(y), \psi_\ell^-(x)\} = \{\psi_\ell^-(x), \psi_m^{\dagger +}(y)\}$

$$\begin{aligned}
\{\psi_\ell^-(x), \psi_m^{\dagger +}(y)\} &= \sum_{s,t=-}^{+} \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3q}{(2\pi)^{3/2}} v_\ell(\vec{p}, s) e^{-ipx} v_m^*(\vec{q}, t) e^{iqy} \{c^\dagger(p,s), c(q,t)\} \\
&= \sum_{s,t=-}^{+} \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3q}{(2\pi)^{3/2}} v_\ell(\vec{p}, s) e^{-ipx} v_m^*(\vec{q}, t) e^{iqy} \delta_{s,t} \delta^{(3)}(\vec{p} - \vec{q}) \\
&= \sum_{s=-}^{+} \int \frac{d^3p}{(2\pi)^3} v_\ell(\vec{p}, s) v_m^*(\vec{p}, s) e^{-ip(x-y)} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left[(-i\gamma^a p_a - m)\, i\gamma^0\right]_{\ell m} e^{ip(x-y)} \\
&= \left[(\gamma^a \partial_a - m)\, i\gamma^0\right]_{\ell m} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ip(x-y)} = \left[(\gamma^a \partial_a - m)\, i\gamma^0\right]_{\ell m} \Delta_+(y-x).
\end{aligned} \tag{33}$$

Putting the two anticommutators together, we find

$$\begin{aligned}\langle 0|\mathcal{T}[\psi_\ell(x)\psi^\dagger_m(y)]|0\rangle &\equiv \langle 0|\theta(x^0-y^0)\psi_\ell(x)\psi^\dagger_m(y) - \theta(y^0-x^0)\psi^\dagger_m(y)\psi_\ell(x)|0\rangle \\ &= \theta(x^0-y^0)\{\psi^+_\ell(x),\psi^{\dagger-}_m(y)\} - \theta(y^0-x^0)\{\psi^{\dagger+}_m(y),\psi^-_\ell(x)\} \\ &= \theta(x^0-y^0)\left[(-\gamma^a\partial_a+m)\,i\gamma^0\right]_{\ell m}\Delta_+(x-y) - \theta(y^0-x^0)\left[(\gamma^a\partial_a-m)\,i\gamma^0\right]_{\ell m}\Delta_+(y-x) \\ &= \theta(x^0-y^0)\left[(-\gamma^a\partial_a+m)\,i\gamma^0\right]_{\ell m}\Delta_+(x-y) + \theta(y^0-x^0)\left[(-\gamma^a\partial_a+m)\,i\gamma^0\right]_{\ell m}\Delta_+(y-x).\end{aligned} \tag{34}$$

We can pull the spatial derivatives and the mass term across the Heaviside functions. To pull the time derivatives, we note that the extra terms actually vanish

$$(\partial_0\theta(x^0-y^0))\Delta_+(x-y) + (\partial_0\theta(y^0-x^0))\Delta_+(y-x) = \delta(x^0-y^0)\Delta_+(x-y) - \delta(x^0-y^0)\Delta_+(y-x) = 0 \tag{35}$$

because at equal times the two invariant functions cancel

$$\Delta_+(y-x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} = \Delta_+(x-y). \tag{36}$$

Thus

$$\begin{aligned}\langle 0|\mathcal{T}[\psi_\ell(x)\psi^\dagger_m(y)]|0\rangle &= \left[(-\gamma^a\partial_a+m)\,i\gamma^0\right]_{\ell m}\left[\theta(x^0-y^0)\Delta_+(x-y)+\theta(y^0-x^0)\Delta_+(y-x)\right] \\ &= \left[(-\gamma^a\partial_a+m)\,i\gamma^0\right]_{\ell m}\left[-i\Delta_F(x-y)\right]\end{aligned} \tag{37}$$

where

$$\Delta_F(x-y) = \int \frac{d^4q}{(2\pi)^4}\frac{\exp(iq(x-y))}{q^2+m^2-i\epsilon}. \tag{38}$$

Cancelling the is, we have

$$\begin{aligned}\langle 0|\mathcal{T}[\psi_\ell(x)\psi^\dagger_m(y)]|0\rangle &= \left[(-\gamma^a\partial_a+m)\,\gamma^0\right]_{\ell m}\Delta_F(x-y) \\ &= \left[(-\gamma^a\partial_a+m)\,\gamma^0\right]_{\ell m}\int \frac{d^4q}{(2\pi)^4}\frac{\exp(iq(x-y))}{q^2+m^2-i\epsilon} \\ &= \int \frac{d^4q}{(2\pi)^4}\frac{\left[(-i\gamma^a q_a+m)\,\gamma^0\right]_{\ell m}}{q^2+m^2-i\epsilon}e^{iq(x-y)}.\end{aligned} \tag{39}$$

As a Thanksgiving vacation homework problem, compute the equal-time anticommutators

$$\{\psi_\ell(t,\vec{x}),\psi_m(t,\vec{y})\} \quad \text{and} \quad \{\psi_\ell(t,\vec{x}),\psi_m^\dagger(t,\vec{y})\} \tag{40}$$

of the Dirac field (4) with itself and with its adjoint (31).