The Uehling effect

$$\Delta E = |\psi(0)|^2 \int d^3r\, \Delta V(r)$$

$$= |\psi(0)|^2 \int d^3r \frac{e_1 e_2}{(2\pi)^3} \int d^3q\, e^{i\vec{q}\cdot\vec{r}} \frac{\Pi(\vec{q}^{\,2})}{\vec{q}^{\,2}}$$

$$= |\psi(0)|^2 e_1 e_2 \lim_{q^2 \to 0} \frac{\Pi(q^2)}{q^2}$$

By (11.2.22),

$$\lim_{q^2\to 0} \frac{\Pi(q^2)}{q^2} = \frac{e^2}{2\pi^2} \int_0^1 x(1-x) \ln\left(1 + \frac{q^2 x(1-x)}{m^2}\right) \frac{dx}{q^2}$$

$$= \frac{e^2}{2\pi^2} \int_0^1 \frac{x^2(1-x)^2}{m^2}\, dx$$

$$= \frac{e^2}{2\pi^2 m^2} \frac{1}{30} = \frac{e^2}{60\pi^2 m^2}.$$

So

$$\Delta E = |\psi(0)|^2 \frac{e_1 e_2\, e^2}{60\pi^2 m^2} \qquad \text{set } e_1 e_2 = -Ze^2$$

$$= -|\psi(0)|^2 \frac{Z e^4}{60\pi^2 m^2} \qquad \alpha = \frac{e^2}{4\pi}$$

$$\Delta E = -|\psi(0)|^2 \frac{Z\,(4\pi)^2\alpha^2}{60\,\pi^2 m^2}$$

$$= -\frac{4Z\alpha^2}{15\,m^2}\,|\psi(0)|^2$$

for hydrogenic atom:

$$\psi(0) = \frac{2}{\sqrt{4\pi}}\left(\frac{Z\alpha m}{n}\right)^{3/2} \quad s\,0$$

$$\Delta E = -\frac{4Z\alpha^2}{15\,m^2}\,\frac{4}{4\pi}\left(\frac{Z\alpha m}{n}\right)^3$$

$$= \frac{-4Z^4\alpha^5 m}{15\,\pi\,n^3}$$

in 2s state of H $\Delta E = -1.122\,10^{-7}$ eV.