Goldstone bosons & the Higgs mechanism 1

First, Goldstone bosons:

$$\mathcal{L} = \sum_{i=1}^{2} -\frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - \lambda\left(\sum \phi_i^2 - \phi_0^2\right)^2$$

Must have

$$\langle \sum \phi_i^2 \rangle_0 = \phi_0^2$$

Suppose $\phi_1 = \phi_0 + \tilde{\phi}_1$, that is

$$\langle \phi_1 \rangle_0 = \phi_0 \quad \text{and} \quad \langle \phi_2 \rangle = 0.$$

Then

$$\phi_1^2 + \phi_2^2 - \phi_0^2 = (\phi_0 + \tilde{\phi}_1)^2 + \phi_2^2 - \phi_0^2$$

$$= 2\phi_0 \tilde{\phi}_1 + \tilde{\phi}_1^2 + \phi_2^2, \text{ so}$$

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \tilde{\phi}_1 \partial^\mu \tilde{\phi}_1 - \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2$$

$$-\lambda\left(2\phi_0\tilde{\phi}_1 + \tilde{\phi}_1^2 + \phi_2^2\right)^2$$

$$= \mathcal{L}_0 - 4\lambda\phi_0^2\tilde{\phi}_1^2 - 4\lambda\phi_0\tilde{\phi}_1\left(\tilde{\phi}_1^2 + \phi_2^2\right)$$

$$-\lambda\left(\tilde{\phi}_1^2 + \phi_2^2\right)$$

The usual mass term is $-\frac{\mu^2}{2}\phi^2$,

so

$$-\frac{\mu^2}{2} = -4\lambda\phi_0^2 \qquad \mu^2 = 8\lambda\phi_0^2$$

So in this theory, ϕ_1 describes a boson of mass $2\sqrt{2}\lambda\phi_0$ and ϕ_2 a boson of mass zero. The massless boson is called a Goldstone boson.

Now, the Higgs mechanism:

$$\mathcal{L}_2 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(D_\mu \phi)_i (D^\mu \phi)_i - \lambda\left(\sum \phi_i^2 - \phi_0^2\right)^2,$$

where

$$D_\mu \phi = \partial_\mu \phi + e A_\mu t$$

and t is the matrix

$$t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ with } t^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -I$$

which generates $O(2) = U(1)$.

$$\phi' = e^{\theta t}\phi = \left[\sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!} + t \sum \frac{\theta^{2n+1}}{(2n+1)!}\right]\phi$$

$$\phi' = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \phi$$

Now the Higgs potential $\lambda(\phi_i^2 - \phi_0^2)^2$ still gives mass $\mu = 8\lambda\phi_0^2$ to $\tilde{\phi}_1$'s boson, but also the $(D_\mu \phi)^2$ term is a mass term for the photon:

$$-\frac{1}{2}(D_\mu\phi)D^\mu\phi = -\frac{1}{2}\phi^T(\overleftarrow{\partial}_\mu + eA_\mu t^T)(\partial^\mu + eA^\mu t)\phi$$

$$= -\frac{1}{2}(\phi_0 + \tilde{\phi}_1, \phi_2)\left(\overleftarrow{\partial}_\mu + eA_\mu\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\left(\partial^\mu + eA^\mu\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\begin{pmatrix} \phi_0 + \tilde{\phi}_1 \\ \phi_2 \end{pmatrix}$$

$$= -\frac{1}{2}\left(\partial_\mu\tilde{\phi}_1 + eA_\mu\phi_2, \partial_\mu\phi_2 - eA_\mu(\phi_0 + \tilde{\phi}_1)\right)\begin{pmatrix} \partial^\mu\tilde{\phi}_1 + eA^\mu\phi_2 \\ \partial^\mu\phi_2 - eA^\mu(\phi_0 + \tilde{\phi}_1) \end{pmatrix}$$

$$= -\frac{1}{2}(\partial_\mu\tilde{\phi}_1 + eA_\mu\phi_2)(\partial^\mu\tilde{\phi}_1 + eA^\mu\phi_2)$$

$$-\frac{1}{2}(\partial_\mu\phi_2 - eA_\mu(\phi_0 + \tilde{\phi}_1))(\partial^\mu\phi_2 - eA^\mu(\phi_0 + \tilde{\phi}_1))$$

in which $-\frac{1}{2}e^2\phi_0^2 A_\mu A^\mu$ is a mass term for A_μ. The full mass term is

$$-\frac{1}{2}e^2\phi_0^2 W_\mu W^\mu \quad \text{where} \quad W_\mu = A_\mu - \frac{1}{e\phi_0}\partial_\mu\phi_2.$$

So the "longitudinal" component of W_μ is $-\frac{1}{e\phi_0}\partial_\mu \phi_2$.

Note that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu W_\nu - \partial_\nu W_\mu$$

because $[\partial_\mu, \partial_\nu]\phi = 0$.

So instead of a theory with 2 spinless bosons, one massive, and one massless photon, we have one massive spinless boson, "the Higgs," and one massive spin-one boson, "the W." This is the "Higgs mechanism." In the standard model the gauge group is $SU_2^L \otimes U_1$ in which the L means that only the "left-handed" fermions are moved by SU_2^L.