Poisson brackets

$$\Pi_a \equiv \frac{\partial L}{\partial \dot{\psi}^a} \qquad \text{where} \quad a = (l, \vec{x}) \quad \text{etc.}$$

$$[A, B]_P \equiv \frac{\partial A}{\partial \psi^a}\frac{\partial B}{\partial \Pi_a} - \frac{\partial B}{\partial \psi^a}\frac{\partial A}{\partial \Pi_a}$$

in which the constraints are ignored during the differentiation. So $[\psi^a, \Pi_b]_P = \delta^a_b$.

P.B.'s are like commutators:

$$[A, B]_P = -[B, A]_P$$

$$[A, BC]_P = [A, B]_P\, C + B\, [A, C]_P$$

and

$$[A, [B, C]_P]_P + [B, [C, A]_P]_P + [C, [A, B]_P]_P = 0$$

which is Jacobi's Identity.

In theories without constraints

$$[A, B] = i\, [A, B]_P.$$

Constraints

Let's write the primary and secondary constraints as $\chi_N = 0$ for $N = 1, 2,$ etc.
The classical equation of motion for A is

$$\dot{A} = [A, H]_P ,$$

and so if the $\chi_N = 0$, then

$$0 = \dot{\chi}_N = [\chi_N, H]_P = 0.$$

A constraint χ_N is <u>first class</u>

if

$$[\chi_N, \chi_M]_P = 0$$

for all M. In E&M both $\Pi_0 = 0$ and $\nabla \cdot E - J^0 = 0$ are first-class constraints.

The full set of first-class constraints $\chi_N = 0$ is related to a group of symmetries under which

$$\delta_\epsilon A \equiv \sum_N \epsilon_N [\chi_N, A]_P .$$

These are symmetries of the hamiltonian:

$$\delta_\epsilon H = \sum \epsilon_N [\chi_N, H]_P = 0$$

Since N contains x, these are local symmetries.
If the χ_N's are first class, $[\chi_N, \chi_M]_P = 0$, then we resolve them by choosing a gauge.

Second-class constraints

After we've solved the first-class constraints by a choice of gauge, the remaining ones χ_N are such that

$$\sum_N u_N [\chi_N, \chi_M]_P \neq 0$$

for all $u_N \neq 0$. That is,

$$C_{NM} \equiv [\chi_N, \chi_M]_P$$

is a non-singular matrix

$$\mathrm{Det}\, C \neq 0.$$

These remaining constraints are <u>second class</u>.

They are always even in number because all $(2n+1)\times(2n+1)$ A.S. matrices are singular. E.g.

$$\mathrm{Det}\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = -a(bc) + b(ac) = 0$$

Example: the massive vector field

$$\chi_{1x} = \chi_{2x} = 0$$

$$\chi_{1x} = \Pi_0(\vec{x}) \qquad \chi_{2x} = \partial_i \Pi_i(x) - m^2 V^0(x) + J^0(x)$$

the P.b. of these is

$$[\chi_{1x}, \chi_{2y}]_P = \sum_a \int d^3z \left(\frac{\partial \Pi_0(x)}{\partial V^a(z)} \frac{\partial(\nabla\cdot\Pi(y) - m^2 V^0(y) + J^0(y))}{\partial \Pi_a(z)} \right.$$

$$\left. - \frac{\partial(\nabla\cdot\Pi(y) - m^2 V^0(y) + J^0(y))}{\partial V^a(z)} \frac{\partial \Pi_0(x)}{\partial \Pi_a(z)} \right)$$

$$= -\sum_a \delta^a_0 \int \delta^3(x-z) \frac{\partial(\nabla\cdot\Pi(y) - m^2 V^0(y) + J^0(y))}{\partial V^a(z)} d^3z$$

$$= - \frac{\partial[\nabla\cdot\Pi(y) - m^2 V^0(y) + J^0(y)]}{\partial V^0(x)} = m^2 \delta^{(3)}(\vec{x}-\vec{y})$$

$$= -C_{2\vec{y},1\vec{x}}$$

Also $[\chi_{1x}, \chi_{1y}]_P = 0$ and $[\chi_{2x}, \chi_{2y}]_P = 0$

So

$$C = m^2 \left(\begin{array}{cccc|cccc} & & & & \delta & & & \\ & 0 & & & & \delta & & \\ & & & & & & \delta & \\ & & & & & & & \delta \\ \hline -\delta & & & & & & & \\ & -\delta & & & & 0 & & \\ & & -\delta & & & & & \\ & & & -\delta & & & & \end{array}\right)$$

$\det C = \delta^\infty \neq 0.$

So these are second-class constraints.

Dirac brackets

Dirac suggested

$$[A,B] = i[A,B]_D$$

where

$$[A,B]_D \equiv [A,B]_P - [A,\chi_N]_P (C^{-1})^{NM} [\chi_M,B]_P .$$

Now

$$[B,A]_D = -[A,B]_P - [\chi_M,A]_P (C^{-1})^{NM} [B,\chi_N]_P$$
$$= -[A,B]_P - [A,\chi_M]_P (C^{-1})^{NM} [\chi_N,B]_P$$
$$= -[A,B]_P + [A,\chi_M]_P (C^{-1})^{MN} [\chi_N,B]_P \qquad (C \text{ is A.S.})$$
$$= -[A,B]_D .$$

Also

$$[A,BC]_D = [A,B]_D C + B[A,C]_D \quad (\text{cf. } 387.1) \quad \text{and}$$

$$[A,[B,C]_D]_D + [B,[C,A]_D]_D + [C,[A,B]_D]_D = 0$$

Also

$$[\chi_N,A]_D = [\chi_N,A]_P - [\chi_N,\chi_M]_P (C^{-1})^{MR} [\chi_R,A]_P$$
$$= [\chi_N,A]_P - C_{NM} (C^{-1})^{MR} [\chi_R,A]_P$$
$$= [\chi_N,A]_P - [\chi_N,A]_P = 0$$

So

$$[\chi_N,A]_D = 0 \qquad \text{for all } A.$$

Thus

$$[\chi_N,A] = i[\chi_N,A]_D = 0$$

which makes the commutators consistent with the constraints $\chi_N = 0$.

$$[A, BC]_D = [A, BC]_P - [A, \chi_N]_P \, C^{-1NM} [\chi_M, BC]_P$$

$$= [A, B]_P C + B[A, C]_P - [A, \chi_N]_P \, C^{-1NM} \left([\chi_M, B]_P C + B[\chi_M, C]_P \right)$$

$$= [A, B]_D C + B[A, C]_D$$

And as long as $\{\chi_N' = 0\}$ and $\{\chi_N = 0\}$ define the same submanifold of phase space, then

$$[A,B]_D = [A,B]_{D'}.$$

Moreover Maskawa and Nakajima have shown that from any set q^a, π_a with second-class constraints $\chi_N = 0$, there is a canonical* transformation to the variables Q^n, q^r and P_n, p_s such that $\{\chi_N = 0\} \Longleftrightarrow q^r = p_s = 0$. In the new coordinates, with $\chi_{1r} = q^r$ $\chi_{2s} = p_s$ we have

$$C_{1r,2s} = [q^r, p_s]_P = \delta^r_s$$

$$C_{1r,1s} = [q^r, q^s]_P = 0 \qquad C_{2r,2s} = [p_r, p_s]_P = 0$$

and for any A, B

$$[A, \chi_{1r}]_P = [A, q^r]_P = -\frac{\partial A}{\partial p_r}$$

$$[A, \chi_{2s}]_P = [A, p_s]_P = \frac{\partial A}{\partial q^s}.$$

Now

$$C = \begin{pmatrix} 0 & \begin{smallmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{smallmatrix} \\ \begin{smallmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{smallmatrix} & 0 \end{pmatrix} \qquad C^{-1} = -C = \begin{pmatrix} 0 & \begin{smallmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{smallmatrix} \\ \begin{smallmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{smallmatrix} & 0 \end{pmatrix}$$

* i.e., Poisson brackets unchanged

$$[\chi_{2n}, B]_P = [p_n, B]_P = -\frac{\partial B}{\partial q_n}$$

$$[\chi_{1n}, B]_P = [q_n, B]_P = \frac{\partial B}{\partial p_n}$$

So the Dirac bracket $[A,B]_D$ is

$$[A,B]_D = [A,B]_P - [A,\chi_N]_P (C^{-1})^{NM} [\chi_M, B]_P$$

$$[A,B]_D = [A,B]_P + [A,\chi_{1n}]_P [\chi_{2n}, B]_P$$

$$- [A, \chi_{2n}]_P [\chi_{1n}, B]_P$$

$$= [A,B]_P + \left(-\frac{\partial A}{\partial p_r}\right)\left(-\frac{\partial B}{\partial q^r}\right) - \frac{\partial A}{\partial q^r}\frac{\partial B}{\partial p_r}$$

$$= \sum_n \frac{\partial A}{\partial Q^n}\frac{\partial B}{\partial P_n} - \frac{\partial B}{\partial Q^n}\frac{\partial A}{\partial P_n} = [A,B]_{PR}$$

OK ↓

In terms of these variables, the Dirac bracket is the Poisson bracket computed with only the <u>restricted</u> set of canonical variables Q_n, P_n.

Example: MVF $\quad \chi_{1\vec{x}} = \Pi_0(\vec{x})$

$$\chi_{2\vec{x}} = \nabla \cdot \vec{\Pi}_0(\vec{x}) - m^2 V^0(\vec{x}) + J^0(\vec{x})$$

$$C_{1x,2y} = -C_{2y,1x} = [\chi_{1x}, \chi_{2y}]_P = m^2 \delta(\vec{x}-\vec{y})$$

$$(C^{-1})^{1x,2y} = -(C^{-1})^{2y,1x} = -m^{-2}\delta^3(\vec{x}-\vec{y})$$

Thus

$$[A,B] = i[A,B]_p - i\int [A, \Pi_0(z)]_p \, (-m^{-2}\delta(z-w))$$

$$[\nabla\cdot\Pi(w) - m^2 V^0(w) + J^0(w), B]_p \, d^3z \, d^3w$$

$$-i\int d^3z\, d^3w\, [A, \nabla\cdot\Pi(w) - m^2V^0(w) + J^0(w)]_p (+m^{-2}\delta(z-w))$$

$$[\Pi_0(z), B]_p$$

$$= i[A,B]_p + i m^{-2}\int d^3z \, [A, \Pi_0(z)]_p \, [\nabla\cdot\Pi(z) - m^2V^0(z) + J^0(z), B]_p$$

$$-i m^{-2}\int d^3z\, [A, \nabla\cdot\Pi(z) - m^2V^0(z) + J^0(z)]_p [\Pi_0(z), B]_p$$

Now $[V^\mu(x), \Pi_\nu(y)]_p = \delta(x-y)\,\delta^\mu_\nu$

$[V^\mu, V^\nu] = [\Pi_\mu, \Pi_\nu] = 0$

So let's compute

$$[V^i(x), V^0(y)] = i[V^i(x), V^0(y)]_p$$

$$+ i m^{-2}\int d^3z \, [V^i(x), \nabla\cdot\Pi(z)]_p [\Pi_0(z), V^0(y)]_p$$

$$= -i m^{-2}\int d^3z \, \delta^i_j \, \partial^z_j \, \delta(x-z)(-1)\,\delta(z-y)$$

$$= i m^{-2}\int d^3z \, \partial^z_i \, \delta(x-z)\,\delta(z-y)$$

$$= -i m^{-2}\int d^3z \, \partial^x_i \, \delta(x-z)\,\delta(z-y)$$

$$= -i m^{-2}\, \partial_i \, \delta(x-y)$$

This agrees with

$$[V^i(x), V^0(y)] = [V^i(x), m^{-2}(\nabla\cdot\pi - j^0)]$$

$$= m^{-2}\nabla^y_j [V^i(x), \pi_j(y)]$$

$$= m^{-2}\partial^y_j\, i\,\delta(x-y) = -i\,m^{-2}\partial_j \delta(x-y).$$

So we just assume $[Q^n, P_m] = i\,\delta^n_m\,\delta(x-y)$ and use constraints to evaluate the other commutators involving the C^n's.