Chapter 7

A classical, given, lagrangian density with a symmetry when canonically quantized leads to a quantum theory that exhibits the symmetry if this is possible.

Canonical Variables

The free fields in chapter 5 are systems of \( q^m(x, t) \) and canonical conjugates \( p_m(x, t) \) that satisfy the canonical (anti) commutation relations:

\[
\begin{align*}
&[q^m(x, t), p_m(y, t)] = i \delta^3(x - y) \\delta^{(4)}_{m, n} \\
&[q^m(x, t), q^n(y, t)] = 0 \\
&[p_m(x, t), p_n(y, t)] = 0
\end{align*}
\]

Note the pairs are at equal times, \( t_x = t_y = t \).

For example, the real scalar field \( \phi(x) \) in the \( \alpha^2 = \alpha \), spin-\( 2 \) case:

\[
\phi(x) = \phi^\uparrow(x) + \phi^\downarrow(x)
\]

satisfies (5.2.6)

\[
\begin{align*}
&[\phi^\uparrow(x), \phi^\uparrow(y)] = \Delta^\uparrow(x - y) \\
&[\phi(x), \phi(y)] = [\phi^\uparrow(x), \phi^\uparrow(y)] + [\phi^\downarrow(x), \phi^\downarrow(y)] -
\end{align*}
\]

\[
\begin{align*}
&= \Delta^\uparrow(x - y) - \Delta^\downarrow(y - x) = \Delta(x - y)
\end{align*}
\]

by the definition (5.2.13) of \( \Delta \),

\[
\Delta(x) = \Delta^\uparrow(x) - \Delta^\downarrow(x) = \int \frac{d^4p}{(2\pi)^3} \left( e^{ipx} - e^{-ipx} \right).
\]
Here \( p^0 = \sqrt{p_o^2 + \mathbf{p}^2} \).

Now \( \Delta(x^0, 0) = 0 \) and

\[
\dot{\Delta}(x^0, 0) = \left. \frac{\partial}{\partial t} \Delta(x^0, t) \right|_{t=0} = \int \frac{d^3 p}{(2\pi)^3} \left( \frac{i p^0}{\sqrt{p_o^2 + \mathbf{p}^2}} \right) e^{-i p^0 x^0 - \mathbf{p} \cdot \mathbf{x}} \left( \frac{-1}{p^0} \right) \left( \frac{1}{\sqrt{p_o^2 + \mathbf{p}^2}} \right) \frac{d^3 p}{(2\pi)^3} = -i \int \frac{d^3 p}{(2\pi)^3} e^{-i p^0 x^0} = -i \delta^{(3)}(\mathbf{x}).
\]

So \( \Delta(x^0, -\mathbf{x}, 0) = 0 \) and thus

\[
[\phi(x^0, t), \phi(y^0, t)] = \Delta(x^0, -\mathbf{y}, 0) = 0.
\]

And

\[
[\phi(x^0, t), \partial_0 \phi(y^0, t)] = \partial_0 \Delta(x^0 - \mathbf{y}, 0)
\]

\[
= -\frac{\partial}{\partial x^0} \Delta(x^0, y^0) \bigg|_{x^0 = y^0} = i \delta^{(3)}(x^0 - \mathbf{y}).
\]

So \( [\phi(x^0, t), \phi(y^0, t)] = i \delta^{(3)}(x^0 - \mathbf{y}). \)

Finally,

\[
[\dot{\phi}(x^0, t), \phi(y^0, x)] = \left. \frac{\partial^2}{\partial x^0 \partial y^0} \Delta(x^0 - \mathbf{y}, 0) \bigg|_{x^0 = y^0} \right. = i p^0 \left( \frac{i p^0}{\sqrt{p_o^2 + \mathbf{p}^2}} \right) e^{-i p^0 x^0 - \mathbf{p} \cdot \mathbf{x}} - i p^0 \left( \frac{-1}{p^0} \right) \left( \frac{1}{\sqrt{p_o^2 + \mathbf{p}^2}} \right) \frac{d^3 p}{(2\pi)^3} = 0.
\]

So \( [\dot{\phi}(x^0, t), \phi(y^0, t)] = 0. \)
So the canonical variables are
\[ q(x, t) = \phi(x, t) \quad \text{and} \quad p(x, t) = \dot{\phi}(x, t), \]

The complex spinor field is
\[ \phi(x, t) = \sqrt{i} \int \frac{d^3 \xi}{(2\pi)^3} e^{i \frac{x \cdot \xi}{\hbar}} \phi(\xi, t) \]

and
\[ i \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x} \cdot \frac{\dot{\phi}}{\hbar} + \frac{\partial}{\partial \xi} \cdot \frac{\phi}{\hbar} \]

becomes
\[ \Delta(x, y) = -i \delta^3(x - y) \]

Also,
\[ \left[ \phi(x, t), \phi^+(y, \tau) \right] = 0. \]

So we may take \( q(x, t) = \phi(x, t) \) and \( p(x, t) = \dot{\phi}(x, t) \).

Or we may use real variables
\[ \phi = \frac{1}{\sqrt{2}} (\phi_i + i \phi_e), \quad \phi_i = \frac{1}{\sqrt{2}} (\phi_i + \phi^+_e), \quad \phi_e = \frac{1}{i \sqrt{2}} (\phi_i - \phi^+_e) \]

Then
\[ \left[ \phi_i(x, \xi), \phi_e(y, \eta) \right] = 0 \]

\[ \left[ \phi_i(x, \xi), \phi_i^+(y, \eta) \right] = \frac{1}{2i} \left[ \phi_i(x, \xi) + \phi_i^+(x, \xi), \phi_i(y, \eta) + \phi_i^+(y, \eta) \right] \]

\[ = i \delta(\vec{x} - \vec{y}) \quad \text{while} \]

\[ \left[ \phi_e(x, \xi), \phi_e(y, \eta) \right] = \frac{\sqrt{2}}{2i} \left[ \phi_e(x, \xi) + \phi_e^+(x, \xi), \phi_e(y, \eta) + \phi_e^+(y, \eta) \right] \]

\[ = \frac{1}{2i} \left[ i \delta(x - y) - 1 \delta(x - y) \right] = 0 \]

\[ \left[ \phi_i(x, \xi), \phi_i^+(y, \eta) \right] = \frac{1}{2i} \left[ \phi_i(x, \xi) - \phi_i^+(x, \xi), \phi_i(y, \eta) - \phi_i^+(y, \eta) \right] = i \delta(\vec{x} - \vec{y}). \]
We may make a complex field \( \phi \) out of two real fields\( \phi = \frac{1}{\sqrt{2}} ( \phi_1 + i \phi_2 ) \) with \( \phi^r = \phi_1 \), \( \phi^i = \phi_2 \).

Then
\[
[\phi(x,t), \phi(y,t)] = \frac{i}{2} \left[ \phi(x,t) : \phi^*(y,t) + \phi^*(x,t) : \phi(y,t) \right] = 0
\]
and
\[
[\phi(x,t), \phi^+(y,t)] = \frac{i}{2} \left[ \phi(x,t) : i \phi^+(y,t) - i \phi^*(y,t) \phi(x,t) \right] = 0
\]
while
\[
[\phi(x,t), \phi^+(y,t)] = \frac{i}{2} \left[ \phi(x,t) : i \phi^+(y,t) - i \phi^*(y,t) : \phi(x,t) \right] = \frac{3}{2} i \delta(x-y) + \frac{i}{2} \delta(x-y) = i \delta(x-y).
\]

But
\[
[\phi(x,t), \phi^+(y,t)] = \frac{i}{2} \left[ \phi^r, \phi^r \right] - \frac{i}{2} \left[ \phi^i, \phi^i \right] = 0.
\]
So the real canonical variables are
\[ \mathbf{q}^1(x, t) = \phi^1(x, t) \quad \text{and} \quad \mathbf{p}^1(x, t) = \phi^0(x, t) \]
which obey
\[ [\mathbf{q}^1(x, t), \mathbf{q}^1(y, t)] = 0 \]
\[ [\mathbf{p}^1(x, t), \mathbf{p}^1(y, t)] = 0 \]
and
\[ [\mathbf{q}^1(x, t), \mathbf{p}^1(y, t)] = i \delta(x - y). \]

The real vector field of spin one, \( \mathbf{v}^\mu(x, t) \),
\[ \mathbf{v}^\mu(x) = \mathbf{v}^{\mu(+)}(x) + \mathbf{v}^{\mu(-)}(x) \]
satisfies by (5.2.1.3)
\[ [\mathbf{v}^{\mu(x, t)}, \mathbf{v}^{\nu(y, t)}] = [\mathbf{V}^{\mu(+)}(x) + \mathbf{V}^{\mu(-)}(x), \mathbf{V}^{\nu(+)}(y) + \mathbf{V}^{\nu(-)}(y)] - \]
\[ = [\mathbf{V}^{\mu(+)}(x), \mathbf{V}^{\nu(+)}(y)] + [\mathbf{V}^{\mu(-)}(x), \mathbf{V}^{\nu(-)}(y)] \]
\[ = (\gamma^{\mu\nu} - \frac{\gamma^{\mu\nu}}{m^2}) \Delta(x - y) = \gamma^{\mu\nu} \Delta(x - y) \]
But by (5.2.1.3)
\[ \Delta(x) - \Delta(-x) = \Delta(x) \], so we have
\[ [\mathbf{v}^{\mu(x, t)}, \mathbf{v}^{\nu(y, t)}] = (\gamma^{\mu\nu} - \frac{\gamma^{\mu\nu}}{m^2}) \Delta(x - y). \]
So
\[ [\mathbf{v}^{1(x, t)}, \mathbf{v}^{1(y, t)}] = (\delta^{1\nu} - \frac{\delta^{1\nu}}{m^2}) \Delta(x - y) \]
\[ = (\delta^{1\nu} - \frac{\delta^{1\nu}}{m^2}) \frac{\partial^1}{\partial x^\nu} \cdot 0 = 0 \]
Let
\[ \mathbf{p}^1(x, t) = \frac{\partial \mathbf{v}^1(x, t)}{\partial t} + \frac{\partial \mathbf{v}^0(x, t)}{\partial x^1} = \partial_0 \mathbf{v}^1 + \partial_1 \mathbf{v}^0 \]
\[ = \partial_1 \mathbf{v}^0 - \partial_0 \mathbf{v}^1. \]
So \( \Phi'(x,t) = V'(x,t) \), then

\[
\Gamma_{\Phi'(x,t), \Phi'(y,t)} = 0 \quad \text{and} \quad \Gamma_{\Phi'(x,t), \Phi'(y,t)} = \left[ \frac{\partial V'(x,t)}{\partial y} + \partial_j V^0(y,t) \right] -
\]

\[
\frac{2}{\partial y_0} \left( \delta^{(3)}(x-y) \right) = \frac{2}{\partial y} \left( \frac{\partial^2}{\partial y^2} \right) \Delta(x-y)
\]

\[
= \frac{2}{\partial y_0} \left( \delta^{(3)}(x-y) \right) + \frac{\partial^2}{\partial y^2} \Delta(x-y)
\]

\[
= \delta^{(3)}(x-y) - \partial_0 \partial^j \Delta(x-y) + \partial^j \partial^0 \Delta(x-y)
\]

And

\[
\Gamma_{\Phi'(x,t), \Phi'(y,t)} = \left[ \partial_0 V'(x,t) + \partial_j V^0(x,t), \partial_0 V_j(y,t) + \partial_j V^0(y,t) \right]
\]

\[
= -\frac{\partial^2}{\partial x^2} \left( \delta^{(3)}(x-y) \right) - \partial_i \partial_0 \left( \frac{\partial^2}{\partial y^2} \right) \Delta(x-y)
\]

\[
- \partial_i \partial_j \left( \frac{\partial^2}{\partial y^2} \right) \Delta(x-y) - \partial_0 \partial^j \left( \frac{\partial^2}{\partial y^2} \right) \Delta(x-y)
\]

\[
= -\delta^{(3)} \partial^2 \Delta + \frac{\partial^2}{\partial x^2} \Delta + \frac{\partial^2}{\partial y^2} \Delta - \frac{\partial^2}{\partial y^2} \Delta - \partial^2 \partial^j \Delta - \partial^2 \partial^j \Delta
\]

\[
= -\delta^{(3)} \partial^2 \Delta(x-y) + \partial_i \partial_j \Delta(x-y) = 0 - 0 = 0 \quad \text{at} \quad x^0 = y^0 = t.
\[ \eta_{(5.3.38)} \quad \partial_m v^m = 0 \implies v^0 = - \nabla \cdot v \]

So \[ \nabla \cdot p = \nabla \cdot v + \Delta v^0 = - \partial_0 v^0 + \Delta v^0 \]

But by \( \eta_{(5.3.36)} \) \[ \Delta v^0 - \partial_0^2 v^0 = \nabla v^0 = m^2 v^0 \]

So \[ \frac{\nabla \cdot p}{m^2} = v^0 \] \[
(\Delta - m^2) V^0(x) = 0
\]

This means that \( v^0 \) is a dependent variable.

The complex spin-one vector field \( V^m(x) \) obeys

\( \eta_{(5.3.35)} \)

\[ [V^m(x, \xi), \bar{V}^n(x', \xi')] = (\eta^{mn} - 2 \eta^{m0} \eta^{n0}) \frac{\Delta (x - x')}{m^2} \]

So \[ \psi'(x, \xi) = V^m(x, \xi) \] and

\[ p_i(x, \xi) = \psi_i(x, \xi) + \frac{\partial v_0^i(x, \xi)}{\partial x^i} \]

On the real fields \( \psi_1^m = \frac{1}{\sqrt{2}} (V^m + V^m) \)

and \( \psi_2^m = \frac{1}{\sqrt{2}} (V^m - V^m) \)

\[ V^m = \frac{1}{\sqrt{2}} \left( V_1^m + i V_2^m \right) \]
\[ (5.5.39) \Rightarrow \left[ \psi^+_2(x, t), \psi^-_2(y, t) \right]^+_t = \left\{ \gamma^0 \delta^m + m \beta \right\} \delta^3_{m \bar{m}} \Delta (x - y), \]

and \[ \left[ \psi^+_2(x, t), \psi^-_2(y, t) \right]^+_t = 0 \quad (t \text{ complex}) \]

Since \[ \left[ \psi^+_2, \psi^-_2 \right]^+_t = 0 \] at equal times, we cannot have \[ \psi^+_2 = \psi^-_2 \] and \[ \psi^+_2 = \psi^-_2 \] instead

\[
\psi^+_m(x) = \psi^-_m(x) \quad \text{and} \quad p^m_m(x) = i \psi^+_m(x),
\]

Then

\[
\left[ \psi^+_m(x, t), \psi^-_{\bar{m}}(y, t) \right]^+_t = 0 \quad \left[ p^m_m, p^\bar{m}_{\bar{m}} \right]^+_t = 0
\]

and

\[
\left[ \psi^+_m(x, t), p^\bar{m}_{\bar{m}}(y, t) \right]^+_t = i \left\{ \gamma^0 \delta^m + m \beta \right\} \delta^3_{m \bar{m}} \Delta (x - y)
\]

\[
= -i \left( \gamma^0 \delta^m + m \beta \right) \delta^3_{m \bar{m}} \Delta (x - y)
\]

\[
\beta = i \gamma^0 \\
\gamma^m = -1
\]

\[
= (\gamma^0) \delta^m \delta^3_{m \bar{m}} \Delta (x - y)
\]

\[
= -2 \Delta (x - y) \delta^3 \delta^m \delta^3_{m \bar{m}} = i \delta^3 (x - y) \delta^3_{m \bar{m}}.
\]
QM functional derivative of a bosonic functional $F$

$$
\frac{\delta F[q(x), p(x)]}{\delta q^m(x,t)} = \frac{i}{\hbar} \left[ \frac{\partial}{\partial x^m} F[q(x), p(x)], \frac{\partial}{\partial p(x)} \right]
$$

and

$$
\frac{\delta F[q(x), p(x)]}{\delta p(x,t)} = \frac{i}{\hbar} \left[ F[q(x), p(x)], \frac{\partial}{\partial q^m(x,t)} \right]
$$

For example, $F = \int d^3x \, q^2(x,t) \, p(x,t)$, then

$$
\frac{\delta F}{\delta q^m(x,t)} = 2 \left[ p_m(x,t), \int d^3y \, \delta(x-y) \right] = 2 \int d^3y \left[ p_m(x,t), \frac{\partial}{\partial q^m(y,t)} \right] p(y,t)
$$

$$
\left[ p, q^2 \right] = (pq - qp) = \frac{\partial}{\partial q} \left( \frac{1}{2} q^2 \right) = -i\hbar \frac{\partial}{\partial q}
$$

So

$$
\frac{\delta F}{\delta p(x,t)} = 2 \int d^3y \left[ p(x,t), \frac{\partial}{\partial q^m(y,t)} \right] = 2 \int d^3y \left[ p(x,t), \frac{\partial}{\partial q^m(x,y)} \right].
$$

But the momenta $L$ & $R$ get interchanged.

For instance, $F = \int d^3y \, \psi^R(x,t) \psi^L(x,t)$

$$
\frac{\delta F}{\delta \psi^m(x,t)} = \frac{i}{\hbar} \left[ \psi^m(x,t), \frac{\partial}{\partial \psi^m(x,t)} \right] = \frac{i}{\hbar} \left[ \int d^3y \, \psi^L(y,t) \psi^R(x,t), \psi^m(x,t) \right]
$$

$$
= \frac{i}{\hbar} \int d^3y \left[ \psi^L(y,t) \psi^R(x,t) \psi^m(x,t) - \psi^m(x,t) \psi^L(y,t) \psi^R(x,t) \right]
$$
\[ \frac{\delta F}{\delta \psi_m(x,t)} = -i \int dy \, \psi(y,t)^+ [\gamma^\mu, \psi_m(x,t)]^+ \]

\[ = +i \int dy \, \delta m \, \delta(x-y) \, \psi(y,t) \]

\[ = +i \, \psi_m(x,t), \quad \because \frac{\partial F}{\partial \psi^+} \text{ from the left} \]

\[ \frac{\delta F}{\delta \psi} = \frac{1}{2} [\psi(x), \psi(x)^+] = \int dy \, [\psi(y), \psi(y)^+], \quad \psi_0 = \psi(x), \quad \because \frac{\partial F}{\partial \psi^+} \text{ from right} \]

\[ \delta F [\psi(x), \psi(x)^+] = \int d^3x \, \sum_m \delta q^m(x,t) \frac{\delta F [\psi(x), \psi(x)^+]}{\delta q^m(x,t)} \]

\[ + \frac{\delta F [\psi(x), \psi(x)^+] \delta p_m(x,t)}{\delta p_m(x,t)} \]

So to get the same result \( F = q^m \, p_m \) for both

\[ \text{but } F = p^m \, q_m \text{ for fermions (with } L \in \mathbb{R} \text{)} \]

For fermions the choices are:

\[ F = \psi^+_m \psi_m \quad \text{or} \quad F = \psi_m \psi^+_m \]

\[ \frac{\delta F}{\delta \psi_m} = -i \gamma^\mu \frac{\partial F}{\partial u^+} \quad \frac{\delta F}{\delta \psi_m} = \frac{\partial F}{\partial u^+} \]

\[ \frac{\delta F}{\delta q^m} = \psi^+_m \frac{\partial F}{\partial u^m} \quad \frac{\delta F}{\delta q^m} = \frac{\partial F}{\partial u^m} \]

Both seem awkward.
\[ H_0 \text{ is the generator of time translations:} \]
\[ \begin{align*}
q^a(x, t) &= e^{\text{i} \omega^a t} q^a(x, 0) e^{-\text{i} \omega^a t} \\
p^a(x, t) &= e^{\text{i} \omega^a t} p^a(x, 0) e^{-\text{i} \omega^a t}
\end{align*} \]

So the free-particle operators obey
\[ \begin{align*}
\frac{\partial}{\partial t} q^a(x, t) &= \frac{\partial}{\partial \omega^a} \tilde{q}^a(x, t) = \frac{
abla H_0}{\nabla p^a(x, t)} \\
p^a(x, t) &= -\text{i} \left[ \tilde{p}^a(x, t), H_0 \right] = -\frac{\nabla H_0}{\nabla \tilde{q}^a(x, t)}
\end{align*} \]

as in Hamilton's mechanics.

The free-particle Hamiltonian is
\[ H_0 = \sum_{\sigma} \int d^3 \chi \, a^\dagger(\mathbf{r}, \sigma, n) a(\mathbf{r}, \sigma, n) \sqrt{\dot{\chi}^2 + m_n^2} \]

Up to a constant, in the real scalar field
\[ H_0 = \int d^3x \left( \frac{1}{2} p^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right) \]
\[ = \frac{1}{4} \int d^3x \int d^3 p \int d^3 p' \left\{ \begin{bmatrix}
\text{i} \phi x & -\text{i} \phi x \\
-\text{i} \phi x & \text{i} \phi x
\end{bmatrix}
\begin{bmatrix}
\text{i} p^0 \phi(p) e^{\text{i} p^0 \phi} \\
\text{i} p^0 \phi(p') e^{\text{i} p^0 \phi(p')}
\end{bmatrix} \right\}
\]
\[ + \left[ i \hat{p} a(p) e^{\text{i} \hat{p} \phi} + i \hat{p} a^\dagger(p') e^{\text{i} \hat{p} \phi(p')} \right] \left[ i \hat{p} a(p') e^{\text{i} \hat{p} \phi} - i \hat{p} a(p') e^{\text{i} \hat{p} \phi(p')} \right] \]
\[ + m^2 \left[ \frac{a(p) e^{\text{i} \hat{p} \phi} + a^\dagger(p') e^{\text{i} \hat{p} \phi(p')}}{2} \right] \left[ a(p) e^{\text{i} \hat{p} \phi} + a^\dagger(p') e^{\text{i} \hat{p} \phi(p')} \right] \]
\[ H_0 = \frac{1}{2} \int \frac{d^3p}{2p^0} \left\{ a(p) a(-p) \left[ -p^2 + \vec{p}^2 + m^2 \right] ight. \\
+ a(p) a^d(p) \left[ p^2 \vec{p}^2 + m^2 \right] \\
+ a^d(p) a(p) \left[ p^2 \vec{p}^2 + m^2 \right] \\
+ a^d(p) a(-p) \left[ -p^2 + \vec{p}^2 + m^2 \right] \left\} \right. \\
= \frac{1}{2} \int \frac{d^3p}{2p^0} \left( -p^2 \left( a(p) a^d(p) + a^d(p) a(p) \right) \right) \\
= \frac{1}{2} \int \frac{d^3p}{p^0} \left( a^d(p) a(p) + \frac{1}{4} \delta^3(0) \right) \\
= \sum_m \frac{p^0}{m} \left[ a^d(p_m) a(p_m) + \frac{1}{4} \frac{(2\pi)^3 a(p_m) a(p)}{V} \right] \\
= \text{zero point energies.} \\

From this \( H_0 \), we derive \( L_0 \) \\
\[ L_0 [\phi(x), \dot{\phi}(x)] = \sum_m \int d^3x \ p(x,x) \hat{\phi}^*(x) \hat{\phi}(x,x) - H_0 \]

For the real scalar field, \( \phi = \dot{\phi} = \nabla \phi \)

\[ L_0 = \int d^3x \ p \dot{\phi} - \frac{1}{2} \nabla^2 \phi - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 \\
= \int d^3x \ \nabla^2 \phi - \frac{1}{2} \nabla \phi \nabla \phi - \frac{1}{2} m^2 \phi^2. \]

This must be \( L_0 \) because it gives the right \( H_0 \).
The 'Heisenberg picture' canonical variables are

\[ Q_n(x,t) = e^{iHt} q_n(x,0) e^{-iHt} \]
\[ \hat{p}_m(x,t) = e^{iHt} \hat{p}_m(x,0) e^{-iHt} \]

where \( H \) is the full Hamiltonian.

\[ \hat{M}[Q, \hat{P}] = e^{iHt} \hat{M}[q(x,0), \hat{p}(0)] e^{-iHt} = \hat{M}[\hat{q}, \hat{p}] \]

\[ e^{iHt} \cdots e^{-iHt} \]

is a similarity transformation so

\[ \hat{E} Q_n(x,t), \hat{P}_m(x,t) \hat{J} = e^{iHt} (q_n(x,0), \hat{p}, (0)) \hat{J} e^{-iHt} \]

\[ = i \delta^3(x-y) \delta^m_n. \]

\[ \{ \hat{Q}^m(x,t), \hat{Q}^n(x',t) \} \hat{J} = 0 = \{ \hat{P}^m(x,t), \hat{P}^n(x',t) \} \hat{J} \]

But they obey

\[ \dot{Q}_n(x,t) = i [\hat{E} Q, \hat{Q}^m(x,t)] = \frac{\delta^m}{\delta P^m(x,t)} \tag{7.133} \]

\[ \dot{P}_m(x,t) = -i [\hat{P}_m(x,t), \hat{E}] = -\frac{\delta \hat{E}}{\delta Q^m(x,t)} \tag{7.134} \]
Ex. \( H = \frac{1}{2} \frac{d^2 x}{dt^2} + \frac{1}{2} (\mathbf{P}^2 + i \frac{1}{2} \mathbf{Q}^2) + \frac{1}{2} m^2 \mathbf{Q}^2 + \mathcal{H}(Q) \)

Here \( \mathbf{P} = \dot{Q} \) as before for the free case.

but in general \( \mathbf{P} \) is not always \( \dot{Q} \). One must use H's equations of motion (7.1.33-34) to infer \( \mathbf{P}(\mathbf{Q}, \dot{Q}) \).

The Lagrangian Formalism

Pick \( L \) with right symmetries and then find \( H \). In fact given \( H \), we could find \( L \).

But it is \( L(x) \) that is a scalar.

In general the lagrangian

\[ L = L[\Phi(x,t), \Phi(x,t)] \]

is a functional of the general fields \( \Phi^I(x,t), \Phi^\phi(x,t) \).

\[ \Pi_\phi(x,t) = \frac{\delta L[\Phi(x,t), \Phi(x,t)]}{\delta \Phi^\phi(x,t)} \quad (7.2.1) \]

where we do what we want to make this sensible.

which may or may not be (7.1.17-18).

The equations of motion are

\[ \Pi_\phi(x,t) = \frac{\delta L[\Phi(x,t), \Phi(x,t)]}{\delta \Phi^\phi(x,t)} \quad (7.2.2) \]
The action is

$$I[\psi] = \int_{-\infty}^{\infty} \text{d}^4x \left[ \frac{\mathcal{L}}{8} \frac{\delta L}{\delta \psi(x)} + \frac{\delta L}{\delta \psi(x)} \frac{\delta}{\delta \psi(x)} \right] \psi^a(x) \psi^a(x) \, dt \, dt \, \text{d}^4x$$

Let $\psi^a = 0$ as $t \to \pm \infty$. Then by parts

$$S I[\psi] = \int d^4x \left[ \frac{\delta L}{\delta \psi(x)} - \frac{\delta L}{\delta \psi(x)} \frac{\delta}{\delta \psi(x)} \right] \psi^a(x) \, \text{d}^4x.$$

So $S I[\psi] = 0$ to lowest order if $\delta \psi^a = 0$ at $t = \pm \infty$ and if

$$\frac{\delta L}{\delta \psi(x)} = \frac{\delta L}{\delta \psi(x)} \frac{\delta}{\delta \psi(x)}$$

which is (7.2.2).

$I[\psi]$ is a scalar usually (a constant).

We usually take $L[\psi(1, \psi(1)]) = \int d^4x \, L(\psi(x), \frac{\partial \psi(x)}{\partial x}, \frac{\partial^2 \psi(x)}{\partial x^2})$

so that the action is

$$I[\psi] = \int d^4x \, L(\psi(x), \frac{\partial \psi(x)}{\partial x}, \frac{\partial^2 \psi(x)}{\partial x^2}),$$

which is the case for all commercial field theories.

Now we vary $\psi^a(x)$ by $\delta \psi^a(x)$ which vanishes on the boundary of $d^4x$;

$$S L = \int d^4x \left[ \frac{\partial}{\partial \psi^a} \psi^a + \frac{\partial}{\partial \psi^a} \nabla \psi^a + \frac{\partial^2}{\partial \psi^a} \psi^a \right]$$

$$= \int d^4x \left[ \frac{\partial}{\partial \psi^a} \big( - \frac{\partial \psi^a}{\partial x} \nabla \psi^a \big) \frac{\delta L}{\delta \psi^a} + \frac{\partial}{\partial \psi^a} \right].$$
\[ \frac{\delta L}{\delta \Phi} = \frac{\partial L}{\partial \Phi} - \nabla \cdot \frac{\partial L}{\partial \nabla \Phi} \]

\[ \frac{\delta L}{\delta \dot{\Phi}} = \frac{\partial L}{\partial \dot{\Phi}} \]

So (1.2.2) now:

\[ \frac{d}{dt} \frac{\delta L}{\delta \Phi} = \frac{\partial L}{\partial \Phi} - \frac{\partial L}{\partial \dot{\Phi}} - \nabla \cdot \frac{\partial L}{\partial \nabla \Phi} \]

or

\[ \frac{\partial}{\partial \Phi} \frac{\partial L}{\partial \Phi} = \frac{\partial L}{\partial \Phi} \]

which are the Euler-Lagrange equations. If \( L \)

is a scalar, then these E-L eqns. are known to be invariant.

We also need \( I = I^* \). Think of the fields as real \((\Phi = \Phi_1 + \Phi_2 \text{ if } \Phi^* \neq \Phi)\). Say here

are \( N \) real fields. If \( I \) were complex, then

there would be \( 2N \) real equations of motion, the

E-L eqns. But we want only \( N \) such E-L equations.
The Legendre transformation is used to find $H$ from $L = \mathcal{L}$.

$$H = \sum_{\alpha} \int d^3 x \, \pi^\alpha (x, t) \Phi^\alpha (x, t) - \mathcal{L} \left[ \Phi^\alpha (x, t), \pi^\alpha (x, t) \right].$$

Now (7.2.1),

$$\pi^\alpha (x, t) = \frac{\delta \mathcal{L} \left[ \Phi^\alpha (x, t), \pi^\alpha (x, t) \right]}{\delta \Phi^\alpha (x, t) \left| \pi^\alpha (x, t) \right|}.$$

does not always allow $\Phi^\alpha (x, t)$ to be expressed uniquely in terms of $\Phi^\alpha$ and $\pi^\alpha$. But $H$ is constructed so that

$$\frac{\delta H}{\delta \Phi^\alpha} \bigg|_\pi = 0.$$

$H$ is taken to be a functional of $\pi^\alpha (x, t)$ and $\Phi^\alpha (x, t)$.

$$\frac{\delta H}{\delta \pi^\alpha (x, t)} \bigg|_\pi = \int d^3 y \sum_{\alpha} \pi^\alpha (y, t) \frac{\delta \mathcal{L} \left[ \Phi^\alpha (y, t), \pi^\alpha (y, t) \right]}{\delta \Phi^\alpha (y, t) \left| \pi^\alpha (y, t) \right|} - \frac{\delta L}{\delta \pi^\alpha (x, t) \left| \pi^\alpha (x, t) \right|} \pi^\alpha (x, t) \bigg|\Phi^\alpha (x, t) \bigg|_{\pi^\alpha (x, t), \Phi^\alpha (x, t)}$$

and

$$\frac{\delta H}{\delta \Phi^\alpha} \bigg|_\pi = \Phi^\alpha (x, t) + \int d^3 y \sum_{\alpha} \pi^\alpha (y, t) \frac{\delta \mathcal{L} \left[ \Phi^\alpha (y, t), \pi^\alpha (y, t) \right]}{\delta \Phi^\alpha (y, t) \left| \pi^\alpha (y, t) \right|} - \frac{\delta L}{\delta \Phi^\alpha (x, t) \left| \pi^\alpha (x, t) \right|} \Phi^\alpha (x, t) \bigg|_{\pi^\alpha (x, t), \Phi^\alpha (x, t)}$$

$$- \int d^3 y \sum_{\alpha} \frac{\delta L}{\delta \pi^\alpha (y, t) \left| \pi^\alpha (y, t) \right|} \pi^\alpha (y, t) \bigg|_{\pi^\alpha (y, t), \Phi^\alpha (y, t)}.$$
\[ \frac{\delta H}{\delta \psi^\ell(x,t)} \bigg|_\pi = - \frac{\delta L}{\delta \psi^\ell(x,t)} \bigg|_\pi \]

and

\[ \frac{\delta H}{\delta \pi^\ell(x,t)} \bigg|_\psi = \psi^\ell(x,t) \]

So (7.2.2),

\[ \pi^\ell(x,t) = \frac{\delta L}{\delta \psi^\ell(x,t)} \]

now simplifies Hamilton's equations

\[ \frac{\delta H}{\delta \psi^\ell(x,t)} \bigg|_\pi = - \pi^\ell(x,t) \quad \text{and} \quad \frac{\delta H}{\delta \pi^\ell(x,t)} \bigg|_\psi = \psi^\ell(x,t). \]

(7.2.12) \quad (7.2.13)

In the simple cases, one may identify

\[ \psi^\ell, \pi^\ell \text{ with } q^\ell, p^\ell, \text{ and one may impose} \]

the canonical commutation relations (7.1.30-32)

\[ [\psi^\ell(x,t), \pi^\ell(x',t')] = i \delta^\ell_{\ell'} \delta(x-t) \]

so that

(7.2.12-13) become like (7.1.33-34)

\[ -\pi^\ell(x,t) = \frac{\delta H}{\delta \psi^\ell(x,t)} = i \left[ \pi^\ell(x,t), H \right] \]

and

\[ \psi^\ell(x,t) = \frac{\delta H}{\delta \pi^\ell(x,t)} = i \left[ H, \psi^\ell(x,t) \right]. \]
In example, if $Z = -\frac{i}{2} \partial \phi \partial^2 \phi - \frac{m^2}{2} \phi^2 - \mathcal{H}(\phi)$,

then the E-L equations

$$\frac{\partial}{\partial (\partial \phi)} \frac{\partial Z}{\partial \phi}$$

are

$$(\Box - m^2)\phi = \mathcal{H}'(\phi).$$

$$\Pi = \frac{\partial Z}{\partial \phi} = \phi$$ (cf. 7.1.36)

$$H = \int d^3x \left( \Pi \phi - Z \right)$$

$$= \int d^3x \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \mathcal{H}(\phi) \right).$$

But $\Pi_\phi$ is often absent from $Z$

and $\Pi^\mu_{\lambda \nu}$ in $\Pi$ has no $\gamma^\mu$. Even if we

write

$$\Pi^\mu_{\lambda \nu} \phi^4 = \frac{1}{2} \Pi^\mu_{\lambda \nu} \phi^4 - \frac{1}{2} \partial \mu \phi \gamma_{\lambda \nu} \phi,$$

by integrating by parts, things are not simpler

because now $\Pi_\phi \sim \phi^4$ but $\Pi_{\lambda \nu} \phi^4 \sim \phi^4$.

Let $\phi^m$ be those canonical variables that have

$\phi^m$ in $Z$. Let $\phi^r$ be those fields that

do not have $\phi^r$ in $Z$. 
Then the conjugates of the $Q^m$ are

$$P_m(x, t) = \frac{S L[Q(x), \dot{Q}(x), C(x)]}{5 \cdot \alpha^n(x, t)}$$

The $C^r$'s have no conjugates because $\frac{\partial L}{\partial \dot{C}^r} = 0$.

So $H$ is given

$$H = \sum \int d^3x \left( P_m \dot{Q}^m \right) - L[Q(x), \dot{Q}(x), C(x)]$$

in which $C^r$ and $\dot{Q}^r$ must be expressed in terms of $Q^r$ and $P_m$.

In some cases (7.6) one may avoid actually solving for $C^r$ & $\dot{Q}^r$. In gauge theories one must either pick a gauge (8) or use the F-P trick of \textit{II}.

To do perturbation theory, we use the interaction picture,

$$H = H_0(Q, P_m) + H I_{\delta x_0} (\dot{Q}^m, P_m)_{t=0}$$

$$= H_0(q, p) + V(q, p).$$
For example, $H = H_0 + V$

$H_0 = \int d^3x \frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \Phi)^2 + \frac{i}{2} m^2 \Phi^2$

$V = \int d^3x \varphi \Phi (\Phi)$.

Pass to interaction rep.

$H_0 \rightarrow i H_0 t \rightarrow H_0 t \phi (x, 0) \rightarrow \Phi (x, 0)$

$\Pi (x, 0) \rightarrow \Pi (x, 0)$

where

$H_0 = \int d^3x \frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \Phi)^2 + \frac{i}{2} m^2 \Phi^2 = \int d^3x H_0 e^{i H_0 t}$

which

$(7.1.21) \quad \phi (x, t) = e^{i H_0 t} \Phi (x, 0)$

$\Pi (x, t) = i \int d^3y \Pi (y, t) (i \delta (x - y) \rightarrow \Pi (x, t))$

$(7.1.22) \quad \phi (x, t) = \int d^3y \Pi (y, t) \pi (y, t)$

$= \int d^3y \int d^3y \int d^3y \int d^3y \int d^3y (\nabla \Phi)^2 + \frac{i}{2} m^2 \Phi^2 \Pi (x, t)$

$= i \Delta \phi - m^2 \phi = \phi$

$\Pi (x, t) = - \delta \Pi_0 = i \int \pi (x, t)$

So $(1 + m^2) \phi = 0$.

The general real solution is

$\phi (x) = \int \frac{d^3p}{2 \pi^{3/2} \sqrt{2 \nu^0}} \left[ e^{i p x} a (p) + e^{-i p x} a^+ (p) \right]$

with

$p^0 = \sqrt{p^2 + m^2}$

$a (p)$ to be determined.

$\Pi (x) = \phi = - i \int \frac{d^3p}{2 \pi^{3/2} \sqrt{2 \nu^0}} \left[ e^{i p x} a (p) - e^{-i p x} a^+ (p) \right]$. 

$\phi (x) = \phi \rightarrow \phi (x, t) = e^{i H_0 t} \Phi (x, 0)$

$\Pi (x, t) = i \int d^3y \Pi (y, t) (i \delta (x - y) \rightarrow \Pi (x, t))$
Now we want

\[ \left[ \phi(x,t), \Pi(q,t) \right] = \delta(x-y) \]

\[ \left[ \phi(x',t), \Pi(q',t) \right] = 0 = \left[ \Pi(x',t), \Pi(q',t) \right] \]

so we choose the \( a \)'s so that

\[ \left[ a(p), a^+(p') \right] = \delta^{(3)}(p-p') \]

and

\[ \left[ a(p), a(p') \right] = 0. \]

These formulae also give

\[ H_0 = \int d^3k \sqrt{m^2 + k^2} \left( a^+(k) a(k) + \frac{1}{2} \right). \]

So in fact we chose \( \lambda \) was okay.

Say \( \delta I = \int \mathcal{M} F^m \). Then

\[ \Delta I = \int \mathcal{M} \frac{\partial F^m}{\partial \phi} = \int \mathcal{M} \frac{\partial F^m}{\partial \Pi} \]

So the action changes only if the fields at the boundary make \( F^m \) antisymmetric.

In any case the equations of motion remain unchanged because they are derived under the assumption that \( \delta \phi = 0 \) on the boundary.

Similarly \( \delta I = \nabla \cdot F \) means that

\[ \Delta L = \int \mathcal{M} \nabla \cdot F = \int \mathcal{M} \cdot F \]

which vanishes if \( F^m = 0 \) on the boundary of space.
Similarly, $\mathbf{S} \cdot \mathbf{F}$ will not affect the field equations.

If $\mathbf{S} \cdot \mathbf{F} = 0$, then clearly the equations of motion are unaffected.

How about the more general term

$$\Delta L(t) = \int d^3 x \ D_m(x) \dot{Q}^m(x, t)$$

i.e.

$$\Delta L = D_m(Q(x)) \dot{Q}^m(x).$$

$$\Delta P_m(x) = \frac{\delta \Delta L(t)}{\delta Q^m(x, t)} = \frac{\partial}{\partial Q^m(x)} \Delta L(x).$$

$$= D_m(Q(x), x).$$

$$\Delta H = \sum_n \int d^3 x \ D_m(x) \dot{Q}^m(x) - \Delta L$$

$$= \int d^3 x \sum_n D_m(x) \dot{Q}^m - \int d^3 x \sum_n \dot{Q}^m \dot{Q}^m = 0.$$ 

So $\Delta H = 0$. But

$$[P_m + \Delta P_m, P_m + \Delta P_m] = [\Delta P_m, P_m] + [P_m, \Delta P_m]$$

$$= [D_m, P_m] + [P_m, D_m]$$

$$= i \frac{S \delta m}{S \delta m} \text{ might not vanish.}$$
although \[ [Q^n, Q^m] = 0 \text{ and} \]
\[ [Q^n, P_m + P_m] = [Q^n, P_m] = i \delta^n_m \]
still work.

Now when

\[
\Delta L = \frac{dL}{dt} = \int dx \frac{\delta G}{\delta Q^n} \dot{Q}^n
\]

\[ u \delta Z = \frac{\delta G}{\delta Q^n} \dot{Q}^n = \frac{dG}{dt} \]

then

\[ \delta L = \frac{\delta G}{\delta Q^n} \dot{Q}^n \]

and so

\[
\delta D_m = \frac{\delta G}{\delta Q^n} \dot{Q}^n - i \left( \frac{\delta^2 G}{\delta Q^n \delta Q^n} - \frac{\delta^2 G}{\delta Q^n \delta Q^n} \right) = 0
\]

So the quantum structure as well as the classical structure of a theory is unaffected by partial integration.
Global Symmetries

Because the dynamics follow from a variational principle, the L. formalism facilitates the implementation of symmetries in the quantum theory.

Suppose under

$$\psi(x) \rightarrow \psi'(x) = \psi(x) + i e F^0(x)$$

the nett

$$I[\psi] = \int dt L[\psi(t)]$$

is invariant,

$$0 = \delta I = i e \int dx \frac{\delta I}{\delta \psi^0(x)} F^0(x)$$

in which e is a constant (hence global).

Now in fact if $$\psi(x)$$ obeys the field
equations, then $$\delta I = 0$$ to order e. Here
we assume $$\delta I = 0$$ even if the $$\psi^0$$s do not
satisfy the field equations. This is a global symmetry.

Consider now the local transformation

$$\psi(x) \rightarrow \psi'(x) = \psi(x) + i e(x) F^0(x)$$

in which $$e(x) \rightarrow 0$$ as $$x \rightarrow \infty$$. That since
due to the symmetry $$\delta I = 0$$ to constant
e, here $$\delta I \neq 0$$ but $$\delta I$$ has the simple
form
\[ \delta I = -\oint d^4x \frac{J^u(x)}{dx^u} \delta e(x) \]

whether or not the fields \( e^u(x) \) satisfy their field equations. This result also follows from the assumption that \( I \) involves only the first derivatives of the fields and is invariant for constant \( e \).

But if the fields do obey their field equations, then

\[ 0 = \delta I = \int d^4x \, e(x) \, \partial_m J^m(x). \]

Since \( e(x) \) is arbitrary, at finite \( x \), we have the conservation law

\[ 0 = \frac{\partial}{\partial x^u} J^u(x), \]

Its integral form is

\[ 0 = \frac{d}{dt} \int d^3x \, J^0(x) = \int d^3x \, \frac{\partial}{\partial t} J^0(x), \]

\[ = \int d^3x \, \nabla \cdot J = -\int d^3x \, J^i, \]

which vanishes if the fields vanish at \( x \to \infty \).

This is Noether's theorem: symmetries imply conservation laws.
If \( L(t) \) and not just \( I \) is invariant under the symmetry transformations, then suppose

\[
\psi^0(x) \to \psi^0(x) = \psi^0(x) + i e^t \chi \mathcal{F}^0(x),
\]

Then

\[
SI = i \int dt \int d^3 x \left[ \frac{\delta L}{\delta \psi^0(x,t)} \frac{\partial}{\partial \chi} \mathcal{F}^0(x,t) \right] + \frac{\delta L}{\delta \psi^0(x,t)} \frac{\partial}{\partial \psi^0(x,t)} \mathcal{F}^0(x,t) \frac{d}{dt} \left( e^t \mathcal{F}^0(x,t) \right)
\]

Now by the symmetry of \( L(t) \)

\[
0 = \int d^3 x \left( \frac{\delta L}{\delta \psi^0(x,t)} \mathcal{F}^0(x,t) + \frac{\delta L}{\delta \psi^0(x,t)} \frac{\partial}{\partial \psi^0(x,t)} \mathcal{F}^0(x,t) \right) \text{ (at each } t) \]

so in general

\[
SI = i \int dt \int d^3 x \frac{\delta L}{\delta \psi^0(x,t)} \mathcal{F}^0(x,t) = -i \int d^3 x J^0 \cdot \frac{\delta \mathcal{F}^0}{\delta \psi^0(x,t)}
\]

so

\[
0 = SI = i \int dt \int d^3 x \frac{d}{dt} \left( \frac{\delta L}{\delta \psi^0(x,t)} \mathcal{F}^0(x,t) \right) = F(t_2) - F(t_1)
\]

\[
F = \int d^3 x J^0 = -i \int d^3 x \frac{\delta L}{\delta \psi^0(x,t)} \mathcal{F}^0(x,t)
\]

This is invariant for any fields that obey their field equations and vanish at \( x^2 \to \infty \).
Examples of such symmetries are the translations and rotations.

Under some symmetry transformations, such asospin, colour, or other internal symmetries, the lagrangian density $L$ is itself invariant.

Then

$$ST = i \int d^4x \frac{\partial L}{\partial \epsilon^m} \frac{\partial}{\partial (\partial_m \epsilon^l)} \left( F^{il} \right)$$

But the symmetry of $L$ means that

$$\frac{\partial L}{\partial \epsilon^m} F^{il} + \frac{\partial}{\partial \epsilon^m} \epsilon_i^m F^{il} = 0 \quad \text{(at each } x \text{)}$$

So for arbitrary fields $\epsilon^m$

$$ST = i \int d^4x \frac{\partial L}{\partial (\partial_m \epsilon^l)} F^{il}$$

$$= -i \int d^4x \epsilon^l G(x) \frac{\partial}{\partial (\partial_m \epsilon^l(x))} \left( \frac{\partial L}{\partial (\partial_m \epsilon^l(x))} F^{il}(x) \right)$$

which vanishes if the fields obey the dynamical equations. So

$$J^m(x) = -i \int \frac{\partial L}{\partial (\partial_m \epsilon^l(x))} F^{il}(x)$$

in agreement; $\int J^m = 0$. 


When \( \mathcal{L}(x) \) is invariant under

\[
\mathcal{L}'^q(x) = \mathcal{L}^q(x) + \epsilon(x) F^q(x)
\]

then by using Lagrange's equations one has

\[
0 = \frac{\partial}{\partial t^q} \mathcal{L}^q + \frac{\partial}{\partial x^i} \frac{\partial}{\partial (\partial_x x^i)} \mathcal{L}^q (i \in F^q)
\]

\[
= \frac{\partial}{\partial \mu} \left( \frac{\partial}{\partial (\partial_x x^i)} \right) \mathcal{L}^q + \frac{\partial}{\partial x^i} \frac{\partial}{\partial (\partial_x x^i)} \mathcal{L}^q (i \in F^q)
\]

\[
= \frac{\partial}{\partial \mu} \left[ \frac{\partial}{\partial (\partial_x x^i)} \mathcal{L}^q \right]
\]

So

\[
j^q = \frac{\partial}{\partial \mu} \mathcal{L}^q (i \in F^q)
\]

is conserved:

\[
0 = \frac{\partial}{\partial \mu} j^q
\]

The conserved quantity is

\[
Q = \int d^3x \, j^q = \int d^3x \, \frac{\partial}{\partial \mu} \mathcal{L}^q (i \in F^q)
\]

\[
= \int d^3x \, \Pi^{0}_q (x) i \in F^0(x)
\]

W. we have \(-Q\), which is also conserved.
The quantum aspects are clearest when the canonical fields transform into $\mathbb{R}^3$-dependent functionals of themselves at the same time,

$$\Psi'(x) = \Psi(x) + i\epsilon(x) F^0[Q(t); \vec{x}]$$

Spatial rotations and translations are examples, as are all infinitesimal internal symmetries. Here $F$ is the conserved generator of the symmetry.

$$F = \int d^3 x \, J^0 = -i \int d^3 x \frac{1}{2} \chi \frac{\partial}{\partial x^\mu} F^\mu(x)$$

$$= -i \int d^3 x \, p_\mu(x) F^\mu[Q(t); \vec{x}']$$

$F$ is time independent. So we may choose the $t'$ in $F'$ to be the same as that of $Q'(x')$.

$$\begin{align*}
\left[ F, Q^\mu(x,t) \right] &= -i \int d^3 x' \left[ p_\mu(x', t), Q^\nu(x', t) \right] F^\nu[Q(x'), \vec{x}'] \\
&= - \int d^3 x' \delta(x-x') \left[ \partial_\nu F^\nu(x', \vec{x}) \right] \\
&= -F^\nu[Q(x); \vec{x}']
\end{align*}$$

So $F$ is the generator, that is,

$$e^{-i e F} e^{i e F} = e^{i e F} e^{-i e F} \approx Q(x, t) + i e F \left[ Q(x, t), P_\mu(\vec{x}, t) \right].$$

On $P_\mu F$ gives

$$\begin{align*}
\left[ F, P_\mu(x, t) \right] &= -i \int d^3 x \, p_\mu(x) \left[ F^0[Q(t); P_\mu(x, t)], \right]
\end{align*}$$
\[ [F, P_n(x, t)] = \int d^3x' P_0(x', t) \frac{\delta F^0(Q, x)}{\delta Q^m(x, t)} \]

If \( F^0 \) is linear in \( Q^m \), as is normally the case, then \( F^0 = -a^m Q^m \) and

\[ [F, P_n(x, t)] = \int d^3x' P_0(x', t) \delta(x - x') a^m \]

\[ = a^m P_0(x, t) \]

while

\[ [F, Q^m(x, t)] = -F^0 = -a^m Q^m. \]

One says that \( P_n \) transforms contragrediently to \( Q^m \).

For example, translations have action invariant,

\[ \psi^0(x) \rightarrow \psi^0(x + \epsilon) = \psi^0(x) + \epsilon^m \partial_m \psi^0(x), \]

Now we have \( \psi^0 \)'s and \( F^0 = -i \partial_m \psi^0 \),

So there are \( J^m \)'s, \( J^m \),

\[ \partial_m J^m = 0 \]

Usually we write \( J^m = T^m \)

\[ \partial_m T^m = 0 \]

The conserved quantities are

\[ P_\nu = \int d^3x J^0 = \int d^3x T^0 \]
Under \( \psi(x) \rightarrow \psi'(x) = \psi(x) + \epsilon^m \partial_m \psi(x) \), we expect \( \psi(x) \) to turn into:

\[
\psi'(x) = \psi(x + \epsilon) = \psi(x) + \epsilon^m \partial_m \psi(x).
\]

So

\[
\epsilon^m \partial_m \psi = \frac{\partial \psi}{\partial \psi^l} \epsilon^m \partial_m \psi^l + \frac{\partial \psi}{\partial \partial_m \psi^l} \partial_m (\epsilon^m \partial_m \psi^l).
\]

So

\[
0 = \partial_l \left[ -\epsilon^m \delta_m^l \frac{\partial \psi}{\partial \psi^l} + \frac{\partial \psi}{\partial \partial_m \psi^l} \partial_m (\epsilon^m \partial_m \psi^l) \right] = \partial_l \left[ T^l_m \right]
\]

setting \( T^l_m = \delta^l_m - \frac{\partial \psi}{\partial \partial_m \psi^l} \partial_m \psi^l \),

we see that

\[
0 = \partial_l T^l_m \quad \text{when}
\]

\[
P_m = \int d^3x T^0_m \quad \text{is conserved.}
\]

\[
\overrightarrow{P} = \int d^3x \frac{\partial \psi}{\partial \psi^l} \overrightarrow{\nabla} \psi^l
\]

momentum

\[
= - \int d^3x \overrightarrow{P} \cdot \overrightarrow{D} \psi^l = - \int d^3x \overrightarrow{P} \cdot D \overrightarrow{Q}^l
\]
\[ -H = P_0 = \int d^3 x \: T^0_0 = \int d^3 x \: \mathcal{T} - \frac{\partial}{\partial \phi} \phi e \]

\[ H = \int d^3 x \: \Pi_\phi \phi^4 - \lambda = \int d^3 x \: P_\phi \phi^4 - \lambda \]

So

\[ [\bar{P}^\phi, Q^\phi(x, \tau)] = -\int d^3 y \: [P_\phi \nabla^\tau Q^\phi(y, \tau), Q^\phi(x, \tau)] \]

\[ = i \int d^3 y \: S^m_n S(x-y) \nabla^\tau, Q^\phi(y, \tau) \]

\[ = i \nabla Q^\phi(x, \tau) \]

\[ [\bar{P}^\phi, P_m(x, \tau)] = -\int d^3 y \: [P_\phi \nabla^\tau P^m(y, \tau)] \]

\[ = -\int d^3 y \: [\nabla P_\phi \phi^0, P_m(x, \tau)] \]

\[ = i \int d^3 y \: S^m_n S(x-y) \nabla P_\phi(y, \tau) \]

\[ = i \nabla P_m(x, \tau) \] Thus

\[ [\bar{P}^\phi, G(Q, P)] = i \nabla G(Q(x, \tau), P(x)) \]

\( \bar{P} \) generates translations in space as \( H \) does in time

\[ [H, G(Q, P)] = -i \dot{G}(Q, P) \]
In which \( P_L = 0 \)

as long as the fields vanish as \( x \to \infty \).

\[ L \text{ is invariant under spatial translations. So } \exists \ P_L \text{ are concerned. (3.11)} \]

\[ P_i = -i \int d^3x \left( \frac{\partial}{\partial x_i} \right) F^0 \]

\[ = -i \int d^3x \left( \frac{\partial}{\partial x_i} \right) (-i) \frac{d^3k}{(2\pi)^3} \]

\[ = -\int d^3x \ P_L \partial_i Q^L \]

\[ P^L = -\int d^3x \ P_L \nabla Q^L \]

Thus

\[ [ P_i, Q^L (x,t) ] = i \nabla \delta(x-x') \]

\[ [ P_L, P_m (x,t) ] = -i \int d^3x' P_L (x',t) \nabla' \delta(x-x') \]

\[ = -i \int d^3x' P_L (x',t) \nabla' \delta(x-x') \]

\[ = i \int d^3x' \delta(x-x') \nabla' P_m (x',t) \]

\[ = i \nabla P_m (x,t) . \]

So any \( G(x,p) \) that does not depend upon \( x \)

explicitly \( [ P_L, G(x) ] = i \nabla G(x) \).

So \( P \) generates spatial translations.
The action is invariant under time translation, as long as the Lagrangian $L(t)$ depends on time $t$ only through the fields $\Psi^\mu(x)$ and not explicitly.

\[ S = \int dt \left[ \frac{\delta L(t)}{\delta \dot{\Psi}^\mu(x)} \dot{\Psi}^\mu(x) + \frac{\delta L(t)}{\delta \Psi^\mu(x)} \Psi^\mu(x) \right] d^3x \]

\[ = \int dt \left[ \frac{\delta L(t)}{\delta \dot{\Psi}^\mu(x)} \dot{\Psi}^\mu(x) + \frac{\delta L(t)}{\delta \Psi^\mu(x)} \Psi^\mu(x) \right] d^3x + \delta L(t) \dot{\Psi}^\mu(x) \left. \frac{d}{dt} \right|_{\Psi^\mu(x)} \]

This vanishes when the fields follow their dynamical equations,

\[ 0 = \int dt \left[ -L + \int d^3x \left( \frac{\partial L}{\partial \Psi^\mu(x)} \right) \Psi^\mu(x) \right] \]

[\text{Thus}]

\[ H = -L + \int d^3x \left( P(x) \Psi^\mu(x) \right) \]

in conserved, $H = 0$. 

$355.5$
Twin translations don't leave $L(E)$ fixed.

$H$ is their generator:

$[\mathbf{H}, G(x)] = -i \dot{G}(x)$

$G(x) = G \left[ Q(x), P(x) \right] ,

Twin translations leave $I$ fixed. Suppose

$I [4] = \int d^4 x \mathcal{A} (x)$

is fixed under

$\psi_- (x) \rightarrow \psi_-' (x) = \psi_- (x + \varepsilon (x)) = \psi_- (x) + \varepsilon ^m \partial _m \psi_- (x)$

when $\varepsilon (x) = \varepsilon$. Then

$S [4] = \int d^4 x \frac{\partial}{\partial \psi_-} \varepsilon ^m \partial _m \psi_- + \frac{\partial ^2}{\partial (\partial \psi_-)^2} \varepsilon ^m \partial _m \psi_- + \frac{\partial ^2}{\partial (\partial \psi_-)^2} \partial _m \psi_- \varepsilon ^m$

$= \int d^4 x \frac{\partial}{\partial \psi_-} \varepsilon ^m (x) + \frac{\partial ^2}{\partial (\partial \psi_-)^2} \partial _m \psi_- \varepsilon ^m$

$= \int d^4 x \left( -\delta ^m _n \partial _n \varepsilon ^m + \frac{\partial ^2}{\partial (\partial \psi_-)^2} \partial _m \psi_- \varepsilon ^m \right)$

$= \int d^4 x \left( \delta ^m _n \partial _n - \frac{\partial ^2}{\partial (\partial \psi_-)^2} \partial _m \psi_- \varepsilon ^m \right)$

$= -\int d^4 x \ T^v _m \partial _v \varepsilon ^m$ with

$T^v _m = \delta ^v _m \partial _v - \frac{\partial ^2}{\partial (\partial \psi_-)^2} \partial _m \psi_- \varepsilon ^m = J^v _m$
For \( \nu = 0 \) and \( \mu = i \) we get as before

\[
\int d^3x J^0_i = \int d^3x T^0_i = -\int d^3x P^\mu_i \partial_{\mu} Q^i = P_i
\]

while for \( \mu = 0 = \nu \) we get

\[
\int d^3x J^0_0 = \int d^3x T^0_0 = \int d^3x \mathcal{L} - P^\mu Q^\mu = -\mu = P_0.
\]

(NB \( T^{\mu\nu} \) is not symmetric. In GR use \( \Theta^{\mu\nu} \) of 7.4.)
Suppose \( X(x) \) is invariant under \( \), then

\[ Q'^n(x) = Q^n(x) + i e^a T^m_a Q^n_m \]

\[ C'^n(x) = C^n(x) + i e^a T^n_a C^s(x) \]

Thus

\[ 0 = \frac{\partial}{\partial Q^n} i e^a T^m_a Q^n_m + \frac{\partial}{\partial (\partial Q^n)} \left( i e^a T^m_a Q^n_m \right) \]

\[ + \frac{\partial}{\partial C^n} i e^a T^n_a C^s + \frac{\partial}{\partial (\partial C^n)} \left( i e^a T^n_a C^s \right) \]

and by Lagrange's equations

\[ 0 = \partial m \left[ \frac{\partial}{\partial (\partial Q^n)} i e^a T^m_a Q^n_m + \frac{\partial}{\partial (\partial C^n)} i e^a T^m_a C^s \right] \]

If the \( e^a \)'s are constants, then the conserved currents are

\[ J^m_a = -i \frac{\partial}{\partial (\partial Q^n)} T^m_a Q^n_m - i \frac{\partial}{\partial (\partial C^n)} T^n_a C^s \]

and

\[ 0 = \partial m J^m_a \]
The conserved charges are

\[ T^a = \int d^3x \mathbf{J}_a \]

\[ = -i \int d^3x \frac{\partial}{\partial \mathbf{Q}^m} \mathbf{t}^a_{mn} \mathbf{Q}^m + \frac{\partial}{\partial \mathbf{\xi}^n} \mathbf{T}^a_{\xi} \mathbf{C}^s \]

\[ = -i \int d^3x \frac{\partial}{\partial \mathbf{Q}^m} \mathbf{t}^a_{mn} \mathbf{Q}^m \]

\[ = -i \int d^3x \mathbf{P}^m \mathbf{t}^a_{mn} \mathbf{Q}^m. \]

The ETCK's now give
Suppose the action invariant under
\[ Q^a(x) = Q^a(x) + i \delta^a_{\lambda} \delta \theta^m Q^m(x) \]
\[ C^r(x) = C^r(x) + i \delta^r_{\lambda} \delta \theta^s C^s(x) \]
on the canonical fields \( Q^a \) and the auxiliary fields \( C^r \). If \( L(x) \) is invariant, then (7.3.11) implies that the operator
\[ T^a = -i \int \delta^3 x \, P_e(x, \xi) \, \delta \theta^m Q^m(x, \xi) \]
\[ \xi \in \text{conserved.} \]
Now the ETCI's give
\[ [T^a, Q^m(x)] = -i \int \delta^3 y \, \delta \theta^m Q^m(x, \xi) \]
\[ = -i \int \delta^3 y \, \delta \theta^m \delta \xi \cdot \delta \phi \cdot \delta \theta^r C^r(x, \xi) \]
\[ = -T^a \theta^m Q^m(x, \xi) \]

And
\[ [T_a, P_m(x)] = P_e \theta^m, \]
If \( T_a \) is diagonal, then \( Q^a \) and \( P_m \) respectively lower and raise the value of \( T^a \) by \( \delta^a_m \).

\[ [T_a, T_b] = -i \int \delta^3 x \, \delta^3 y \, \delta \theta^m Q^m(x, \xi) \]
\[ = -i \int \delta^3 x \, \delta \theta^m \theta^m \delta \theta^a \delta \theta^b \delta \theta^c \delta \theta^d \left( i \delta^m_{\xi \cdot \delta \phi} \right) \]
\[ = -i \theta^m \delta^m \theta^a \delta \theta^b \delta \theta^c \delta \theta^d \]
\[ = -i \delta^a_{\delta \phi} \theta^m \theta^m \delta \theta^d \]
If the ta's form a lie algebra with structure constants $f_{abc}$, that is, 

$$[t_a, t_b] = i f_{abc} t_c,$$

then

$$[T_a, T_b] = i \int d^3 x \ P_m Q^n t_b \ t^e \ t^m - P_0 Q^n t^e t_0,$$

$$= -i \int d^3 x \ P^m (t_a t_b - t_b t_a) Q,$$

$$= -i \int d^3 x \ P^m (t a b c t_e) Q,$$

$$= \int d^3 x \ f_{abc} P_e t^e_m Q^n,$$

$$= i f_{abc} (-i) \int d^3 x \ P_e t^e c m Q^n,$$

$$= i f_{abc} T_c.$$

The $T_a$'s generate the group.

If $L(x)$ is invariant under these

transformations, then

$$0 = \frac{\partial L}{\partial Q^m} \ i e^a \ t^m Q^n + \frac{\partial L}{\partial (t^m Q^n)} \ d_m (i e^a \ t^m Q^n),$$

$$+ \frac{\partial L}{\partial (C^a)} \ i e^a T^a_5 C^5 + \frac{\partial L}{\partial (d_m C^a)} (i e^a T^a_5 C^5).$$

Using the dynamics we have

$$0 = \frac{\partial L}{\partial (t^m Q^n)} \ i e^a \ t^m Q^n + \frac{\partial L}{\partial (d_m C^a)} i e^a T^a_5 C^5,$$

$$= d_m J^m e^a$$
Then the current
\[
J^a_m = -i \frac{\partial}{\partial x^a} \epsilon_{mnr} Q^n = -i \frac{\partial}{\partial (x^a C^a)} T^a \Phi^s
\]
is conserved
\[0 = \partial_m J^m_a \]
for each \(a\).

Suppose \( \Phi(x) \) is invarient under
\[
\Phi'(x) = \exp \left[ i \frac{\phi \theta^a}{2} \right] \Phi(x)
\]
where \( \Phi(x) = (\phi_1(x)) \) both \( \phi_i \)s being complex.

For instance
\[
L = -\partial_m \Phi^+ \partial^m \Phi - m^2 \Phi^+ \Phi - \phi_1 \Phi^+ \Phi.
\]
The conserved currents are
\[
J^m_a = -i \frac{\partial}{\partial x^a} \epsilon_{mnr} \phi^i = i \partial^m \Phi^+ \frac{\partial}{2} \phi^i.
\]

In general the time components of the currents are
\[
J^0_a = -i \partial^m \epsilon_{mnr} \phi^i = -i \partial^m \Phi^+ \frac{\partial}{2} \phi^i.
\]

In this example\( \phi^+ = \phi^+ = -P \phi \)
\[
J^0_a = -i \partial^m \epsilon_{mnr} \phi^i = -i \phi_1 \phi^+ \frac{\partial}{2} \phi^i.
\]
In general

\[ [J_0^a (x, t), Q^m (y, t)] = [ -i \eta^a_0 \gamma^m \delta^3 (x - y), Q^m ] \]

\[ = - \eta^a_0 \gamma^m Q^m (x, t) \delta^3 (x - y) \]

And

\[ [J_0^a (x, t), P^m (y, t)] = [ -i \eta^a_0 \gamma^m \Delta^m, P^m ] \]

\[ = \delta^3 (x - y) \eta^a_0 \gamma^m P^m (x, t) \]

If the auxiliary fields \( C^m (x, t) = C^m (Q(x, t), P(x, t)) \)
and transform correctly, then also

\[ [J_0^a (x, t), C^m (y, t)] = - \delta^3 (x - y) \eta^a_0 \gamma^m C^m (x, t) \]

One may write these rules as

\[ [J_0^a (x, t), \phi^0 (y, t)] = - \delta^3 (x - y) \eta^a_0 \gamma^0 \phi^0 (x, t) \]

Such relations are used in QCD Ward identities.
Locally invariance

The action is invariant under

\[ x'^\mu = \Lambda^\mu_\nu x^\nu \]

with \( \Lambda_{\mu\nu} = -\Lambda_{\nu\mu} \). So we expect it to be conserved currents

\[ M^{\mu\nu} \]

with

\[ \partial_\mu M^{\mu\nu} = 0 \]

\[ M^{\mu\nu} = -M^{\nu\mu} \]

and conserved currents

\[ J^{\mu\nu} = \int d^3x \, M^{\mu\nu} \]

\[ \frac{d}{dt} J^{\mu\nu} = 0. \]

How does \( \partial_\lambda \Phi(x) \) transform?

\[ \partial_\lambda \Phi(x^\prime) = \partial_\lambda \Phi(x) \frac{\partial x^\prime}{\partial x^\nu} = \Lambda^\nu_\lambda \partial_\nu \Phi(x) \]

So

\[ \delta \partial_\lambda \Phi(x^\prime) = \omega^\lambda_\nu \partial_\nu \Phi(x) \]

\[ u(\Lambda) \psi^e_x(x) \tilde{u}(\Lambda) = D_{\lambda\lambda'}(\Lambda^{-1}) \psi^e_{\lambda'}(\Lambda x) \]

\[ u(\Lambda) \psi^e_x(x) \tilde{u}(\Lambda) = D_{\lambda\lambda'}(\Lambda) \psi^e_{\lambda'}(\Lambda^{-1} x) \]

In this case

\[ D_{\lambda\lambda'}(\Lambda) \psi^e_{\lambda'}(\Lambda^{-1} x) = (\delta_{\lambda\lambda'} + \frac{i}{2} \omega^{\mu\nu}_{\lambda\lambda'} \psi^e_{\lambda'})(\Lambda^{-1} x) \]
So on we switch to $\Lambda^{-1}$

\[ \Lambda^{-1} \nu = \Lambda^{-1} \nu = \delta \nu^m + \omega \nu^m \]

So derivatives transform as

\[ \delta \partial_\lambda \phi(x') = \omega^\lambda_k \partial_k \phi(x') \]

where $x' = \Lambda^{-1}x$.

Hence

\[ \delta (\partial_\lambda \psi^e) = \frac{i}{2} \omega^\lambda_k \partial_k \psi^m - \omega_k^\lambda \partial_k \psi^e \]

\[ U(\Lambda^{-1}) \mathcal{L}(x) U(\Lambda^{-1}) = \mathcal{L}(\Lambda^{-1}x) \]

So apart from $x' = x$, $\mathcal{L}$ is invariant.

That is under $(x)$ with no change in $x$, $\mathcal{L}(x)$ is invariant, whereas

\[ 0 = \frac{\partial L}{\partial \psi^e} \frac{i}{2} \omega^\lambda_m \partial_\lambda \psi^m \]

\[ + \frac{\partial L}{\partial \psi^e} \left[ \frac{i}{2} \omega^\lambda_m \partial_\lambda \psi^m + \omega_k^\lambda \partial_k \psi^e \right] \]

\[ = \frac{\partial L}{\partial (\partial_\lambda \psi^e)} \frac{i}{2} \omega^\lambda_m \partial_\lambda \psi^m \]

\[ + \frac{\partial L}{\partial (\partial_\lambda \psi^e)} \left[ \frac{i}{2} \omega^\lambda_m \partial_\lambda \psi^m + \omega_k^\lambda \partial_k \psi^e \right] \]
Now \[ w_k \cdot \Sigma_\ell \psi^\ell = \eta_k \sigma \cdot w_\sigma \cdot \partial \cdot \psi \]
\[ = \frac{1}{2} \eta_k \sigma \left( w^\sigma \cdot \not{\partial} - w^\partial \cdot \not{\sigma} \right) \psi \]
\[ = \frac{1}{2} \left( \eta_k \sigma w^{\mu \nu} \partial_\mu - \eta_{k \nu} w^{\mu \nu} \partial_\mu \right) \psi \]
\[ = \frac{1}{2} \left( \eta_{k \nu} w^{\mu \nu} \partial_\mu - \eta_{k \mu} w^{\mu \nu} \partial_\nu \right) \psi \]

So cancelling \( w^{\mu \nu} \), we have
\[ 0 = 2k \left( \frac{\partial \ell}{\partial (\partial \psi)} \right)^2 \frac{1}{2} \partial_{\mu \nu} \partial^\ell \psi^\ell \]
\[ + \frac{2k}{2 (\partial \psi)} \left[ \frac{1}{2} i \partial_{\mu \nu} \partial_k \psi^\ell + \frac{1}{2} \left( \eta_{k \mu} \partial_\nu - \eta_{k \nu} \partial_\mu \right) \psi \right] \]
\[ = 2k \left[ \frac{1}{2} \frac{\partial \ell}{\partial (\partial \psi)} \partial_{\mu \nu} \partial^\ell \psi^\ell \right] \]
\[ + \frac{1}{2} \frac{2k}{2 (\partial \psi)} \left( \eta_{k \mu} \partial_\nu - \eta_{k \nu} \partial_\mu \right) \psi \]

Recall (7.3.34) \[ T^\ell \mu = \delta^\ell \mu - \frac{\alpha k}{\partial (\partial \psi)} \partial \psi \]
Then
\[ T^\ell \mu - T^\ell \nu = - \frac{\partial \psi}{\partial (\partial \psi)} \partial_k \psi^\ell + \frac{\partial \psi}{\partial \psi} \partial_k \psi^\ell \]
So,
\[ 0 = \partial_k \left[ \frac{i}{2} \frac{\partial}{\partial (\partial_k \psi^k)} (\partial^{\mu} \psi^\mu) \partial^k \psi^k + \frac{1}{2} \left( \frac{\partial}{\partial (\partial^\mu \psi^\mu)} \psi^k - \frac{\partial}{\partial (\partial^k \psi^k)} \psi^\mu \right) \right] \]

\[ 0 = \partial_k \left[ J + \frac{1}{2} (T_{mn} - T_{nm}) \right] \quad (7.4.10) \]

So the A.S. part of $T_{mn}$ is a total divergence.

The Belinfante tensor $\Theta$ is
\[ \Theta^{\mu\nu} = T^{\mu\nu} - \frac{i}{2} \partial_k \left[ \frac{\partial}{\partial (\partial_k \psi^k)} (\partial^{\mu} \psi^\mu) \partial^k \psi^k - \frac{\partial}{\partial (\partial^k \psi^k)} (\partial^\mu \psi^\mu) \partial_k \psi^k \right] \]

in which $\partial_k \frac{1}{2} (T_{mn} - T_{nm})$ A.S. in $\mu, \nu, k$.

Hence $\partial_k \Theta^{\mu\nu} = \partial^m T^{m\nu} - \frac{1}{2} \partial^m \partial_k \left[ J \right]$

\[ = \partial^m T^{m\nu} = 0. \]

And
\[ \int \Theta^{\mu\nu} d^3x = \int d^3x (T^{\mu\nu} - \frac{1}{2} \partial_k \left[ \partial_{[\mu} \partial_{\nu]} \psi^k \right]) \]

\[ = \int d^3x \left( T^{\mu\nu} - \frac{1}{2} \partial^m \partial_k \left[ \partial_{[\mu} \partial_{\nu]} \psi^k \right] \right) = \int d^3x \partial_{[\mu} T^{\nu]} = \rho^\nu. \]

Hence $\rho^\nu = H$. 

\[ \quad 2. \left[ \text{AS} \right] = 0 \]
So $\Theta^{\mu v}$ is as good as $T^{\mu v}$. But $\Theta^{\mu v}$ is symmetric:

$$\Theta^{\mu v} - \Theta^{v \mu} = T^{\mu v} - T^{v \mu} - i \partial_k \left[ \frac{\partial}{\partial \partial_k \eta^p} \right] F^{\mu \nu} F_{\mu \nu} + i \partial_k \left[ \frac{\partial}{\partial \partial_k \eta^p} \right] F^{\mu v} F_{\mu v}$$

because the second two terms in $\Theta = T - [ ]$ are $\mu, \nu$ symmetric. But by (7.4.10) from

$$\Theta^{\mu v} = \Theta^{v \mu}.$$

This is the source of the quark internal field.

Since $\Theta$ is symmetric,

$$M^{\lambda \mu \nu} = x^\lambda \Theta^{\lambda \mu} - x^\nu \Theta^{\lambda \mu}$$

is conserved:

$$\partial_\lambda M^{\lambda \mu \nu} = x^\lambda \partial_\lambda \Theta^{\lambda \mu} - x^\nu \partial_\lambda \Theta^{\lambda \mu} + 8 \lambda \Theta^{\lambda \mu} - 8 \lambda \Theta^{\lambda \mu} = \Theta^{\mu v} - \Theta^{v \mu} = 0.$$

So the density

$$J^{\mu v} = \int M^{0 \mu \nu} d^3 x = \int d^3 x \left( x^\mu \Theta^{\mu v} - x^v \Theta^{\mu v} \right)$$

is constant,

$$J^{ij} = \frac{i}{2} \epsilon^{ijk} J^{ij}$$

is both constant and free of any explicit time dependence. So it obeys

$$0 = \left[ H, J \right].$$
By 7.3.28 \[ [\hat{p}^2, G(x)] = i \hat{\nabla} G(x) \]

\[ [p_j, J_i] = \frac{i}{2} \varepsilon_{ijk} [p_j, \Theta^{0k}] \]

\[ = \frac{i}{2} \varepsilon_{ijk} \int d^3x \ x^0 [p_j, \Theta^{0k}] - x^k [p_j, \Theta^{0k}] \]

\[ = \frac{i}{2} \varepsilon_{ijk} \int d^3x \ x^0 \delta_j^0 \Theta^{0k} - x^k \delta_j^0 \Theta^{0k} \]

\[ = \frac{i}{2} \int d^3x \ \delta_j^0 \Theta^{0k} - \delta^0_j \Theta^{0k} \]

\[ = \frac{i}{2} \int d^3x \ \varepsilon_{ijk} \Theta^{0k} = -\varepsilon_{ijk} P_k \]

which we usually write as

\[ [p_i, J_j] = i \varepsilon_{ijk} P_k \]

The boost \( K \) does not involve \( \varepsilon \) explicitly but is time independent.

\[ K = \int d^3x \ (x^k \Theta^{0k} - x^0 \Theta^{0k}) \]

\[ = -t P_k + \int d^3x \ x^k \Theta^{00}(x,t) \]

\[ \Theta = K = -\hat{p}^2 + i [\hat{H}, K] \]

So \[ [H, K] = -i \hat{p}^2 \] (2.4.24)
\[ (1.3.28) \Rightarrow [ P^j, G ] = i \Delta G. \quad \text{So} \]
\[
[ P_j, K_k ] = \int d^3 y \ y_k [ P_j, \Theta^{00} ] = i \int d^3 y \ y_k \partial \Theta^{00}
\]
\[
= i \int d^3 y \ y_k \frac{\partial \Theta^{00}}{\partial y_j} = -i \int d^3 y \ S^k_j \Theta^{00}
\]
\[
= -i \delta^k_j H
\]

Thus:
\[
[ P_j, K_k ] = -i \delta^k_j H. \quad (2.4.22)
\]

In general \( K \) is smooth, i.e.,
\[
\text{Hot} \int d^3 x \ x^i \Theta^{00} = \text{Hot}
\]
\[
\langle \rho \left( d^3 x \right)^i \Theta^{00} \rangle \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0
\]

where \( \Theta^{00} \) is the non-Ho part of \( \Theta^{00} \).

This resotheser and \( [ M, k ] = -i P^j \) causes \( \rho_0 \)

Her Lorents in d'cance of the \( S \)-matiex.

The \( J^{ij} \)'s are:

\[
J^{ij} = \int d^3 x \left( x^i \Theta^{0j} - x^j \Theta^{0i} \right)
\]
\[
= \int d^3 x \left( x^i \left[ \Theta^{0j} - \frac{i}{\hbar} \partial \Theta^{0j} \right] \partial_x \left[ \Theta^{0j} \right] \right. \\
\left. - \frac{2 i}{\hbar} \left[ \Theta^{0j} \right] \partial_x \left[ \Theta^{0j} \right] \right)
\]

\[ (1.3.28) \]
\[ J^{i j} = \int d^3x \, \psi^i \Gamma^{0 j} + \frac{i}{2} \left[ \frac{\partial \psi^i}{\partial (\partial \psi^j)} \begin{pmatrix} \partial \psi^j & \partial \psi^j \\ \partial \psi^j & \partial \psi^j \end{pmatrix} - \psi^i \Gamma^{0 j} \right] - \psi^i \Gamma^{0 j} \]
Note there were no $\phi$'s here. But $\phi^0$ mixes $\phi^1$ with $\phi^0$ and so $\phi^1$ is with $\phi^0$'s. So the
self-dimension of $[J^0, J^1]$ requires a case-by-case
dimension.

The Interaction Picture

Scalar field with derivative coupling

$$\mathcal{L} = -\frac{1}{2} \frac{d}{d^3x} \phi^2 \nabla^2 \phi - \frac{i}{2} m^2 \phi - \vec{J} \cdot \nabla \phi - V(\phi)$$

where $J^\mu$ may be external or composed of other
fields.

$$\Pi = \frac{\partial L}{\partial \dot{\phi}^*} = \dot{\phi} - J^0$$  \hspace{1cm} (H's fields)

$$N_{\mu \nu}
\Pi = \int d^3x \left( \Pi J^0 - \chi \right)$$

$$= \int d^3x \left( \Pi J^0 \right) + \frac{i}{2} \left( \nabla \phi \right)^2 - \frac{1}{2} \left( \Pi + J^0 \right)^2 + \frac{1}{2} m^2 \phi^2$$

$$+ \vec{J} \cdot \nabla \phi + J^0 \left( \Pi + J^0 \right) + V(\phi)$$

$$= \int d^3x \left( \frac{1}{2} \Pi^2 + \Pi J^0 + \frac{i}{2} \left( \nabla \phi \right)^2 + \frac{1}{2} m^2 \phi^2 + V(\phi) \right)$$

$$+ \frac{1}{2} J^0 + \vec{J} \cdot \nabla \phi$$

We choose to do pert. theory. We know $\Pi_0$.

$$\mathcal{H} = \mathcal{H}_0 + V$$

$$\mathcal{H}_0 = \int d^3x \left( \frac{1}{2} \Pi^2 + \frac{i}{2} \left( \nabla \phi \right)^2 + \frac{1}{2} m^2 \phi^2 \right)$$

So $V = \int d^3x \left( \Pi J^0 + \nabla \phi \cdot \vec{J} + \frac{i}{2} J^0 \right) + V(\phi)$.

All fields on this page are Heisenberg fields.
The interaction fields $\phi(x,t)$ and $\pi(x,t)$ are the Heisenberg fields at $t=0$ and evolve with $t$ via $H_0$. $H_0$ is fixed in the int. picture.

\[
H_0 = H_0(t) = \int d^3x \left\{ \frac{1}{2} \pi^2 + \frac{i}{2} (\partial \phi)^2 + \frac{i}{2} \bar{\pi}^2 \right\}, \quad \{ \phi, \pi \}
\]

and

\[
V(t) = \int d^3x \left[ \pi \mathcal{J}^0 + \partial \phi \cdot \mathcal{J} + \frac{i}{2} \mathcal{J}^0 \mathcal{J} + \mathcal{L}(\phi) \right]. \quad \{ \text{int. picture} \}
\]

Now

\[
\Pi = \dot{\phi} - \mathcal{J}^0 = i \left[ H_0, \phi \right] - \mathcal{J}^0 = i \left[ H_0, \phi \right]
\]

\[
\Pi = \frac{\partial}{\partial t} \left[ \phi \right] = i \left[ H_0, \phi \right] = \dot{\phi} \quad \text{at } t=0.
\]

So in $V(t)$ we may set

\[
\Pi(x,0) = i \left[ H_0, \phi(x,0) \right] = \dot{\phi}
\]

and then evolve $\Pi$ with $H_0$. Then $\Pi(x,t) = \dot{\phi}(x,t)$ in which both follow $H_0$'s time evolution.

\[
V(t) = \int d^3x \partial \phi \mathcal{J}^0 + \frac{i}{2} \mathcal{J}^2 + \mathcal{L}(\phi),
\]

in which the need for the non-covariant term

\[
\frac{1}{2} \mathcal{J}^2
\]

cancels a non-covariant term in the $\phi$ propagator as explained in 6.2, Eq. (6.2.27).
Vector field of spin one (H's fields)

\[ \mathcal{L} = -\frac{1}{2} \alpha \partial_\mu V^\mu \partial^\nu V_\nu - \frac{1}{2} \beta \partial_\mu V^\mu \partial^\nu V_\nu - \frac{i}{2} m^2 V^\mu V_\mu + j_\mu V^\mu \]

\[ \frac{\partial}{\partial V^\mu} \left( \left( -\alpha \partial^\nu V_\nu - \beta \partial_\nu (\partial_\mu V^\mu) \right) \right) = \frac{\partial \lambda}{\partial V^\mu} = -m^2 V_\mu + j_\mu \]

\[ -\alpha \partial_\mu V^\mu - \beta \partial_\nu (\partial_\mu V^\mu) + m^2 V_\mu = j_\mu \]

\[- (\alpha + \beta) \partial_\mu \partial_\nu V^\mu + m^2 \partial_\mu V^\mu = \partial_\mu j^\mu \]

This describes a scalar field \( \partial_\mu V^\mu \) of mass \( m^2 (\alpha + \beta) \) and source \( \partial_\mu j^\mu / (\alpha + \beta) \).

To prevent this field from being dynamical, we set \( \alpha = -\beta = 1 \) so that

\[ \mathcal{L} = -\frac{1}{2} \partial_\mu V^\mu \partial^\nu V_\nu + \frac{i}{2} m^2 V^\mu V_\mu + j_\mu V^\mu \]

We set \( F_{\mu \nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \), whence

\[ F_{\mu \nu} F^{\mu \nu} = \partial_\mu V_\nu \partial^\nu V^\mu + \partial_\nu V_\mu \partial^\mu V^\nu - \frac{1}{2} \partial_\mu V^\mu \partial^\nu V_\nu - \frac{1}{2} \partial_\nu V^\nu \partial^\mu V_\mu \]

\[ = 2 \partial_\mu V_\nu \partial^\mu V^\nu - 2 \partial_\mu V_\nu \partial^\nu V^\mu \]

So

\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} m^2 V_\mu V^\mu + j_\mu V^\mu \]

Now \( \frac{\partial \lambda}{\partial V^\mu} = -F^{\mu \nu} = 0 \) if \( m = 0 \).

\[ \frac{\partial}{\partial V^\mu} \]
So the \( V^i \)'s \( i = 1, 2, 3 \) are canonical fields with

\[
\Pi^i = \partial \xi^i - F^i_0 = F^{i0} = \partial_i V^0 - \partial^0 V^i \tag{10}
\]

and \( F^{00} = 0 = m^2 V^0 \) in \( \Phi \). So \( V^0 \) is an auxiliary field. Its field equation is

\[
\partial_0 V^0 = -m^2 V^0 + J^0 = \partial_i \partial_i \xi^0 = -\partial_i F^{i0} \quad \partial_0 \xi^0
\]

\[
\partial^0 \partial_0 = m^2 V^0 - J^0 \quad \text{like Gauss's law}
\]

which has no \( V^m \) terms at all. It's a constraint, which we use to solve for \( V^0 \)

\[
V^0 = \frac{1}{m^2} (J^0 + \partial_i F^{i0})
\]

\[
= \frac{1}{m^2} (\nabla \cdot \Pi + J^0).
\]

Now

\[
\Pi = \int d^3 x \ \Pi \cdot \nabla - 2
\]

By (10), \( \Pi = \nabla V^0 + \dot{V}^0 \), so \( \dot{V}^0 = \Pi - \nabla V^0 \) or

\[
\dot{V}^0 = \Pi - \frac{1}{m^2} \nabla (\nabla \cdot \Pi + J^0).
\]
\[ H = \int d^3x \, \pi \cdot \nabla \cdot \nabla (\nabla \cdot \pi + J^0) \]
\[ = \int d^3x \, \left( \pi - \frac{1}{m^2} \nabla \cdot \left( \nabla \pi + J^0 \right) \right) - \frac{\pi^2}{2} + \frac{1}{2} \left( \nabla \pi \cdot \nabla \pi \right) + \frac{1}{2 m^2} \left( \nabla \cdot \left( \nabla \pi + J^0 \right) \right) \]
\[ + \frac{1}{2} m^2 V^2 - J \cdot V + \frac{J^0 \left( \nabla \cdot \pi + J^0 \right)}{m^2} \]
\[ = \int d^3x \, \frac{1}{2} \pi^2 + \frac{1}{2} \left( \nabla \pi \cdot \nabla \pi \right) + \frac{1}{2 m^2} V^2 + \frac{1}{2} \nabla \pi \]
\[ + \int d^3x \, \frac{J^0 \nabla \cdot \pi - J \cdot V + J^0}{m^2} \]
\[ = H_0 + V \quad \text{So far all fields are Heisenberg fields.} \]

In terms of L.P. fields \( \pi \neq 0 \)
\[ H_0 = \int d^3x \, \frac{1}{2} \pi^2 + \frac{1}{2 m^2} \left( \nabla \pi \cdot \nabla \pi \right) + \frac{1}{2} \left( \nabla \pi \right)^2 + \frac{1}{2} m^2 \nu^2 \]
\[ V = \int d^3x \, -J \cdot V + m^2 \frac{J^0 \nabla \cdot \pi}{m^2} + \frac{1}{2 m^2} J^0 \nu^2. \]

So
\[ \nu = i \left[ H_0, \nu \right] = \pi^2 - \frac{1}{m^2} \nabla \left( \nu, \pi \right) \text{in the L.P.} \quad (7.5.20) \]

To evaluate
\[ \pi = i \left[ H_0, \pi \right] - m^2 \nu \]

we first write
\[ \int d^3x \, \left( \nabla \pi \cdot \nabla \pi \right) = \int d^3x \, \epsilon_{ij} \epsilon_{kl} \frac{d_j V_k + d_k V_i}{m^2} \]
\[ = \int d^3x \, \left( \epsilon_{ij} \delta_{km} - \delta_{jm} \epsilon_{kl} \right) \frac{d_j V_k + d_k V_i}{m^2} = \int d^3x \, \frac{d_j V_k + d_k V_i}{m^2} - \frac{d_j V_k + d_k V_i}{m^2}, \]
\[ \int d^3x \left( \nabla \cdot \mathbf{v} \right)^2 = \int d^3x \left( \mathbf{v}_k \mathbf{\nabla} \cdot \mathbf{v}_k - \partial_k \left( \mathbf{v}_k \right) \partial_j \mathbf{v}_j \right) \]

\[ = \int d^3x \left( \mathbf{v}_k \mathbf{\nabla} \cdot \mathbf{v}_k + \mathbf{v}_k \partial_k \partial_j \mathbf{v}_j \right) \]

\[ = \int d^3x \left( \mathbf{v}_k \mathbf{\nabla} \cdot \mathbf{v}_k \right) + \mathbf{v} \cdot \mathbf{\nabla} \left( \nabla \cdot \mathbf{v} \right) \]

Then

\[ \mathbf{\Pi} = i \left[ H_0, \mathbf{\Pi} \right] = -m^2 \mathbf{\Pi} - \nabla \left( \mathbf{\Pi} \cdot \mathbf{\Pi} \right) \]

\[ \mathbf{\Pi} \text{ is the I. P. at } t = 0 \]

\[ \mathbf{\Pi} = \mathbf{\Pi} = \left[ H_0, \mathbf{\Pi} \right] + \nabla \left( \mathbf{\Pi} \cdot \mathbf{\Pi} \right) \]

So \[ \mathbf{\Pi} = \mathbf{\Pi} - \nabla \left( \mathbf{\Pi} \cdot \mathbf{\Pi} \right) \]

\[ \text{by } (\star) \]

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\[ \mathbf{\Pi} = \mathbf{\Pi} = \mathbf{\Pi} \]

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\[ \mathbf{\Pi} = \mathbf{\Pi} = \mathbf{\Pi} \]

\[ \text{by } (\star \star \star \star \star \star \star \star \star \star \star \star \star \star \star) \]

\[ \mathbf{\Pi} = \mathbf{\Pi} = \mathbf{\Pi} \]

\[ \text{by } (\star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star) \]

\[ \mathbf{\Pi} = \mathbf{\Pi} = \mathbf{\Pi} \]

\[ \text{by } (\star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star) \]

\[ \mathbf{\Pi} = \mathbf{\Pi} = \mathbf{\Pi} \]

\[ \text{by } (\star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star) \]

\[ \mathbf{\Pi} = \mathbf{\Pi} = \mathbf{\Pi} \]

\[ \text{by } (\star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star \star) \]
\[ \int d^3x \left( \nabla u \right)^2 = \int d^3x \left( -u_k \partial^2_v + u \partial (\nabla u) \right) \]

Then
\[ \pi = i \left[ H_0, \pi \right] \]

\[ = i \int d^3x \left[ -\frac{1}{2} u \cdot \nabla^2 u + \frac{1}{2} \nabla (\nabla u) + \frac{m^2 v^2}{2} \right] \pi \]

\[ = \nabla^2 u - \nabla (\nabla u) - m^2 u \quad (7.5.21) \]

Now \( V^0 \) does not appear in \( H_0 \) in \( v \). We may define
\[ v^0 = m^2 \nabla \pi. \quad (7.5.22) \]

Then \( (7.5.20) \) gives
\[ \dot{\pi} = \pi - \nabla v^0 \quad (7.5.23) \]

by \( (22) \)

\[ \nabla \cdot \dot{\pi} = \nabla \cdot \pi - \nabla^2 v^0 = m^2 v^0 - \nabla^2 v^0 \quad \Rightarrow \]

\[ \nabla^2 v^0 + \nabla \cdot \dot{\pi} - m^2 v^0 = 0 \quad (\text{con}) \]

and \( (21) \) & \( (23) \) give
\[ \pi = \dot{\pi} + \nabla v^0 = \nabla^2 u - \nabla (\nabla u) - m^2 u \quad \Rightarrow \]

\[ \nabla^2 u - \nabla (\nabla u) - \dot{u} - \nabla v^0 - m^2 u = 0 \quad \Rightarrow \]

\[ \nabla v^i - \partial_j u_j - \partial_i \partial_j v^0 - m^2 v^0 = 0 \quad \Rightarrow \]

\[ \nabla v^i - \partial_j \partial_j v^i - m^2 v^i = 0 \quad (\text{con}) \]
\((\text{con})_0\) : \(\Box v^0 + \Box (v^\nu v^\nu) - m^2 v^0 = 0\) \(\Box\) 

\(\Box v^0 - \Box (v^\nu v^\nu) - m^2 v^0 = 0\) \(\text{(con')}\) 

So \((\text{con}) \text{ & } (\text{con'})\) now give 

\(\Box v^\mu - \Box (v^\nu v^\nu) - m^2 v^\mu = 0\) \(\text{(24)}\) 

\(\Box \partial^\mu v^\nu - \Box (v^\nu v^\nu - m^2) \partial^\mu v^\mu = 0\) \(\text{(25)}\) 

whence 

\((\Box - m^2) v^\mu = 0\) \(\text{(26)}\) 

and 

\(\partial^\mu v^\mu = 0\) \(\text{(25')}\)
So \[ \nabla u^m - \partial^m u^\nu \nabla_\nu - m^2 u^m = 0 \]

\[ \square u^m - \partial \partial u^m - m^2 \partial u^m = 0 \]

where \( m^2 = 0 \).

(26)

A real vector field obeying (25-26) may be written as

\[ u^m(x) = \sum_s \int \frac{d^3 p}{2 \sqrt{2 \pi} \hbar} \left\{ \begin{array}{cc} e^m(p, \nu) a(p, 0) e_{\nu} & \nu = \sigma + 1, 0, -1, \\
 \end{array} \right. \]

with \( p^0 = \sqrt{p^2 + m^2} \) and \( \sigma = 1, 0, -1, \)

\[ p^m e^m(p, \nu) = 0 \]

and

\[ \sum_{\nu} e^m(p, \nu) e^{\nu}(p, 0) = \eta^m_n + \frac{p^m p^n}{m^2}, \]

which is (5.3. 28-29).

In fact \( u^m \) and \( a^\dagger(p, 0) \) act by at equal times

\[ [v_i(x, t), \pi^j(x', t')] = i \delta_{ij} \delta(x - x') \delta(t - t') \]

\[ [v_i, v_i] = [\pi^i, \pi^i] = 0 \]

\[ [a(p, \nu), a^\dagger(p', \nu')] = \delta_\nu^{\nu'} \delta(p - p') \]

\[ [a(p, \nu), a(p', 0)] = 0. \]
Since the expansion (7.5.27) for $V^0(x)$ was derived in 5, these results show that $H_0$ is constant. One may show that

$$H_0 = \sum_o \int d^3p \ p^0 \left( a^2(p.r) a(p.r) + \frac{1}{2} \right).$$

Finally, by using $V^0 = 17.11/\text{m}^2$, we may write $V$ as

$$V = \int d^3x - S_m V^0 + \frac{1}{2m^2} (F^0)^2$$

in which the term $F^0$ cancels a running kerne in the propagator of $V^0$ as shown in 6.
\( \chi = - \frac{i}{\hbar} (\gamma^\mu \partial_\mu + m) \psi - \mathcal{H}(\psi, \bar{\psi}) \).

\( \chi \) is real apart from a total divergence.

\( \chi - \chi^+ = - \frac{i}{\hbar} \sum_{\alpha} \gamma^\alpha \partial_\alpha \psi + \partial_\mu \gamma^\mu \gamma^\alpha (i \gamma^\beta \gamma^\alpha + i \gamma^\beta + \gamma^\alpha) \psi \)

\( (i \gamma^\beta - \beta - \beta) \gamma^\mu \gamma^\beta = - \gamma^\mu \)

\( \chi - \chi^+ = - \frac{i}{\hbar} \sum_{\alpha} \gamma^\alpha \partial_\alpha \psi - \partial_\mu \gamma^\mu \gamma^\alpha \psi \)

\( = - \partial_\mu (\gamma^\mu \psi) \).

So \( \chi \) and \( \chi^+ \) generate the same field equations.

\( \Pi = \frac{\partial \mathcal{H}}{\partial \psi} = - \frac{i}{\hbar} \gamma^0 = - \frac{i}{\hbar} i \gamma^0 \gamma^0 = i \gamma^+ \)

\( \mathcal{H} = \int d^3 x \, \Pi \psi - \chi = \int d^3 x \, \Pi \psi + \frac{i}{\hbar} \gamma^0 \psi + \gamma^0 (\gamma^0 \mathcal{H} + m) \psi + \mathcal{H} \)

\( = \int d^3 x \, \Pi \psi - \Pi \psi + \frac{i}{\hbar} (\gamma^0 \gamma^0 + m) \psi + \mathcal{H} \)

\( = \int d^3 x \, \Pi \gamma^0 (\gamma^0 \gamma^0 + m) \psi + \mathcal{H} \)

\( = H_0 + \mathcal{V} \) \quad \Pi \gamma^0 \overset{\text{means}}{=} \Pi \gamma^0 \)

\( H_0 = \int d^3 x \, \Pi \gamma^0 (\gamma^0 \gamma^0 + m) \psi \)

\( \mathcal{V} = \int d^3 x \, \Pi \gamma^0 (\bar{\psi}, \psi) \).

\( \Pi = - \frac{\bar{\psi} \gamma^0}{\mathcal{M}^2} \quad \text{means} \quad \Pi = - \frac{i}{\hbar} \gamma^0 \).
\[ \dot{\psi}_\alpha = i [H_0, \psi_\alpha] = i \int d^3x \left[ \Gamma^0 (\gamma \cdot D + m) \psi, \psi_\alpha(x) \right] \]

\[ = i \int d^3x' \delta^3(x - x') \left( \psi(x) \left( \gamma^0 \gamma \cdot (D + m) \psi(x') \right) \right)_\beta \]

\[ = \int d^3x' \delta(x - x') \left( \gamma^0 \gamma \cdot (D + m) \psi(x') \right) \beta \]

\[ = \left[ \gamma^0 \gamma \cdot (D + m) \right]_\alpha \gamma \psi_\alpha(x) \]

\[ \psi = \gamma^0 (\gamma \cdot D + m) \psi \]

\[ \gamma^0 \psi_\alpha = - (\gamma \cdot D + m) \psi \]

\[ (\gamma^0 \partial^\alpha + m) \psi = 0 \]

At equal times \[ \left[ \psi(x), \pi_\beta(x) \right] = \frac{i}{\hbar} \int d^3x' \left( \pi \gamma^0 \gamma \cdot (D + m) \psi(x') \right) \pi_\beta(x) \]

\[ \pi = i \left[ \chi_0, \pi \right] = i \int d^3x' \left( \pi \gamma^0 \gamma \cdot (D + m) \psi(x') \right) \pi_\beta(x) \]

\[ = \pi \gamma^0 \gamma \cdot (D + m) \psi \]

which, since \[ \pi = - i \gamma^0 \psi \] is

\[ - \dot{\psi}^0 = - \gamma^0 \gamma \cdot (D + m) \]

\[ i \dot{\psi}^+ = + \gamma^0 \gamma \cdot (D + m) \]

\[ = - \gamma^0 \beta \left( \beta \gamma^0 \gamma \cdot (D + m) \right) = - \gamma^0 \beta \left( \gamma^0 \gamma \cdot (D + m) \right) \beta \]

\[ i \dot{\psi}^\beta = - \gamma^0 \gamma \cdot (D + m) \beta \]

\[ \dot{\psi}^0 = - \gamma^0 \gamma \cdot (D + m) \]

\[ 0 = \gamma^0 \gamma \cdot (D + m) \]

which in \( \left[ 0 = (\gamma^m \partial_m + m) \psi \right] \)}
The general solution of \( \psi = (\psi_1 \psi_2)^T \) is

\[
\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{Z} \left[ u(p_0) e^{ip_0 x} + v(p_0) e^{-ip_0 x} \right]
\]

in which \( p_0 = \sqrt{p^2 + m^2} \), \( \alpha \) and \( \alpha^* \) are separation coefficients, and \( u(p_0) \) and \( v(p_0) \) are two independent solutions of

\[
(i\sigma_y \gamma^m + m) u(p_0) = 0
\]

and \( v(p_0) \) are two \( \gamma \)

\[
(-i\sigma_y \gamma^m + m) v(p_0) = 0
\]

normalized so that

\[
\sum_{p_0} u(p_0) \bar{u}(p_0) = \frac{1}{Z} \frac{1}{2p_0}
\]

\[
\sum_{p_0} v(p_0) \bar{v}(p_0) = -\frac{(i\gamma_y \gamma^m + m)}{2p_0}
\]

Here \( i\gamma_y \gamma^m \) has even's \( m \).

\[
-i\gamma_y \gamma^m \bar{u} = m \bar{u}
\]

\[
-i\gamma_y \gamma^m v = m v
\]

\[
(-i\gamma_y + m) \bar{v} = 2m \bar{v}
\]

\[
(i\gamma_y + m) v = 2m v
\]

\[
2m \Sigma v \bar{u} = (-i\gamma_y + m) \Sigma \bar{v} u
\]

\[
2m \Sigma \bar{u} v = (i\gamma_y + m) \Sigma u \bar{v}
\]

\[
\Sigma \bar{u} v = \frac{-i\gamma_y + m}{2m} \Sigma \bar{u} \bar{v} = \frac{i\gamma_y + m}{2m} \Sigma \bar{u} v = \frac{2m (\bar{u} \bar{v})}{2p_0}
\]

\[
\Rightarrow \bar{u} v = \frac{i\gamma_y}{2p_0}
\]
\[ t \sum \sin^2 \beta = \sum u^2 u = 2 = \frac{t \left( -i p_0 Y^0 + m \right) Y^0}{2p^0} \]

\[ = \frac{t - p_0 I}{2p^0} = \frac{1}{2} t I = 2 \]

\[ t \sum \nu \nu = \sum v^2 v = -t \left[ \frac{1}{2p^0} \right] \left( -i p_0 Y^0 + m \right) Y^0 \]

which verify the work done in 5, \((-\rho = 4)\)

To have the ETAC's : \(\pi = -4Y^0 p\)

\[ \left[ \psi_\alpha (x, t), \psi_\beta (x, t) \right]_+ = \left[ \psi_\alpha (x, t), \pi \psi_\beta (x, t) \right]_+ Y^0 \]

\[ = i \delta(x - x') \delta_\alpha \beta Y^0 \]

\[ = i \delta(x - x') \delta_\alpha \beta \]

With \(\left[ \psi_\alpha (x, t), \psi_\beta (x, t) \right]_+ = \delta_\alpha \beta \delta(x - x') \)

\[ \left[ \psi_\alpha (x, t), \psi_\beta (x, t) \right]_+ = 0 \]

\[ \text{where} \quad \left[ a(p, \rho), a^\dagger (p', \rho') \right]_+ = \delta \omega \delta(p - p') \]

\[ \left[ a(p, \rho), a^\dagger (p', \rho') \right]_+ = \delta \omega \delta(p - p') \]

\[ \left[ a, a^\dagger \right]_+ = \left[ a, a^\dagger \right] = 0 < 0 = \left[ a, a^\dagger \right] = \left[ a, a^\dagger \right] \]

so \(\pi \leq 5\) : \(7, 5, 37\) is a gauge \(H_0\).

\[ H_0 = \sum p^0 \left( a^\dagger (p, \rho) a(p, \rho) - a(p, \rho) a^\dagger (p, \rho) \right) \]

\[ = \sum p^0 \left( a^\dagger (p, \rho) a(p, \rho) + a(p, \rho) a(p, \rho) - \delta^3 (p - p') \right) \]

\[ \text{only quantum} \quad 5 \leq \gamma \]
Dirac Brackets

Primary constraints are imposed (e.g. as gauge conditions) or arise from $L$. Thus if $f$ does not appear in $L$, then

$$\Pi_f \cdot \frac{\partial L}{\partial \Pi_f} = 0.$$

In general primary constraints arise when the equations

$$\Pi_f = \frac{\partial L}{\partial \dot{f}}$$

cannot be solved for the $\dot{f}$'s, i.e., when the matrix

$$M_{nm} = \frac{\partial^2 L}{\partial \dot{f}_n \partial \dot{f}_m} - \frac{\partial \Pi_n}{\partial \Pi_m} \frac{\partial \Pi_m}{\partial \Pi_n}$$

is singular, i.e., $\det M = 0$. Such $L$ are irregular.

Secondary constraints arise from the field equations and the primary constraints. For the massive vector field, since $\Pi^a = 0$, the 0th field equation is

$$D_i T^i = m^2 V^0 - J^0.$$

In $5D$ we have Gauss's law $\nabla \cdot E = \rho = J^0$, which is a secondary constraint.