1 Masses

Masses occur in the action density as the coefficients of terms quadratic in the fields. Thus, in the case of a neutral, spin-zero field $\phi$ the action density is

$$L = -\frac{1}{2} \partial_a \phi \partial^a \phi - \frac{1}{2} \mu^2 \phi^2,$$

and the mass is $\mu$. The equation of motion is

$$\left( \partial_a \partial^a - \mu^2 \right) \phi(x) = 0.$$

The field obeying this equation is

$$\phi(x) = \int \left[ a(k)e^{ikx} + a^\dagger(k)e^{-ikx} \right] \frac{d^3k}{\sqrt{(2\pi)^32k^0}}.$$ 

The charged spin-zero field is a complex linear combination of two equal-mass real fields

$$\phi = \frac{1}{\sqrt{2}} \left( \phi^{(1)} + i\phi^{(2)} \right).$$
Its action density is

\[ L = -\partial_a \phi^* \partial^a \phi - \mu^2 |\phi|^2, \] (5)

and its equation of motion is

\[ (\partial_a \partial^a - \mu^2) \phi(x) = 0. \] (6)

It is

\[ \phi(x) = \int \left[ a(k)e^{2\pi k x} + b^\dagger(k)e^{-2\pi k x} \right] \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}}. \] (7)

For a Dirac spin-one-half field, the action density is

\[ L = -\overline{\psi} (\gamma^a \partial_a + m) \psi \equiv -\overline{\psi} (\partial + m) \psi \] (8)

in which \( \overline{\psi} = i\psi^\dagger \gamma^0 = \psi^\dagger \beta \). The equation of motion is

\[ (\gamma^a \partial_a + m) \psi = (\partial + m) \psi = 0. \] (9)

The field is

\[ \psi_j(x) = \frac{1}{\sqrt{2}} \left( \psi_j^{(1)}(x) + i\psi_j^{(2)}(x) \right) \]

\[ = \sum_{s=\pm} \int \left[ u_j(\vec{p}, s)b(p, s)e^{ipx} + v_j(\vec{p}, s)c^\dagger(p, s)e^{-ipx} \right] \frac{d^3p}{(2\pi)^{3/2}} \] (10)

in which the fields \( \psi^1 \) and \( \psi^2 \) are the Majorana fields that make the Dirac field, and the Dirac index \( j \) runs from 1 to 4, \( px = \vec{p} \cdot \vec{x} - p^0 t, \ p^0 = \sqrt{\vec{p}^2 + m^2} \),

\[ b(p, s) = \frac{1}{\sqrt{2}} \left( a(p, s, 1) + ia(p, s, 2) \right) \] (11)

\[ c^\dagger(p, s) = \frac{1}{\sqrt{2}} \left( a^\dagger(p, s, 1) + ia^\dagger(p, s, 2) \right), \] (12)
and the annihilation $a_i$ and creation $a_j^\dagger$ operators satisfy the anticommutation relations
\[
\{a(p, s, i), a(p', s', j)\} \equiv a(p, s, i) \ a(p', s', j) + a(p', s', j) \ a(p, s, i) = 0
\]
\[
\{a(p, s, i), a^\dagger(p', s', j)\} = \delta_{ij} \delta_{ss'} \delta^{(3)}(\vec{p} - \vec{p'}) .
\]

For a massive vector field, the action density is
\[
L = -\frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} m^2 A_a A^a
\]
in which $F_{ab} = \partial_a A_b - \partial_b A_a$. The equation of motion is
\[
\partial_a F^{ab}(x) = m^2 A^b(x) .
\]
This field contains a part that is spin zero. The spin-zero part is the divergence $\partial_b A^b$, and the spin-one part has zero divergence
\[
\partial_b A^b = 0 .
\]
So the equation of motion of the spin-one part is
\[
(\Box - m^2) A_b(x) = (\Delta - \partial_t^2 - m^2) A_b(x) = 0 .
\]
The spin-one field is
\[
A_b(x) = \sum_{s=-1}^{1} \int [e_b(p, s)a(p, s)e^{ipx} + e^*_b(p, s)a^\dagger(p, s)e^{-ipx}] \frac{d^3p}{\sqrt{(2\pi)^32p^0}}
\]
in which the sum is over $s = -1, 0, 1$,
\[
p^a e_a(p, s) = 0 ,
\]
and the spin sum is
\[
\sum_{s=-1}^{1} e_a(p, s)e^*_b(p, s) = \eta_{ab} + \frac{p_a p_b}{m^2} .
\]
Homework problem 1 of set 2: Use Lagrange’s equations to derive the equation of motion (15) from the action density (14).

Homework problem 2 of set 2: Use the condition $\partial b A^b = 0$ to convert the equation of motion (15) to its spin-one form (17).

Homework problem 3 of set 2: Show that the zero-divergence condition (16) implies the spin condition (19).

2 Spontaneous Symmetry Breaking

The usual $\phi^4$ theory has action density

$$L = -\frac{1}{2} \partial_a \phi \partial^a \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda \phi^4.$$  \hspace{1cm} (21)

In the limit $\lambda \to 0$, this theory is that of equations (1–3). If we flip the sign of the mass term in (22), then we have

$$L = -\frac{1}{2} \partial_a \phi \partial^a \phi + \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda \phi^4.$$  \hspace{1cm} (22)

Now both action densities are symmetric under the reflection $\phi(x) \to -\phi(x)$, which is a **discrete symmetry**.

To the extent that we understand such theories, the vacuum of the first theory has $\langle 0 | \phi | 0 \rangle = 0$. This vacuum is invariant under the reflection $\phi(x) \to -\phi(x)$. But there are two vacua of the second theory. The potential energy of that theory is

$$V = -\frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4,$$  \hspace{1cm} (23)

and it has two minima at

$$\phi_\pm = \pm \frac{m}{\sqrt{\lambda}} \equiv \pm v.$$  \hspace{1cm} (24)

These vacua are not invariant under the reflection $\phi(x) \to -\phi(x)$; they transform into each other $\phi_+ \to \phi_-$. So if the states of the universe are clustered about $\phi_+$, then in that universe, the mean value of the field in its vacuum is

$$\langle 0_+ | \phi | 0_+ \rangle = \phi_+.$$  \hspace{1cm} (25)
We say, “the vacuum spontaneously beaks the reflection symmetry.”

If we set

$$\phi(x) = v + \sigma(x),$$

then the action density of the theory near $\phi_+$ is

$$L = -\frac{1}{2} \partial_a \sigma \partial^a \sigma + \frac{1}{2} m^2 (v + \sigma)^2 - \frac{1}{4} \lambda (v + \sigma)^4$$

$$= -\frac{1}{2} \partial_a \sigma \partial^a \sigma + \frac{1}{2} m^2 (v^2 + 2v \sigma + \sigma^2) - \frac{1}{4} \lambda (v^4 + 4v^3 \sigma + 6v^2 \sigma^2 + 4v \sigma^3 + \sigma^4)$$

$$= -\frac{1}{2} \partial_a \sigma \partial^a \sigma + \frac{1}{2} m^2 \left[ \left( \frac{m}{\sqrt{\lambda}} \right)^2 + 2 \frac{m}{\sqrt{\lambda}} \sigma + \sigma^2 \right]$$

$$- \frac{1}{4} \lambda \left[ \left( \frac{m}{\sqrt{\lambda}} \right)^4 + 4 \left( \frac{m}{\sqrt{\lambda}} \right)^3 \sigma + 6 \left( \frac{m}{\sqrt{\lambda}} \right)^2 \sigma^2 + 4 \frac{m}{\sqrt{\lambda}} \sigma^3 + \sigma^4 \right]$$

$$= -\frac{1}{2} \partial_a \sigma \partial^a \sigma - m^2 \sigma^2 - \sqrt{\lambda} m \sigma^3 - \frac{1}{4} \lambda \sigma^4 + \frac{1}{4} \frac{m^4}{\lambda} \tag{27}$$

in which the last term is a constant (and so is relevant only in gravitational theories where it might represent dark energy). In the limit $\lambda \to 0$, this theory is that of a particle of mass $\sqrt{2} m$.

If we replace the single field $\phi$ by an $n$-vector of fields $\phi_i$, then we get the linear sigma model with action density

$$L = -\frac{1}{2} \sum_{i=1}^{n} \partial_a \phi_i \partial^a \phi_i + \frac{1}{2} m^2 \sum_{i=1}^{n} \phi_i^2 - \frac{1}{4} \lambda \left( \sum_{i=1}^{n} \phi_i^2 \right)^2 \tag{28}$$

or with $\phi^2 \equiv \phi_1^2 + \cdots + \phi_n^2$

$$L = -\frac{1}{2} \sum_{i=1}^{n} \partial_a \phi_i \partial^a \phi_i + \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda \left( \phi^2 \right)^2. \tag{29}$$

Again the mass term has the wrong sign. In what follows, we will not bother to indicate sums over a repeated index $i$ from 1 to $n$. 

5
This $L$ is invariant when the fields change by

$$\phi'_i = O_{ik} \phi_k$$

in which $O$ is an $n \times n$ orthogonal matrix. That is, the squared length

$$\phi_i'^2 = (O_{ik} \phi_k)^2 = \phi_k^2.$$  

(31)

The action density is invariant under the nonabelian Lie group $O(n)$. This is a continuous symmetry.

What are the minima of the potential energy

$$V = -\frac{1}{2}m^2 \phi^2 + \frac{1}{4} \lambda \phi^4$$

(32)

in which $\phi^2 = \phi_i^2$? Now they are the points on the sphere

$$\phi^2 = \frac{m^2}{\lambda} \quad \text{of radius} \quad \phi = v = \frac{m}{\sqrt{\lambda}}.$$  

(33)

Whereas in the discrete case there were two degenerate vacua, here there are infinitely many.

As in the discrete case, we pick one vacuum. We imagine that in the physical vacuum $|0\rangle$ the mean values of the $n$ fields $\phi_i$ are

$$\langle 0 | \phi_i | 0 \rangle = \frac{m}{\sqrt{\lambda}} \delta_{in} = v \delta_{in}.$$  

(34)

Now we write call the components of the field

$$\phi_i = (\pi_1, \pi_2, \ldots, \pi_{n-1}, v + \sigma).$$  

(35)

So now

$$\phi^2 = \phi_1^2 \ldots \phi_n^2 = \pi_1^2 + \cdots + \pi_{n-1}^2 + (v + \sigma)^2 \equiv \pi^2 + (v + \sigma)^2,$$

(36)
and the action density (29) is
\[
L = -\frac{1}{2} \partial_a \phi_i \partial^a \phi_i + \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda (\phi^2)^2
\]
\[
= -\frac{1}{2} \partial_a \pi_i \partial^a \pi_i - \frac{1}{2} \partial_a \sigma \partial^a \sigma + \frac{1}{2} m^2 \left[ \pi^2 + (v + \sigma)^2 \right] - \frac{1}{4} \lambda \left[ \pi^2 + (v + \sigma)^2 \right]^2
\]
\[
= -\frac{1}{2} \partial_a \pi_i \partial^a \pi_i - \frac{1}{2} \partial_a \sigma \partial^a \sigma + \frac{1}{2} m^2 \left[ \pi^2 + v^2 + 2v\sigma + \sigma^2 \right] - \frac{1}{4} \lambda \left[ \pi^2 + v^2 + 2v\sigma + \sigma^2 \right]^2.
\]
In this expression, the coefficient of \( \pi^2 \) vanishes while that of \( \sigma^2 \) is \(-m^2\)
\[
L = -\frac{1}{2} \partial_a \pi_i \partial^a \pi_i - \frac{1}{2} \partial_a \sigma \partial^a \sigma - m^2 \sigma^2 - m\sqrt{\lambda} \sigma \pi^2 - m\sqrt{\lambda} \sigma^3 - \frac{1}{4} \lambda \sigma^4 - \frac{1}{2} \lambda \pi^2 \sigma^2 + \frac{1}{4} \lambda \pi^4 - \frac{1}{4} m^4 \lambda.
\]
So the theory describes one field \( \sigma \) of mass \( \sqrt{2}m \) and \( n - 1 \) massless fields \( \pi_k \). These massless fields are called Goldstone bosons.

3 Goldstone’s Theorem

Let \( V(\phi) \) be a potential that is bounded below and that depends upon \( n \) fields \( \phi_j \). Assume that the action density
\[
L = -\frac{1}{2} \partial_a \phi_j \partial^a \phi_j - V(\phi)
\]
is invariant under the global linear transformation
\[
\phi_j' = O_{jk} \phi_k
\]
in which the \( n \times n \) unitary matrix \( O \) is a member of a representation of a continuous Lie group \( G \). Since \( V(\phi) \) is bounded below and invariant under the symmetry (40), it has several minima \( \phi_0 \). At these minima, the first-order partial derivatives must vanish
\[
\left. \frac{\partial V(\phi)}{\partial \phi_k} \right|_{\phi=\phi_0} = 0
\]
and the mixed second-order partial derivatives must be nonnegative

\[
\left. \frac{\partial^2 V(\phi)}{\partial \phi_k \partial \phi_\ell} \right|_{\phi = \phi_0} \equiv m_{k\ell} \geq 0. \tag{42}
\]

Near each minimum, the potential looks like

\[
V(\phi) = V(\phi_0) + \frac{1}{2}(\phi_k - \phi_{k0})(\phi_\ell - \phi_{\ell0}) m_{k\ell}^2. \tag{43}
\]

The matrix \( m_{k\ell}^2 \) is real and symmetric. So it can be diagonalized by an orthogonal transformation. Its eigenvalues are the squares of the masses of the scalar bosons of the theory.

Near the identity, the matrix \( O \) looks like

\[
O = I + i \theta_r t^r \tag{44}
\]

in which the generators \( t^r \) are hermitian, \( t^r_\dagger = t^r \). Since \( V(O\phi) = V(\phi) \), the derivatives with respect to the \( \theta_r \)'s must vanish

\[
\frac{\partial V(\phi)}{\partial \phi_j} \frac{\partial \phi_j}{\partial \theta_r} = \frac{\partial V(\phi)}{\partial \phi_j} \frac{\partial \phi_j}{\partial \theta_r} = 0. \tag{45}
\]

Now

\[
\phi_j(\theta) = \phi_j(0) + it^r_{jk} \phi_k(0) \tag{46}
\]

so the derivative (45) is

\[
\frac{\partial V(\phi)}{\partial \phi_j} \frac{\partial \phi_j}{\partial \theta_r} = \frac{\partial V(\phi)}{\partial \phi_j} it^r_{jk} \phi_k = 0 \tag{47}
\]

because of the symmetry. We now differentiate this vanishing quantity with respect to \( \phi_\ell \)

\[
\frac{\partial^2 V(\phi)}{\partial \phi_j \partial \phi_\ell} it^r_{jk} \phi_k + \frac{\partial V(\phi)}{\partial \phi_j} it^r_{j\ell} \phi_k = 0 \tag{48}
\]
and then set all the fields equal to their values $\phi_{k_0}$ at one of the vacua

$$\frac{\partial^2 V(\phi_0)}{\partial \phi_j \partial \phi_\ell} t^r_{jk} \phi_{k_0} + \frac{\partial V(\phi_0)}{\partial \phi_j} i t^r_{\ell j} = \frac{\partial^2 V(\phi_0)}{\partial \phi_\ell \partial \phi_j} i t^r_{jk} \phi_{k_0} = 0$$

(49)

in which we used the vanishing (41) of the first derivatives at $\phi_0$. But the second derivatives are just the elements $m_{\ell j}$ of the mass matrix (42). So we have for each generator $t^r$ an eigenvector $t^r_{jk} \phi_{k_0}$ of the mass matrix with eigenvalue zero

$$m_{\ell j} t^r_{jk} \phi_{k_0} = 0$$

(50)

unless the generator annihilates the vector $\phi_0$. That is, if $I + i \theta_r t^r$ leaves the vector $\phi_0$ unchanged

$$O \phi_0 = \phi_0,$$

(51)

then

$$\theta_r t^r \phi_0 = 0$$

(52)

and we don’t get an eigenvector of the mass matrix $m_{\ell j}$ with eigenvalue zero. There are as many massless bosons as there are generators of the symmetry group that don’t annihilate the vacuum vector $\phi_0$. If the symmetry group $G$ has a subgroup $H$ that leaves $\phi_0$ invariant, then one says that there is a massless boson for every generator of the quotient group $G/H$.

### 4 Gauge Invariance

The reason we could generalize our formulas for muon pair production to tau pair production is that all the charged leptons are coupled to the photon in the same way. Although electrodynamics is an abelian gauge theory, we might as well consider the general case of a nonabelian gauge theory.

The action density of a Yang-Mills theory is unchanged when a space-time dependent unitary matrix $U(x)$ changes a vector $\psi(x)$ of matter fields to $\psi'(x) = U(x)\psi(x)$. Terms like $\psi^\dagger \psi$ are invariant because $\psi^\dagger(x) U^\dagger(x) U(x) \psi(x) = \psi^\dagger(x) \psi(x)$, but how can kinetic terms like $\partial_i \psi^\dagger \partial^i \psi$ be made invariant? Yang and
Mills introduced matrices $A_i$ of gauge fields, replaced ordinary derivatives $\partial_i$ by covariant derivatives $D_i \equiv \partial_i + A_i$, and required that $D'_i \psi' = UD_i \psi$ or that
\begin{equation}
(\partial_i + A'_i) U = \partial_i U + U \partial_i + A'_i U = U (\partial_i + A_i). \tag{53}
\end{equation}

Their nonabelian gauge transformation is
\begin{align*}
\psi'(x) &= U(x)\psi(x), \\
A'_i(x) &= U(x)A_i(x)U^\dagger(x) - (\partial_i U(x)) U^\dagger(x).
\end{align*} \tag{54}

One often writes the unitary matrix as $U(x) = \exp(-ig \theta_a(x) t_a)$ in which $g$ is a coupling constant, the functions $\theta_a(x)$ parametrize the gauge transformation, and the generators $t_a$ belong to the representation that acts on the vector $\psi(x)$ of matter fields.

In the case of electrodynamics, the unitary matrix is a member of the group $U(1)$; it is just a phase factor $U(x) = \exp(-ie \theta_a(x))$. The abelian gauge transformation is
\begin{align*}
\psi'(x) &= U(x)\psi(x) = e^{-ie \theta(x)} \psi(x), \\
A'_b(x) &= U(x)A_b(x)U^\dagger(x) + i (\partial_b U(x)) U^\dagger(x) = A_b(x) + ie \partial_b \theta(x). \tag{55}
\end{align*}

I have been using a notation in which $A_i$ is antihermitian to simplify the algebra. So if $A_i = iA_b$, then the abelian gauge transformation is
\begin{align*}
\psi'(x) &= U(x)\psi(x) = e^{-ie \theta(x)} \psi(x), \\
A'_b(x) &= U(x)A_b(x)U^\dagger(x) + i (\partial_b U(x)) U^\dagger(x) = A_b(x) + e\partial_b \theta(x). \tag{56}
\end{align*}

Similarly, with real gauge fields, $A_b = -iA_i$, the nonabelian gauge transformation is
\begin{align*}
\psi'(x) &= U(x)\psi(x), \\
A'_b(x) &= U(x)A_b(x)U^\dagger(x) + i (\partial_b U(x)) U^\dagger(x). \tag{57}
\end{align*}
5 Abelian Higgs Mechanism

A theory with action density

\[
L = -\frac{1}{4} F_{ab} F^{ab} - (D_a \phi)^* D^a \phi - m^2 \phi^* \phi - \frac{1}{2} \lambda (\phi^* \phi)^2
\]

(58)
in which the complex field \( \phi \) is

\[
\phi = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2)
\]

(59)
and it covariant derivative is

\[
D_b \phi = (\partial_b + ieA_b) \phi
\]

(60)
describes charged bosons of mass \( m \) interacting with themselves directly and through the massless electromagnetic field \( A_b \). This theory has an abelian gauge symmetry. That is, the action density is invariant under the spacetime-dependent \( U(1) \) transformation \( U(x) = \exp(-ie\theta(x)) \)

\[
\psi'(x) = U(x) \psi(x) = e^{-ie\theta(x)} \psi(x)
\]

\[
A'_b(x) = U(x) A_b(x) U^\dagger(x) + i (\partial_i U(x)) U^\dagger(x) = A_b(x) + e\partial_b \theta(x).
\]

(61)

But if we flip the sign of the mass term from \(-m^2 \phi^* \phi\) to \(m^2 \phi^* \phi\)

\[
L = -\frac{1}{4} F_{ab} F^{ab} - (D_a \phi)^* D^a \phi + m^2 \phi^* \phi - \frac{1}{2} \lambda (\phi^* \phi)^2
\]

(62)
then things get more interesting. The complex field \( \phi \) now minimizes the energy of the ground state of the theory by assuming a mean value

\[
|\phi| = \frac{m}{\sqrt{\lambda}}.
\]

(63)
The various possible vacua lie on a circle of radius \( m/\sqrt{\lambda} \) in the complex \( \phi \)-plane. We choose the one in which \( \langle 0 | \phi | 0 \rangle \equiv \phi_0 = m/\sqrt{\lambda} \) is real and set

\[
\phi = \phi_0 + \frac{1}{\sqrt{2}} (\sigma_1 + i \sigma_2) = \frac{m}{\sqrt{\lambda}} + \frac{1}{\sqrt{2}} (\sigma_1 + i \sigma_2) = \frac{m}{\sqrt{\lambda}} + \sigma.
\]

(64)
Now the potential energy is

\[
V = -m^2 \phi^* \phi + \frac{1}{2} \lambda (\phi^* \phi)^2
\]

\[
= -m^2 \left[ (\phi_0 + \sigma_1/\sqrt{2})^2 + \sigma_2^2/2 \right] + \frac{1}{2} \lambda \left[ (\phi_0 + \sigma_1/\sqrt{2})^2 + \sigma_2^2/2 \right]^2
\]

\[
= -m^2 \left[ \phi_0^2 + \sqrt{2} \phi_0 \sigma_1 + \sigma_1^2/2 + \sigma_2^2/2 \right] + \frac{1}{2} \lambda \left[ \phi_0^2 + \sqrt{2} \phi_0 \sigma_1 + \sigma_1^2/2 + \sigma_2^2/2 \right]^2
\]

\[
= -m^2 \left[ \phi_0^2 + \sqrt{2} \phi_0 \sigma_1 + \sigma_1^2/2 + \sigma_2^2/2 \right] + \frac{1}{2} \lambda \left[ \phi_0^2 (\sigma_1^2 + \sigma_2^2) + 2 \phi_0^2 \sigma_1 + 2 \sqrt{2} \phi_0^2 \sigma_1 \right]
\]

\[
+ \sqrt{2} \lambda \phi_0 \sigma_1 \sigma_2^2 + \frac{\lambda}{2} (\sigma_2^2)^2 + \frac{1}{2} \lambda \phi_0^4.
\]

(65)

The terms linear in \(\sigma_1\) cancel, as do the terms quadratic in \(\sigma_2\). We then have

\[
V = m^2 \sigma_1^2 + \sqrt{2} \lambda m \sigma_1 \sigma_2^2 + \frac{\lambda}{2} (\sigma_2^2)^2 - \frac{m^4}{2 \lambda}.
\]

(66)

So the theory seems to have a spinless boson \(\sigma_1\) of mass \(\sqrt{2}m\) and a massless spinless boson \(\sigma_2\).

But wait. What about the kinetic action of the scalar fields? It is

\[
L_{\phi,K} = - (D_a \phi)^* D^a \phi = - (\partial_a - ieA_a) (\phi_0 + \sigma^* ) (\partial^a + ieA^a) (\phi_0 + \sigma)
\]

\[
= - ( -ie \phi_0 A_a + D_a^* \sigma^* ) (ie \phi_0 A^a + D^a \sigma)
\]

\[
= - \left[ -ie A_a \phi_0 + (\partial_a - ie A_a ) \left( \frac{\sigma_1}{\sqrt{2}} - i \frac{\sigma_2}{\sqrt{2}} \right) \right] \left[ ie A^a \phi_0 + (\partial^a + ie A^a ) \left( \frac{\sigma_1}{\sqrt{2}} + i \frac{\sigma_2}{\sqrt{2}} \right) \right]
\]

\[
= - e^2 \phi_0 A_a A^a - (D_a \sigma)^* D^a \sigma + ie A_a \phi_0 D^a \sigma - ie A^a \phi_0 (D_a \sigma)^*
\]

\[
= - e^2 \phi_0^2 A_a A^a - (D_a \sigma)^* D^a \sigma - \sqrt{2} e^2 A_a A^a \phi_0 \sigma_1 - \sqrt{2} e A^a \phi_0 \partial_a \sigma_2
\]

in which

\[
\sigma = \frac{1}{\sqrt{2}} (\sigma_1 + i \sigma_2).
\]

(68)
The gauge field $A_a$ has acquired a mass

$$M = \sqrt{2} e \phi_0 = \sqrt{2} \frac{e m}{\sqrt{\lambda}}.$$  

(69)

It makes sense to change the name of this field to

$$B_a \equiv A_a + \frac{1}{M} \partial_a \sigma_2 = A_a + \frac{1}{\sqrt{2} e \phi_0} \partial_a \sigma_2 = A_a + \frac{\sqrt{\lambda}}{\sqrt{2} e m} \partial_a \sigma_2.$$  

(70)

Note that the extra gradient of $\sigma_2$ does not change the Faraday tensor

$$F_{ab} = \partial_a A_b - \partial_b A_a = \partial_a B_b - \partial_b B_a.$$  

(71)

In these terms, the action density is

$$L = -\frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} M^2 B_a B^a - (D_a \sigma)^* D^a \sigma + \text{a constant plus cubic and quartic terms}$$  

(72)

in which

$$-(D_a \sigma)^* D^a \sigma = -\frac{1}{2} (\partial_a \sigma_1 \partial^a \sigma_1 + \partial_a \sigma_2 \partial^a \sigma_2) + e A^a (\sigma_1 \partial_a \sigma_2 - \sigma_2 \partial_a \sigma_1) - \frac{1}{2} e^2 A_a A^a \left( \sigma_1^2 + \sigma_2^2 \right).$$  

(73)

But it is more straightforward to write the action as

$$L = -\frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} M^2 A_a A^a - (D_a \sigma)^* D^a \sigma - m^2 \sigma_1^2$$  

$$- \frac{\sqrt{2}}{\sqrt{\lambda}} e^2 m A_a A^a \sigma_1 - \frac{\sqrt{2}}{\sqrt{\lambda}} e m A^a \partial_a \sigma_2 - \frac{1}{2} \sqrt{2} \lambda m \sigma_1 \sigma_2 - \frac{1}{2} \lambda (\sigma^2)^2 + m^4 2\lambda.$$  

(74)

Apart from the quadratic term $-\sqrt{2} e m A^a \partial_a \sigma_2 / \sqrt{\lambda}$, this theory would describe a vector boson $A_b$ of mass $M = e m \sqrt{2/\lambda}$ interacting with a scalar boson $\sigma_1$ of mass $\sqrt{2} m$, at least at low energies and low temperatures.
But this is a $U(1)$ gauge theory, so we can rotate the complex field $\phi$ at every point of space-time so as to make it real. Then, instead of

$$L = -\frac{1}{4}F_{ab}F^{ab} - (D_a\phi)^* D^a\phi + m^2\phi^2 - \frac{1}{2}\lambda (\phi^*\phi)^2,$$  \hspace{1cm} (75)

we have

$$L = -\frac{1}{4}F_{ab}F^{ab} - (D_a\phi)^* D^a\phi + m^2\phi^2 - \frac{1}{2}\lambda\phi^4$$

$$= -\frac{1}{4}F_{ab}F^{ab} - (\partial_a - ieA_a)\phi (\partial^a + ieA^a)\phi + m^2\phi^2 - \frac{1}{2}\lambda\phi^4$$  \hspace{1cm} (76)

$$= -\frac{1}{4}F_{ab}F^{ab} - \partial_a\phi \partial^a\phi - e^2 A_a A^a \phi^2 + m^2\phi^2 - \frac{1}{2}\lambda\phi^4.$$

Now, we have the simplest kind of spontaneous symmetry breaking in which the real field $\phi$ assumes a mean value $\phi_0$ whose square is $m^2/\lambda$. We choose

$$\langle 0|\phi|0 \rangle = \phi_0 = \frac{m}{\sqrt{\lambda}}$$  \hspace{1cm} (77)

and set

$$\phi = \frac{m}{\sqrt{\lambda}} + \frac{\sigma}{\sqrt{2}}$$  \hspace{1cm} (78)

where now both $\phi$ and $\sigma$ are real. In these terms, $L$ is

$$L = -\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}\partial_a\phi \partial^a\phi - e^2 A_a A^a \phi^2 + m^2\phi^2 - \frac{1}{2}\lambda\phi^4$$

$$= -\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}\partial_a\phi \partial^a\phi - e^2 A_a A^a \left( \frac{m}{\sqrt{\lambda}} + \frac{\sigma}{\sqrt{2}} \right)^2 + m^2 \left( \frac{m}{\sqrt{\lambda}} + \frac{\sigma}{\sqrt{2}} \right)^2 - \frac{1}{2}\lambda \left( \frac{m}{\sqrt{\lambda}} + \frac{\sigma}{\sqrt{2}} \right)^4$$

$$= -\frac{1}{4}F_{ab}F^{ab} - \frac{e^2 m^2}{\lambda} A_a A^a - \frac{1}{2}\partial_a\phi \partial^a\phi - m^2\sigma^2$$

$$- \sqrt{\frac{2}{\lambda}} me^2 A_a A^a \sigma - \frac{1}{2}e^2 A_a A^a \sigma^2 - \sqrt{\frac{\lambda}{2}} m \sigma^3 - \frac{\lambda}{8} \sigma^4 + \frac{m^4}{\lambda}.$$  \hspace{1cm} (79)
In this **unitary gauge**, the theory has a real scalar boson of mass $\sqrt{2}m$ interacting with a massive vector boson $A_a$ of mass $M = em\sqrt{2/\lambda}$. In the quadratic part of $L$, there are no terms coupling $\sigma$ to $A_b$. If both $e$ and $\lambda$ are small, then perturbation theory should describe $\sigma$ interacting with $A_b$ and with itself through the cubic and quartic terms in the second line of the last form of this equation. This is the abelian **Higgs mechanism**.

One may wonder whether one can transform to the unitary gauge even when the mass term $-m^2|\phi|^2$ has the “right” sign so that the $U(1)$ symmetry is unbroken. The phase $e\theta = \text{atan} (\phi_2/\phi_1)$ of the required gauge transformation is not defined where it is most needed, namely in the vicinity of the vacuum where $\langle 0|\phi|0 \rangle = 0$, and has derivatives that are singular there.

### 6 $O(n)$ Nonabelian Higgs Mechanism

Let’s start with our $O(n)$ theory (29)

$$L = -\frac{1}{2} \sum_{i=1}^{n} \partial_a \phi_i \partial^a \phi_i + \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda (\phi^2)^2.$$  \hspace{1cm} (80)

in which the sign of the mass term induces spontaneous symmetry breaking and

$$\phi^2 = \sum_{i=1}^{n} \phi_i \phi_i.$$  \hspace{1cm} (81)

We can make this **global** $O(n)$ symmetry **local** by introducing $n(n-1)$ gauge fields $A_b^f$, one for each generator $t^f$ of the group $O(n)$. The antihermitian gauge-field matrix is

$$A_b(x) = i e \sum_{f=1}^{n(n-1)} t^f A_b^f(x)$$  \hspace{1cm} (82)

in which the generators obey the commutation relations

$$[t^f, t^g] = if_{fgk} t^k$$  \hspace{1cm} (83)
with totally antisymmetric \textbf{structure constants} \( f_{fgk} \). The generators are orthogonal but not normalized

\[
\text{Tr}(t^{f\dagger}t^g) = k \delta_{fg}
\]  

(84)

in which the positive constant \( k \) depends upon the representation to which the generators belong. In the defining representation of \( O(n) \), the generators are \( n \times n \) imaginary antisymmetric matrices. The covariant derivative is

\[
D_b = \partial_b + A_b
\]

(85)

The nonabelian Faraday tensor

\[
F_{ik} = [D_i, D_k] = \partial_i A_k - \partial_k A_i + [A_i, A_k]
\]

(86)

transforms covariantly

\[
F'_{ik} = UF_{ik}U^{-1}
\]

(87)

The action of the nonabelian Faraday tensor is invariant

\[
\text{Tr} \left( UF_{ik}U^{-1}UF^{ik}U^{-1} \right) = \text{Tr} \left( F_{ik}F^{ik} \right) = \text{Tr} \left( F_{ik}^t F^{ik} \right)
\]

(88)

The action density of this theory is

\[
L = \frac{1}{4} \text{Tr}(F_{ik}F^{ik}) - \frac{1}{2} \langle D_a \phi \rangle^T (D^a \phi) + \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda (\phi^2)^2
\]

(89)

in which the sign of the trace is due to the \( i \) in \( A_b = i e^f t^f A^f_b \) (summed over \( f \)).

Once again, we have spontaneous symmetry breaking. In the vacuum state \(|0\rangle\), the field \( \phi \), which is a real \( n \)-vector assumes a value on the sphere of radius \( |\phi| = \phi_0 = m/\sqrt{\lambda} \). As before, we write

\[
\phi = (\pi_1, \pi_2, \ldots, \pi_{n-1}, \phi_0 + \sigma)
\]

(90)

We have one scalar field of mass \( m_\sigma = \sqrt{2} m \) and, at least apparently, \( n - 1 \) massless scalar fields \( \pi_i \), one for each generator of the quotient group \( G/H \) in which \( H \) is the subgroup of rotations that leave the vector

\[
|0\rangle \phi_i |0\rangle = (0, 0, \ldots, 0, \phi_0)
\]

(91)
invariant. This subgroup is $O(n-1)$. So there are

$$\frac{1}{2}n(n-1) - \frac{1}{2}(n-1)(n-2) = n-1$$

(92)

apparently massless scalar fields $\pi_i$

Look now at the kinetic action of the fields $\phi_i$

\[ L_{K\phi} = -\frac{1}{2}(D_a\phi)^T (D^a\phi) = -\frac{1}{2} \sum_{i,k,\ell=1}^{n} \left( \delta_{ik} \partial_a + \sum_{f=1}^{n(n-1)/2} i\epsilon_{ik} A_{fa}^f \right) \phi_k \left( \delta_{\ell\ell} \partial^a + \sum_{g=1}^{n(n-1)/2} i\epsilon_{\ell\ell} A^{ga} \right) \phi_{\ell}. \tag{93} \]

Since $\langle 0|\pi_i|0 \rangle = \phi_0 \delta_{in}$, the part of this that involves $\phi_0$ quadratically is

\[ L_{K\phi_0} = -\frac{1}{2} \epsilon_{ik} A_{fa}^f \phi_0 \delta_{ik} \phi_0 \delta_{\ell\ell} \phi_0 \ell = \frac{1}{2} \sum_{i=1}^{n} e^2 \epsilon_{ik} \epsilon_{\ell\ell} \phi_0^2 A_{fa}^f A^{ga} \tag{94} \]

in which we sum over $i$ but not $n$. The $n \times n$ generators $t^f$ are hermitian and imaginary so that $it^f \phi$ is real; they are imaginary and antisymmetric. Thus

\[ L_{K\phi_0} = -\frac{1}{2} e^2 \phi_0^2 \epsilon_{in} t^f_{in} t^g_{in} A_{fa}^f A^{ga} = -\frac{1}{2} e^2 \phi_0^2 \epsilon_{in} t^f_{in} t^g_{in} \phi_0^2 A_{fa}^f A^{ga} = -\frac{1}{2} M_{fg}^2 A_{fa}^f A^{ga} \tag{95} \]

in which

\[ M_{fg}^2 = e^2 \phi_0^2 \epsilon_{in} t^f_{in} t^g_{in} \tag{96} \]

is real and symmetric. The mass-squared matrix $M_{fg}^2$ also is nonnegative since for any real vector $V_g$, the vector

\[ R_i = \epsilon_{in} V_f \tag{97} \]

is real, and we have

\[ V_f M_{fg}^2 V_g = e^2 \phi_0^2 \epsilon_{in} t^f_{in} t^g_{in} V_g = e^2 \phi_0^2 R_i^2 \geq 0. \tag{98} \]

In the present theory, this matrix does have zero eigenvalues because the $(n-1)(n-2)/2$ generators $t^h$ of the subgroup $H$ leave the vector

\[ \langle 0|\phi_i|0 \rangle = (0,0,\ldots,0,\phi_0) \tag{99} \]

17
invariant, and so
\[ t_{ik}^h \langle 0 | \phi_k | 0 \rangle = t_{in}^h \langle 0 | \phi_n | 0 \rangle = 0 \quad \text{or} \quad t_{in}^h = 0. \] (100)
That is,
\[ M_{fh}^2 = M_{hg}^2 = M_{hh'}^2 = 0. \] (101)
So the only part of the mass-squared matrix $M_{fg}$ that can have positive eigenvalues is an $(n-1) \times (n-1)$ block. And in fact these $n-1$ eigenvalues are all positive because if we choose the generators so that $t_{in}^f = i \delta_{fi} \sqrt{k/2}$ for $1 \leq f, g \leq n - 1$, then
\[ M_{fg}^2 = e^2 \phi_0^2 \langle i \delta_{fi} \phi_n \rangle = e^2 \phi_0^2 \delta_{fg} k/2 \] (102)
for $1 \leq f, g \leq n - 1$, in which $k$ is the positive constant that appears in the orthogonality relation (84). So there are $n-1$ massive vectors bosons. These are the ones that absorb the $n - 1$ apparently massless scalar bosons $\pi_i$. They all get the same mass $e \phi_0 \sqrt{k}$, which depends upon the parameter $k$ as well as upon $e$ and $\phi_0$.

As in the theory in which a nonzero mean value in the vacuum of a scalar field broke a local $U(1)$ symmetry, we can here transform to the unitary gauge in which the scalar field points in the $n$th direction at every point of spacetime
\[ \phi = (0, 0, \ldots, 0, \phi_0 + \sigma). \] (103)

In this gauge, the kinetic action of the fields $\phi_i$

\[ L_{K\phi} = -\frac{1}{2} (D_a \phi)^T (D^a \phi) = -\frac{1}{2} \sum_{i,k,\ell=1}^n \left( \delta_{ik} \partial_a + \sum_{f=1}^{n(n-1)/2} i e t_{ik}^f A_a^f \right) \phi_k \left( \delta_{i\ell} \partial^a + \sum_{g=1}^{n(n-1)/2} i e t_{i\ell}^g A^g_a \right) \phi_\ell \] (104)

reduces to
\[ L_{K\phi_\pi} = -\frac{1}{2} \sum_{i=1}^n \left( \delta_{in} \partial_a + \sum_{f=1}^{n(n-1)/2} i e t_{in}^f A_a^f \right) \phi_n \left( \delta_{in} \partial^a + \sum_{g=1}^{n(n-1)/2} i e t_{in}^g A^g_a \right) \phi_n \] (105)
in which the sum is only over the \( n - 1 \) generators for which \( t^f_{in} \neq 0 \) for some \( i \). Thus, summing over repeated indices, we have

\[
L_{K_{\phi_ug}} = -\frac{1}{2} \left( \delta_{in} \partial_a + i e t^f_{in} A^f_a \right) \left( \phi_0 + \sigma \right) \left( \delta_{in} \partial^a + i e t^g_{in} A^g_a \right) \left( \phi_0 + \sigma \right)
\]

\[
= -\frac{1}{2} \left( \delta_{in} \partial_a \sigma + i e t^f_{in} A^f_a \right) \left( \delta_{in} \partial^a \sigma + i e t^g_{in} A^g_a \right) \sigma - i e \phi_0 t^f_{in} A^f_a \partial^a \sigma - \frac{1}{2} e^2 i e t^f_{in} i e t^g_{in} \phi_0^2 A^f_a A^g_a
data(106)
\]

because \( t^f_{nn} = 0 \) since the generators of \( O(n) \) are antisymmetric.

7 \( SU(2) \) Higgs Mechanism

Let’s now consider a theory of a 2-component complex scalar field \( \phi \) with action density

\[
L = -(D_a \phi)^\dagger D^a \phi + \frac{1}{4k} \text{Tr}(F_{ab} F^{ab}) + m^2 |\phi|^2 - \lambda |\phi|^4
\]

in which \( |\phi|^2 = |\phi_1|^2 + |\phi_2|^2 \). Here the antihermitian gauge-field matrix is

\[
A_b = i e \frac{1}{2} \sigma \cdot A_b
\]

(108)

in which the \( \sigma \) are the Pauli matrices

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(109)

The covariant derivative is

\[
(D_b \phi)_i = \partial_b \phi_i + (A_b)_{ij} \phi_j = \partial_b \phi_i + (i e \frac{1}{2} \sigma \cdot A_b)_{ij} \phi_j = \partial_b \phi_i + i e \frac{1}{2} (\sigma^k)_{ij} A^k_b \phi_j.
\]

(110)
This action density is invariant under $SU(2)$ transformations

$$
\phi'(x) = U(x)\phi(x) = \exp(-i\theta(x) \cdot \sigma/2)\phi(x)
$$

$$
A_b'(x) = U(x)A_b(x)U^\dagger(x) + i (\partial_b U(x)) U^\dagger(x) = \exp(-i\theta(x) \cdot \sigma/2)A_b(x) \exp(i\theta(x) \cdot \sigma/2) + i (\partial_b \exp(-i\theta(x) \cdot \sigma/2)) \exp(i\theta(x) \cdot \sigma/2)
$$

that depend upon the space-time point $x$. The trace relation for the generators $\sigma/2$ is

$$
\text{Tr}(\frac{1}{2}\sigma_i \frac{1}{2}\sigma_j) = \frac{1}{2}\delta_{ij},
$$

so the constant $k = 1/2$.

Going again to the unitary gauge, we rotate the field $\phi$ to

$$
\phi = \begin{pmatrix} 0 \\ v + s \end{pmatrix} \quad \text{with} \quad \phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}
$$

in which $v = m\sqrt{2/\lambda}$ is real. The kinetic action of the Higgs field $\phi$ has a term quadratic in $\phi$

$$
\left[ ie \frac{1}{2}(\sigma^k)_{ij} A_b^k \phi_j \right] \bar{\phi} \left[ ie \frac{1}{2}(\sigma^m)_{\ell\ell} A^m_{\ell\ell} \right]
$$

which is simpler in matrix notation

$$
e^2 \frac{1}{4}\phi_0^\dagger \sigma^k \sigma^m \phi_0 A_b^k A^m_{\ell\ell}.
$$

Symmetrizing and using the relation

$$
\sigma^k \sigma^m = \delta_{km} I + i\epsilon_{km\ell} \sigma^\ell,
$$

we find at $\phi = \phi_0$

$$
e^2 \frac{1}{8} \phi_0^\dagger \{\sigma^k, \sigma^m\} \phi_0 A_b^k A^m_{\ell\ell} = e^2 \frac{1}{4} \phi_0^\dagger \delta_{km} I \phi_0 A_b^k A^m_{\ell\ell} = \frac{e^2 v^2}{4} A_b^k A^{kb} \equiv \frac{1}{2}M^2 A_b^k A^{kb}.
$$
So all three gauge bosons get the same mass $M = ev/\sqrt{2}$. Why are none left massless? Because the $SU(2)$ symmetry is completely broken; $H$ is just the trivial group consisting of an identity element.

What if we had put $\phi$ in the adjoint representation of $SU(2)$? In this case, its mean value would have been a real three vector of some length pointing in some direction. The unbroken subgroup $H$ would have consisted of rotations about that direction. So one gauge boson would have remained massless. In fact, this is a model we already have studied: it is just the $O(3)$ gauge theory with the Higgs in the defining representation.

## 8SU(3) Pure Gauge Theory

The Gell-Mann matrices are

\[
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \text{and} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\]

The generators $t_a$ of the $3 \times 3$ defining representation of $SU(3)$ are these Gell-Mann matrices divided by 2

\[
t_a = \lambda_a/2 = t^a = \lambda^a/2
\]

(Murray Gell-Mann, 1929–).

The eight generators $t_a$ are orthogonal with $k = 1/2$

\[
\text{Tr} (t_a t_b) = \frac{1}{2} \delta_{ab}
\]
and satisfy the commutation relation
\[ [t_a, t_b] = i f_{abc} t_c. \]  
(121)

A trace formula gives us the **SU(3) structure constants** as
\[ f_{abc} = (-i/k) \text{Tr}([t_a, t_b]t_c) = -2i \text{Tr} ([t_a, t_b]t_c). \]  
(122)

They are real and totally antisymmetric with \( f_{123} = 1, f_{458} = f_{678} = \sqrt{3}/2, \) and \( f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2. \)

While no two generators of \( SU(2) \) commute, two generators of \( SU(3) \) do. In the representation (118,119), \( t_3 \) and \( t_8 \) are diagonal and so commute
\[ [t_3, t_8] = 0. \]  
(123)

They generate the **Cartan subalgebra** of \( SU(3) \).

The gauge-field matrix is
\[ A_\mu(x) = ig \sum_{b=1}^{8} t^b A^b_\mu(x) \]  
(124)
in the defining representation. The covariant derivative in that representation is
\[ D_\mu = I \partial_\mu + A_\mu(x) = I \partial_\mu + ig \sum_{b=1}^{8} t^b A^b_\mu(x). \]  
(125)

The Faraday matrix is
\[ F_{\mu\nu} = [D_\mu, D_\nu] = [I \partial_\mu + A_\mu(x), I \partial_\nu + A_\nu(x)] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \]  
(126)
in matrix notation. With more indices exposed, it is
\[ (F_{\mu\nu})_{cd} = (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])_{cd} = ig \sum_{b=1}^{8} t^b A^b_\mu(x) \left( I \partial_\nu + ig \sum_{e=1}^{8} t^e A^e_\nu(x) \right)_{cd}. \]  
(127)
To avoid the sum signs, we sum over repeated indices from 1 to 8

\[
\begin{align*}
(F_{\mu\nu})_{cd} &=igt_{cd} (\partial_\mu A^b_\nu - \partial_\nu A^b_\mu) - g^2 A^b_\mu A^c_\nu \left[t^b, t^c\right]_{cd} =igt_{cd} (\partial_\mu A^b_\nu - \partial_\nu A^b_\mu) - g^2 A^b_\mu A^c_\nu i f_{bef} t^f_{cd} \\
&=igt_{cd} (\partial_\mu A^b_\nu - \partial_\nu A^b_\mu) - ig^2 A^b_\mu A^c_\nu f_{bef} t^f_{cd} =igt_{cd} (\partial_\mu A^b_\nu - \partial_\nu A^b_\mu) - ig^2 A^b_\mu A^c_\nu f_{bef} t^f_{cd} \\
&=igt_{cd} (\partial_\mu A^b_\nu - \partial_\nu A^b_\mu - g A^f_\mu A^e_\nu f_{efb}) =igt_{cd} (\partial_\mu A^b_\nu - \partial_\nu A^b_\mu - g f_{bef} A^f_\mu A^e_\nu) \\
&=igt_{cd} F^b_{\mu\nu} 
\end{align*}
\]

(128)

where

\[ F^b_{\mu\nu} = \partial_\mu A^b_\nu - \partial_\nu A^b_\mu - g f_{bef} A^f_\mu A^e_\nu \]

(129)

is the Faraday tensor.

The action density of this tensor is

\[ L_F = -\frac{1}{4} F^b_{\mu\nu} F^b_{\mu\nu} \]

(130)

in which raising and lowering an index of a compact group is of cosmetic, not cosmic, significance. The trace of the square of the Faraday matrix is

\[
\text{Tr} \left[ F^b_{\mu\nu} F^b_{\mu\nu} \right] = \text{Tr} \left[ igt^b_{\mu\nu} F^b_{\mu\nu} \right] = -g^2 F^b_{\mu\nu} F^b_{\mu\nu} \text{Tr} \left[t^b, t^c\right] - kg^2 F^b_{\mu\nu} F^b_{\mu\nu} k. \]

(131)

So the Faraday action density is

\[ L_F = -\frac{1}{4} F^b_{\mu\nu} F^b_{\mu\nu} = \frac{1}{4kg^2} \text{Tr} \left[ F^b_{\mu\nu} F^b_{\mu\nu} \right] = \frac{1}{2g^2} \text{Tr} \left[ F^b_{\mu\nu} F^b_{\mu\nu} \right]. \]

(132)

The theory described by this action density, without scalar or spinor fields, is called pure gauge theory.

## 9 Quantum Chromodynamics

If we add massless quarks in the fundamental or defining representation, then we get a theory of the strong interactions called Quantum Chromodynamics. Thus, let \( \psi \) be a complex 3-vector of Dirac fields

\[ \psi = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix} \]

(133)
(so 12 fields in all). This complex 12-vector could represent \( u \) or “up” quarks. We use the covariant derivative

\[
D_\mu = I \partial_\mu + A_\mu (x) = I \partial_\mu + ig \sum_{b=1}^{8} t^b A^b_\mu (x).
\]  

(134)

The action density then is

\[
L = \frac{1}{2g^2} \text{Tr} \left[ F_{\mu\nu} F^{\mu\nu} \right] - \bar{\psi} (\gamma^\mu D_\mu + m) \psi.
\]  

(135)

Nonperturbative effects are supposed to “confine” the quarks and massless gluons. There are 6 known “flavors” of quarks—\( u, d, c, s, t, b \).

10 \( SU(3) \) Higgs Mechanism

Let’s now add a triplet of complex scalar fields that transform according to the defining representation

\[
\phi^t_b (x) = U_{bc}(x) \phi_c (x) = \left[ e^{-i\theta^a(x) t^a} \right]_{bc} \phi_c (x).
\]  

(136)

The \( SU(3) \) gauge fields will transform as

\[
A'_\mu (x) = U(x) A_\mu (x) U^\dagger (x) + i (\partial_\mu U(x))U^\dagger (x) = e^{-i\theta^a(x) t^a} A_\mu (x) e^{i\theta^a(x) t^a} + i (\partial_\mu e^{-i\theta^a(x) t^a}) e^{i\theta^a(x) t^a}
\]  

(137)

in which the gauge-field matrix is

\[
A_\mu (x) = ig \sum_{b=1}^{8} t^b A^b_\mu (x)
\]  

(138)

in the defining representation.

Suppose the action density is

\[
L = \frac{1}{4kg^2} \text{Tr} \left[ F_{\mu\nu} F^{\mu\nu} \right] - (D_\mu \phi)^\dagger D^\mu \phi + m^2 |\phi|^2 - \frac{1}{2} \lambda^2 |\phi|^4.
\]  

(139)
I have fiddled with the coefficients so as to avoid extra factors and roots of 2. Once again, we have spontaneous breaking of the local $SU(3)$ symmetry as the vacuum arranges itself so as to give the scalar field a mean value

$$\phi_0 \equiv \langle 0 | \phi_a(x) | 0 \rangle = v \delta_{a3} = \frac{m}{\lambda} \delta_{a3}$$

(140)

so that

$$\phi_3 = \frac{1}{\sqrt{2}}(\sqrt{2}v + s + i\phi_{3i}) = v + \frac{1}{\sqrt{2}}(s + i\phi_{3i})$$

(141)

in which the choice of the third direction was arbitrary. The complex fields $\phi_1$ and $\phi_2$, and the imaginary part of $\phi_3$ remain massless, but the real part of $\phi_3$ acquires the mass $\sqrt{2}m$.

Now instead of (117), we have

$$\frac{e^2}{8} \phi_0^\dagger \{ \lambda^k, \lambda^m \} \phi_0 A_b^k A^{mb} \equiv \frac{1}{2} M^2 A_b^k A^{kb}.$$ 

(142)

The gauge fields that don’t move $\phi_0$, that is, the ones that have

$$\lambda^m \phi_0 = 0$$

(143)

remain massless. So $A^1$, $A^2$, and $A^3$ remain massless. The other five gauge bosons, $A^4 \ldots A^8$ absorb the five massless scalar fields and acquire masses.

Homework set 3: Find those masses.

Let’s put the scalar fields in the adjoint representation of $SU(3)$. Now there are 8 real scalar fields, and we can write them as an 8-vector $\phi$ or as a $3 \times 3$ matrix

$$\Phi = \sum_{a=1}^{8} t^a \phi^a.$$ 

(144)

The covariant derivative now is

$$D_\mu \phi = (\partial_\mu + igA_\mu)\phi = (\partial_\mu + igA_\mu^a T^a)\phi$$ \hspace{1cm} (145)
where the generators in the adjoint representation are
\[ T^b_{ac} = if_{abc} \]
in which the structure constants \( f_{abc} \) are real and totally antisymmetric. Thus, we have
\[ (D_\mu \phi)_a = (\delta_{ac} \partial_\mu + igA_{\mu ac})\phi_c = \partial_\mu \phi_a - gA^b_\mu f_{abc}\phi_c. \] (147)

We also can write this as
\[
D_\mu \Phi = \partial_\mu \Phi + g[A_\mu, \Phi] = t^a \partial_\mu \phi_a + ig[t^b, t^c]A^b_\mu \phi^c = t^a \partial_\mu \phi_a + igif_{bac}t^a A^b_\mu \phi^c \\
= t^a \partial_\mu \phi_a - gt^a A^b_\mu f_{abc} \phi_c = t^a \left( \partial_\mu \phi_a - gA^b_\mu f_{abc} \phi_c \right) = t^a (D_\mu \phi)_a.
\] (148)

Now the gauge-boson mass term inside \( \frac{1}{2}(D_\mu \phi)^a(D_\mu \phi)_a \) is the proportional to the trace
\[ g^2 \text{Tr} \left( [t^a, \Phi][t^b, \Phi] \right) A_\mu^a A_\mu^b. \] (149)

So is the vacuum gives \( \Phi \) the mean value
\[ \Phi_0 = \langle 0|\Phi|0 \rangle, \] (150)

then the gauge-boson mass term is proportional to the trace
\[ g^2 \text{Tr} \left( [t^a, \Phi_0][t^b, \Phi_0] \right) A_\mu^a A_\mu^b. \] (151)

So those linear combinations of gauge fields times generators that commute with \( \Phi_0 \) remain massless. For instance, if
\[ \Phi_0 \propto \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \]
then the gauge bosons \( A^1, A^2, A^3 \) and \( A^8 \) remain massless, while \( A^4, A^5, A^6, \) and \( A^7 \) become massive. Interestingly, the \( SU(3) \) symmetry is broken to \( SU(2) \times U(1) \).

On the other hand, if
\[ \Phi_0 \propto \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \] (153)
then only \( A^3 \) and \( A^8 \) remain massless, and the unbroken symmetry is just \( U(1) \times U(1) \).
11 The GSW Electroweak Model

The local gauge group is $SU(2)_\ell \times U(1)$. What’s weird is that the $SU(2)_\ell$ symmetry applies only to the “left-handed” quarks and leptons and to a complex doublet (or 2-vector) of scalar fields $H$, the Higgs.

11.1 The Higgs Mechanism

Let’s leave out the fermions for the moment, and focus just on the Higgs and the gauge fields. The gauge transformation is

$$
H'(x) = U(x)H(x) \\
A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + (\partial_\mu U(x))U^\dagger(x)
$$

in which the $2 \times 2$ unitary matrix $U(x)$ is

$$
U(x) = \exp \left[ ig \frac{\sigma^a}{2} \alpha^a(x) + ig' \frac{Y I}{2} \beta(x) \right].
$$

The generators here are the 3 Pauli matrices and the matrix $I/2$, where $I$ is the $2 \times 2$ identity matrix.

The action density of the theory (without the fermions) is

$$
L = - (D_\mu H)^\dagger D^\mu H + \frac{1}{4k} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + m^2|H|^2 - \lambda|H|^4
$$

in which the covariant derivative for the Higgs doublet is

$$
D_\mu H = \left( i\partial_\mu + i g \frac{\sigma^a}{2} A^a_\mu + i g' \frac{Y I}{2} B_\mu \right) H.
$$

The minimum of the Higgs potential is where

$$
0 = \frac{\partial V}{\partial|H|^2} = 2\lambda|H|^2 - m^2.
$$
So
\[
\langle 0|H|0 \rangle = \frac{m}{\sqrt{2} \lambda} = \frac{v}{\sqrt{2}}.
\] (159)

Going to unitary gauge, we transform this mean value to
\[
H_0 = \langle 0|H(x)|0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}.
\] (160)

In unitary gauge, the Higgs potential is
\[
V(v) = -\frac{1}{2} m^2 v^2 + \frac{1}{4} \lambda v^4,
\] (161)

and its second derivative is
\[
V''(v) = 3 \lambda v^2 - m^2 = 2 m^2 = m_H^2.
\] (162)

The mass of the Higgs then is
\[
m_H = \sqrt{2} m = \sqrt{2} v \lambda.
\] (163)

Experiments at LEP2 (see below) revealed value of \(v\) to be
\[
v = 246 \text{ GeV}.
\] (164)

In 2012, experiments at the LHC showed the Higgs’s mass to be
\[
m_H = 126 \text{ GeV}.
\] (165)

The self coupling \(\lambda\) therefore is
\[
\lambda = \frac{m_H}{\sqrt{2} v} = \frac{126}{\sqrt{2} 246} = 0.36.
\] (166)
In unitary gauge and after spontaneous symmetry breaking, the mass terms for the gauge bosons that emerge from \(- (D_\mu H)^\dagger D^\mu H\) are

\[
L_M = -\frac{1}{2} (0, v) \left( g \frac{\sigma^a}{2} A^\mu_a + g' \frac{I}{2} B_\mu \right) \left( g \frac{\sigma^a}{2} A^\mu_a + g' \frac{I}{2} B_\mu \right) \left( 0 \right)
\]

\[
= -\frac{1}{8} (0, v) \left( g A^3_\mu + g' B_\mu \right) \left( g A^3_\mu + g' B_\mu \right) \left( 0 \right)
\]

\[
= -\frac{v^2}{8} \left[ g \left( A^1_\mu + i A^2_\mu \right) + g' B_\mu \right] \left( 0 \right)
\]

\[
= -\frac{v^2}{8} \left[ g \left( A^1_\mu + i A^2_\mu \right) + g' B_\mu \right] \left( 0 \right)
\]

\[
= -\frac{v^2}{8} \left[ g \left( A^1_\mu + i A^2_\mu \right) + g' B_\mu \right] \left( 0 \right)
\]

(167)

The normalized complex, charged gauge bosons are

\[
W^\pm_\mu = \frac{1}{\sqrt{2}} \left( A^1_\mu \mp i A^2_\mu \right)
\]

(168)

and the normalized neutral one is

\[
Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left( g A^3_\mu - g' B_\mu \right)
\]

(169)

In terms of these properly normalized fields, the mass terms are

\[
L_M = -\frac{g^2 v^2}{4} W^\mu_\mu W^{+\mu} - \frac{(g^2 + g'^2) v^2}{8} Z_\mu Z^\mu
\]

(170)

So the \(W^+\) and the \(W^-\) get the same mass

\[
M_W = g \frac{v}{2} = 80.385 \text{ GeV}/c^2.
\]

(171)

while the \(Z\) (also called the \(Z^0\)) has mass

\[
M_Z = \sqrt{g^2 + g'^2} \frac{v}{2} = 91.1876 \text{ GeV}/c^2.
\]

(172)
The orthogonal, normalized gauge boson

\[ A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g' A^3_\mu + g B_\mu) \]  

remains massless. It is the photon.

Why do three gauge bosons become massive? Because there are three Goldstone bosons corresponding to three ways of moving \( \langle 0 | H | 0 \rangle \) without changing the Higgs potential. Why does one gauge boson stay massless? Because one linear combination of the generators of \( SU_L(2) \otimes U(1) \) maps \( \langle 0 | H | 0 \rangle \) to zero, and so does not make an eigenstate of the gauge-boson mass matrix with eigenvalue zero.

In terms of these mass eigenstates, the original gauge bosons are

\[ A^1_\mu = \frac{1}{\sqrt{2}} (W^+ + W^-) \]
\[ A^2_\mu = \frac{1}{i\sqrt{2}} (W^- - W^+) \]
\[ A^3_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g' A_\mu + g Z_\mu) \]
\[ B_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g A_\mu - g' Z_\mu). \]  

Thus, the covariant derivative for a fermion of \( U(1) \) charge \( Y \) is

\[
D_\mu = I \partial_\mu + i g \frac{\sigma^a}{2} A^a_\mu + i g' Y I B_\mu \\
= I \partial_\mu + i g \left[ \frac{\sigma_1}{2} \sqrt{2} (W^+_\mu + W^-_\mu) + \frac{\sigma_2}{2} \frac{1}{i\sqrt{2}} (W^-_\mu - W^+_\mu) + \frac{\sigma_3}{2} \frac{1}{\sqrt{g^2 + g'^2}} (g' A_\mu + g Z_\mu) \right] \\
+ i g' Y I \frac{1}{\sqrt{g^2 + g'^2}} (g A_\mu - g' Z_\mu) \\
= I \partial_\mu + i \frac{g}{\sqrt{2}} (W^+_\mu T^+ + W^-_\mu T^-) + i \frac{1}{\sqrt{g^2 + g'^2}} Z_\mu (g^2 T^3 - g'^2 Y I) + i \frac{g g'}{\sqrt{g^2 + g'^2}} A_\mu (T^3 + Y I) 
\]
in which
\[ T^\pm = \frac{1}{2} (\sigma_1 \pm i \sigma_2) . \]  

(176)

We call the interaction strength or coupling constant of the photon \( A_\mu \)
\[ 0 < e = \frac{gg'}{\sqrt{g^2 + g'^2}} , \]  

(177)

and set the charge operator equal to
\[ Q = T^3 + Y I . \]  

(178)

When acting on the doublet
\[ E_\ell = \begin{pmatrix} \nu_e \\ e \end{pmatrix} \]  

(179)

to which we assign \( Y = -1/2 \), the charge \( Q \) gives 0 as the charge of the neutrino and \(-1\) as the charge of the electron. The photon-lepton term then is
\[ \frac{gg'}{\sqrt{g^2 + g'^2}} A_\mu \left( T^3 + Y I \right) E_\ell = e A_\mu \left( T^3 + Y I \right) E_\ell = e A_\mu \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix} = -e A_\mu e \]  

(180)
in which the first \( e \) is the absolute value of the charge (177) of the electron and the second is the field of the electron.

The **weak mixing angle** \( \theta_w \) is defined by
\[ \begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} \cos \theta_w & -\sin \theta_w \\ \sin \theta_w & \cos \theta_w \end{pmatrix} \begin{pmatrix} A^3 \\ B \end{pmatrix} . \]  

(181)

Our equations (169 and 173) identify these trigonometric values as
\[ \cos \theta_w = \frac{g}{\sqrt{g^2 + g'^2}} \quad \text{and} \quad \sin \theta_w = \frac{g'}{\sqrt{g^2 + g'^2}} . \]  

(182)
Since the charge is $Q = T^3 + Y I$, we can use $Q - T^3$ instead of $Y I$, so that the coupling to the $Z$ is

$$
\frac{1}{\sqrt{g^2 + g'^2}} Z_\mu \left[ g^2 T^3 - g'^2 Y \right] = \frac{1}{\sqrt{g^2 + g'^2}} Z_\mu \left[ (g^2 + g'^2)T^3 - g'^2 Q \right]
$$

(183)

and

$$
\frac{g g'}{\sqrt{g^2 + g'^2}} A_\mu (T^3 + Y) = \frac{g g'}{\sqrt{g^2 + g'^2}} A_\mu Q.
$$

(184)

We also have

$$
\frac{g^2 + g'^2}{\sqrt{g^2 + g'^2}} = \sqrt{g^2 + g'^2} = \frac{g}{\cos \theta_w} \quad \text{and} \quad \frac{g'^2}{\sqrt{g^2 + g'^2}} = \sqrt{g^2 + g'^2} \frac{g'^2}{g^2 + g'^2} = \frac{g}{\cos \theta_w} \sin^2 \theta_w.
$$

(185)

So the coupling to the $Z$ is

$$
\frac{1}{\sqrt{g^2 + g'^2}} Z_\mu \left[ (g^2 + g'^2)T^3 - g'^2 Q \right] = \frac{g}{\cos \theta_w} Z_\mu \left( T^3 - \sin^2 \theta_w Q \right)
$$

(186)

and, if we set

$$
e = g \sin \theta_w,
$$

(187)

then the coupling to the photon $A$ is

$$
\frac{g g'}{\sqrt{g^2 + g'^2}} A_\mu Q = g \sin \theta_w A_\mu Q = e A_\mu Q.
$$

(188)

In these terms, the covariant derivative (175) is

$$
D_\mu = I \partial_\mu + i \frac{g}{\sqrt{2}} \left( W_\mu^+ T^+ + W_\mu^- T^- \right) + i \frac{1}{\sqrt{g^2 + g'^2}} Z_\mu \left( g^2 T^3 - g'^2 Y \right) + i \frac{g g'}{\sqrt{g^2 + g'^2}} A_\mu \left( T^3 + Y \right)
$$

\[= I \partial_\mu + i \frac{g}{\sqrt{2}} \left( W_\mu^+ T^+ + W_\mu^- T^- \right) + i \frac{g}{\cos \theta_w} Z_\mu \left( T^3 - \sin^2 \theta_w Q \right) + ie A_\mu Q \]

(189)
in which the generators $T^\pm$ and $T^3$ are those of the representation to which the fields they act on belong. When acting on left-handed fermions, they are half the Pauli matrices, $T = \frac{1}{2}\sigma$. When acting on right-handed fermions, they are zero, $T = 0$. Since $g = e/\sin\theta_w$, the couplings involve one new parameter $\theta_w$.

Our mass formulas (171 and 172) for the $W$ and the $Z$ show that their masses are related by

$$M_W = g \frac{v}{2} = \frac{g}{\sqrt{g^2 + g'^2}} \sqrt{g^2 + g'^2} \frac{v}{2} = \cos \theta_w \sqrt{g^2 + g'^2} \frac{v}{2} = \cos \theta_w M_Z.$$  \hspace{1cm} (190)

Experiments have determined the masses and shown that

$$\sin^2 \theta_w = 0.231 \quad \text{or} \quad \theta_w = 0.233$$  \hspace{1cm} (191)

and that

$$v = 246.22 \text{ GeV}.$$  \hspace{1cm} (192)

11.2 Quark and Lepton Interactions

The right-handed fermions $u_r, d_r, e_r,$ and $\nu_{e,r}$ are singlets under $SU_L(2) \otimes U_Y(1)$. So they have $T^3 = 0$. The definition (178) of the charge $Q$

$$Q = T^3 + YI$$  \hspace{1cm} (193)

then implies that

$$Y_r = Q_r.$$  \hspace{1cm} (194)

That is, $Y_{\nu_{e,r}} = 0$, $Y_{e_r} = -1$, $Y_{u_r} = 2/3$, and $Y_{d_r} = -1/3$.

The left-handed fermions are in doublets

$$E_\ell = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \text{and} \quad Q_\ell = \begin{pmatrix} u \\ d \end{pmatrix}$$  \hspace{1cm} (195)

with $T^3 = \pm 1/2$. So the choices $Y_E = -1/2$ and $Y_Q = 1/6$ and the definition (178) of the charge $Q$ give the right charges:

$$QE_\ell = Q \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} = \begin{pmatrix} 0 \\ -1e^- \end{pmatrix} \quad \text{and} \quad QQ_\ell = Q \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} 2u/3 \\ -d/3 \end{pmatrix}.$$  \hspace{1cm} (196)
Fermion-gauge-boson interactions are due to the covariant derivative (189) acting on either the left- or right-handed fields. On right-handed fermions, the covariant derivative is just

\[ D^r_\mu = I \partial_\mu + i \frac{g}{\cos \theta_w} Z_\mu \left(-\sin^2 \theta_w Q\right) + ie A_\mu Q = I \partial_\mu - i g \frac{\sin^2 \theta_w}{\cos \theta_w} Z_\mu Q + ie A_\mu Q \]

(197)

So the covariant derivative of a neutral right-handed fermion is just the ordinary derivative.

On left-handed fermions, the covariant derivative is

\[ D^\ell_\mu = I \partial_\mu + i \frac{g}{\sqrt{2}} \left(W^+ + W^-\right) + i \frac{g}{\cos \theta_w} Z_\mu \left(T^3 - \sin^2 \theta_w Q\right) + ie A_\mu Q \]

(198)

For the first family or generation of quarks and leptons, the kinetic action density is

\[ L_k = -\mathcal{D}^\ell \mathcal{D}^\ell E_\ell - \mathcal{D}^r \mathcal{D}^r E_r - \mathcal{Q}^\ell \mathcal{D}^r Q_\ell - \mathcal{Q}^r \mathcal{D}^\ell Q_r \]

(199)

in which \( \mathcal{D} \equiv \gamma^\mu D_\mu \). The 4 \times 4 matrix \( \gamma_5 = \gamma^5 \) plays the role of a fifth (spatial) gamma matrix \( \gamma^4 = \gamma_5 \) in 5-dimensional space-time in the sense that the anticommutator

\[ \{\gamma^a, \gamma^b\} = 2\eta^{ab} \]

(200)

in which \( \eta \) is the 5 \times 5 diagonal matrix with \( \eta^{00} = -1 \) and \( \eta^{aa} = 1 \) for \( a = 1, 2, 3, 4 \). In Weinberg’s notation, \( \gamma_5 \) is

\[ \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

(201)

The combinations

\[ P_\ell = \frac{1}{2} \left(1 + \gamma_5\right) \quad \text{and} \quad P_r = \frac{1}{2} \left(1 - \gamma_5\right) \]

(202)
are projection operators onto the left- and right-handed fields. That is,

\[ P_\ell Q = Q_\ell \quad \text{and} \quad P_\ell Q_\ell = Q_\ell \]  

(203)

with a similar equation for \( P_r \). We can write \( L_k \) as

\[
L_k = -\frac{1}{2} \left[ E_\ell \gamma^\ell \frac{1}{2} (1 + \gamma_5) E - E_r \gamma^r \frac{1}{2} (1 + \gamma_5) E - Q_\ell \gamma^\ell \frac{1}{2} (1 - \gamma_5) Q - Q_r \gamma^r \frac{1}{2} (1 - \gamma_5) Q \right].
\]

(204)

Homework set 4, problem 1: Show that

\[
\bar{E}_\ell \gamma^\ell E_\ell = [\frac{1}{2} (1 + \gamma_5) E]^\dagger \gamma^0 \gamma^0 \gamma^\ell \frac{1}{2} (1 + \gamma_5) E = E_\ell \gamma^\ell \frac{1}{2} (1 + \gamma_5) E.
\]

(205)

Recall that in Weinberg’s notation

\[
\bar{\psi} = \psi^\dagger i\gamma^0 = \psi^\dagger \beta = \psi^\dagger \left( \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right)
\]

(206)

in which \( I \) is the \( 2 \times 2 \) identity matrix.

11.3 Quark and Lepton Masses

The Higgs mechanism also gives masses to the fermions, but somewhat arbitrarily. Dirac’s action density (8) has as its mass term

\[
-m \bar{\psi} \psi = -im \psi^\dagger \gamma^0 \psi = -im \psi^\dagger \gamma^0 (P_\ell + P_r) \psi = -im \psi^\dagger \gamma^0 (P_\ell^2 + P_r^2) \psi.
\]

(207)

Since \( \{ \gamma^0, \gamma^5 \} = 0 \), this mass term is

\[
-m \bar{\psi} \psi = -im \psi^\dagger P_r \gamma^0 P_\ell \psi - im \psi^\dagger P_\ell \gamma^0 P_r \psi = -im (P_\ell \psi)^\dagger \gamma^0 P_\ell \psi - im (P_r \psi)^\dagger \gamma^0 P_r \psi = -im \psi^\dagger \gamma^0 \psi_\ell - im \psi^\dagger \gamma^0 \psi_r = -m \bar{\psi}_r \psi_\ell - m \bar{\psi}_\ell \psi_r.
\]

(208)
Incidentally, because the fields $\psi_\ell$ and $\psi_r$ are independent, we can redefine them

\begin{align}
\psi_\ell' &= e^{i\theta} \psi_\ell \\
\psi_r' &= e^{i\phi} \psi_r
\end{align}

(209) (210)

at will. Such a redefinition changes the mass term to

\[ -m' \overline{\psi}_r \psi_\ell - m^{*} \overline{\psi}_\ell \psi_r = -m e^{i(\theta - \phi)} \overline{\psi}_r \psi_\ell - m e^{-i(\theta - \phi)} \overline{\psi}_\ell \psi_r. \tag{211} \]

So the phase of a Dirac mass term has no significance.

The definition (206) of $\overline{\psi}$ shows that the Dirac mass term is

\[ -m \overline{\psi} \psi = -m \psi_\ell \beta \psi = -m \begin{pmatrix} \psi_\ell^\dagger, \psi_r^\dagger \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \psi_\ell \\ \psi_r \end{pmatrix} = -m \left( \psi_\ell^\dagger \psi_r + \psi_r^\dagger \psi_\ell \right). \tag{212} \]

These mass terms are invariant under the Lorentz transformations

\[ \psi_\ell' = \exp(-z \cdot \sigma) \psi_\ell \]
\[ \psi_r' = \exp(z^* \cdot \sigma) \psi_r \tag{213} \]

because

\[ \psi_\ell^\dagger \psi_r' = \psi_\ell^\dagger \exp(-z^* \cdot \sigma) \exp(z \cdot \sigma) \psi_r = \psi_\ell^\dagger \psi_r \]
\[ \psi_r^\dagger \psi_\ell' = \psi_r^\dagger \exp(z \cdot \sigma) \exp(-z^* \cdot \sigma) \psi_\ell = \psi_r^\dagger \psi_\ell. \tag{214} \]

They are not invariant under rigid, let alone local, $SU(2)_\ell$ transformations. But we can make them invariant by using the Higgs field $H(x)$. For instance, the quantity $\overline{Q}_\ell H d_r$ is invariant under local $SU(2)_\ell$ transformations. In unitary gauge, its mean value in the vacuum is

\[ \langle 0 | \overline{Q}_\ell H d_r | 0 \rangle = \frac{1}{\sqrt{2}} v \langle 0 | d_\ell^\dagger d_r | 0 \rangle. \tag{215} \]
So the term

\[ -c_d \bar{Q}_\ell H d_r - c_d^* \bar{d}_r H^\dagger Q_\ell \]  

(216)
is invariant, and in the vacuum it is

\[ \langle 0 | -c_d \bar{Q}_\ell H d_r - c_d^* \bar{d}_r H^\dagger Q_\ell |0 \rangle = \frac{1}{\sqrt{2}} v \langle 0 | -c_d d^\dagger_\ell d_r - c_d^* d^\dagger_r d_\ell |0 \rangle \]  

(217)

which gives to the d quark the mass

\[ m_d = \frac{|c_d|}{\sqrt{2}} v. \]  

(218)

Note that we must add one new parameter \( c_d \) to get one new mass \( m_d \). This parameter \( c_d \) is complex in general, but the mass \( m_d \) depends only upon the absolute value and not upon its phase of \( c_d \).

Similarly, the term

\[ -c_e \bar{E}_\ell H e_r - c_e^* \bar{e}_r H^\dagger E_\ell \]  

(219)
is invariant, and in the vacuum it is

\[ \langle 0 | -c_e \bar{E}_\ell H e_r - c_e^* \bar{e}_r H^\dagger Q_\ell |0 \rangle = \frac{1}{\sqrt{2}} v \langle 0 | -c_e e^\dagger_\ell e_r - c_e^* e^\dagger_r e_\ell |0 \rangle \]  

(220)

which gives to the electron the mass

\[ m_e = \frac{|c_e|}{\sqrt{2}} v. \]  

(221)

Again, we must add one new (complex) parameter \( c_e \) to get one new mass \( m_e \).

The mass of the up quark requires a new trick. The Higgs field \( H \) transforms under \( SU(2)_\ell \times U(1) \) as

\[ H'(x) = \exp \left[ ig \frac{\sigma^a}{2} \alpha^a(x) + ig' \frac{1}{2} \beta(x) \right] H(x). \]  

(222)

If for clarity, we leave aside the \( U(1) \) part for the moment, then the Higgs field \( H \) transforms under \( SU(2)_\ell \) as

\[ H'(x) = \exp \left[ ig \frac{\sigma^a}{2} \alpha^a(x) \right] H(x). \]  

(223)

37
Let us use $H^*$ to be the complex column vector whose components are $H_1^\dagger$ and $H_2^\dagger$. How does $\sigma_2 H^*$ transform under $SU(2)_\ell$? Suppressing our explicit mention of the space-time dependence and using the asterisk to mean hermitian conjugation when applied to operators but complex conjugation when applied to matrices and vectors, we have, since $\sigma_2$ is imaginary with $\sigma_2^2 = I$ while $\sigma_1$ and $\sigma_3$ are real,

$$
\begin{align*}
(\sigma_2 H^*)' &= \sigma_2 \left[ \exp \left( i g \frac{\sigma_2^a}{2} \alpha^a \right) H \right]^* = \sigma_2 \exp \left( -i g \frac{\sigma_2^a}{2} \alpha^a \right) H^* \\
&= \sigma_2 \exp \left( -i g \frac{\sigma_2^*}{2} \alpha^a \right) \sigma_2 \sigma_2 H^* = \exp \left( i g \frac{\sigma_2^a}{2} \alpha^a \right) \sigma_2 H^*.
\end{align*}
$$

Thus, the term

$$
- c_u \overline{Q}_\ell \sigma_2 H^* u_r - c_u^* \overline{\nu}_r H^T \sigma_2 Q_\ell
$$

is invariant under $SU(2)_\ell$. In the vacuum of the unitary gauge, it is

$$
\langle 0 | - c_u \overline{Q}_\ell \sigma_2 H^* u_r - c_u^* \overline{\nu}_r H^T \sigma_2 Q_\ell | 0 \rangle = \frac{1}{\sqrt{2}} v \langle 0 | i c_u \overline{\nu}_r u_r - i c_u^* \overline{Q}_\ell u_\ell | 0 \rangle = \frac{1}{\sqrt{2}} v \langle 0 | i c_u u_\ell^T u_r - i c_u^* u_r^T u_\ell | 0 \rangle
$$

which gives the up quark the mass

$$
m_u = \frac{|c_u|}{\sqrt{2}} v.
$$

But there are three families of generations of quarks and leptons on which the gauge fields act simply:

$$
F_1 = \begin{pmatrix} u \\ d \\ \nu_e \end{pmatrix}', \quad F_2 = \begin{pmatrix} c \\ s \\ \nu_\mu \end{pmatrix}', \quad \text{and} \quad F_3 = \begin{pmatrix} t \\ b \\ \nu_\tau \end{pmatrix}'.
$$

These are called the flavor eigenstates or more properly flavor eigenfields; I’m designating them with primes. They are the ones that the $W^\pm$ act on most simply. The weak interactions use $W^-_{\mu} T^-$ to map the flavor up fields $u'$, $c'$, $t'$ into the flavor down fields $d'$, $s'$, $b'$ and $W^+_{\mu} T^+$ to map the flavor down fields into the flavor up fields.
The action density
\[
\sum_{i,j=1}^{3} -c_{dij} \bar{Q}_{ti} H d'_{rj} - c^*_{dij} \bar{d}'_{rj} H^\dagger Q'_{ti}
\] (229)
gives for the \(d', s', \) and \(b'\) quarks the mixed mass terms
\[
\frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} -c_{dij} \bar{d}'_{ti} d'_{rj} - c^*_{dij} \bar{d}'_{rj} d'_{ti} = \frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} -c_{dij} d'^\dagger_{ti} d'_{rj} - c^*_{dij} d'^\dagger_{rj} d'_{ti}.
\] (230)

The \(3 \times 3\) mass matrix \(M_d\) with entries
\[
[M_d]_{ij} = \frac{v}{\sqrt{2}} c_{dij}
\] (231)
need have no special properties. It need not be hermitian because for each \(i\) and \(j\), the term (230) is hermitian. But every \(3 \times 3\) complex matrix has a **singular value decomposition**
\[
M_d = L_d \Sigma_d R_d^\dagger
\] (232)
in which \(L_d\) and \(R_d\) are \(3 \times 3\) unitary matrices, and \(\Sigma_d\) is a \(3 \times 3\) diagonal matrix with nonincreasing positive singular values on its main diagonal.

The singular value decomposition works for any \(N \times M\) (real or) complex matrix. Every complex \(M \times N\) rectangular matrix \(A\) is the product of an \(M \times M\) unitary matrix \(U\), an \(M \times N\) rectangular matrix \(\Sigma\) that is zero except on its main diagonal which consists of its nonnegative singular values \(S_k\), and an \(N \times N\) unitary matrix \(V^\dagger\)
\[
A = U \Sigma V^\dagger.
\] (233)
This singular-value decomposition is a key theorem of matrix algebra. One can use the *Matlab* command “[U,S,V] = svd(A)” to perform the svd \(A = USV^\dagger\).

The singular values of \(\Sigma_d\) are the masses \(m_b\), \(m_s\), and \(m_d\):
\[
\Sigma_d = \begin{pmatrix} m_b & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_d \end{pmatrix}.
\] (234)

39
So the mass eigenfields of the left and right down-quark fields are

\[ d_{ri} = R_{dij}^\dagger d_{rj} \quad \text{and} \quad d_{\ell i}^\dagger = d_{\ell j}^\dagger L_{dij} \quad \text{or} \quad d_{\ell i} = L_{dij}^\dagger d_{\ell j}. \]

(235)

The inverse relations are

\[ d_{ri}^\dagger = R_{dij} d_{rj} \quad \text{and} \quad d_{\ell i}^\dagger = d_{\ell j}^\dagger R_{dij}^\dagger \quad \text{or} \quad d_{\ell i}^\dagger = L_{dij} d_{\ell j}. \]

(236)

or in matrix notation

\[ d_{\ell i}^\dagger = L_{dij} d_{\ell j} \]

in which

\[ d_{\ell} = \begin{pmatrix} b \\ s \\ d \end{pmatrix} \]

are the down-quark fields of definite masses.

Similarly, the up quark action density

\[ \sum_{i,j=1}^{3} -c_{uij} Q'_{\ell i} \sigma_2 H^* u_{rj} - c_{uij}^* \bar{u}_{rj} H^T \sigma_2 Q'_{\ell i} \]

(239)

gives for the three known families the mixed mass terms

\[ \frac{iv}{\sqrt{2}} \sum_{i,j=1}^{3} c_{uij} \bar{u}_{rj}^\dagger u_{\ell i} - c_{uij}^* u_{rj}^\dagger \bar{u}_{\ell i} = \frac{iv}{\sqrt{2}} \sum_{i,j=1}^{3} c_{uij} u_{\ell i}^\dagger u_{rj}^\dagger - c_{uij}^* u_{\ell i} u_{rj}^\dagger. \]

(240)

The 3 × 3 mass matrix \( M_u \) with entries

\[ [M_u]_{ij} = \frac{iv}{\sqrt{2}} c_{uij} \]

(241)
need have no special properties. It need not be hermitian because for each \( i \) and \( j \), the term \( (240) \) is hermitian. But every \( 3 \times 3 \) complex matrix \( M_u \) has a singular value decomposition

\[
M_u = L_u \Sigma_u R_u^\dagger
\]

in which \( L_u \) and \( R_u \) are \( 3 \times 3 \) unitary matrices, and \( \Sigma_u \) is a \( 3 \times 3 \) diagonal matrix with nonincreasing positive singular values on its main diagonal. These singular values are the masses \( m_t, m_c, \) and \( m_u \):

\[
\Sigma_u = \begin{pmatrix} m_t & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_u \end{pmatrix}.
\]

So the mass eigenfields of the left and right up-quark fields are

\[
u_{ri} = R_{uij}^\dagger u_{rj} \quad \text{and} \quad u_{\ell i}^\dagger = u_{\ell j}^\dagger L_{uij}, \quad \text{or} \quad u_{\ell i}^\dagger = L_{uij}^\dagger u_{\ell j}.
\]

The inverse relations are

\[
u_{ri}^\dagger = R_{uij} u_{rj} \quad \text{and} \quad u_{\ell i}^\dagger = u_{\ell j}^\dagger L_{uij}^\dagger, \quad \text{or} \quad u_{\ell i}^\dagger = L_{uij} u_{\ell j}
\]

or in matrix notation

\[
u_{r}^\dagger = R_u \nu_r, \quad \nu_{\ell}^\dagger = u_{\ell}^\dagger R_u^\dagger \quad \text{and} \quad u_{\ell}^\dagger = L_u^\dagger u_{\ell}
\]

in which

\[
u_{\ell} = \begin{pmatrix} t \\ c \\ u \end{pmatrix}
\]

are the up-quark fields of definite masses.
11.4 The CKM Matrix

We will use the labels $u$, $c$, $t$ and $d$, $s$, $b$ for the states that are eigenstates of the quadratic part of the hamiltonian after the Higgs mechanism has given a mean value to the real part of the neutral Higgs boson in the unitary gauge. The $u$, $c$, $t$ quarks have the same charge $2e/3 > 0$ and the same $T^3 = 1/2$, so they all have the same electroweak interactions. Similarly, the $d$, $s$, $b$ quarks have the same charge $-e/3 < 0$ and the same $T^3 = -1/2$, so they also all have the same electroweak interactions.

The right-handed covariant derivative (197)

$$D^r_\mu = I \partial_\mu - ie \tan \theta_w Z_\mu Q + ie A_\mu Q$$

(248)

just sends the fields of these mass eigenstates into themselves multiplied by their charge and either a $Z$ or a photon. That is,

$$\bar{u}^r_\ell D^r_\mu u^\ell_r = \bar{u}^r_\ell R^r_u D^r_u R^r_u u^\ell_r = \bar{u}^r_\ell D^r_\mu u^\ell_r$$

$$\bar{d}^r_\ell D^r_\mu d^\ell_r = \bar{d}^r_\ell R^r_d D^r_d R^r_d d^\ell_r = \bar{d}^r_\ell D^r_\mu d^\ell_r$$

(249)

Thus, the $Z$ and the photon don’t distinguish between right-handed flavor and mass fields.

But the left-handed covariant derivative (198) is

$$D^\ell_\mu = I \partial_\mu + i \frac{e}{\sqrt{2} \sin \theta_w} (W^+_\mu T^+ + W^-_\mu T^-) + i \frac{e}{\cos \theta_w} Z_\mu \left( \frac{T^3}{\sin \theta_w} - \sin \theta_w Q \right) + ie A_\mu Q.$$  

(250)

So we have

$$\left( \begin{array}{c} \bar{u}^\ell \ \bar{d}^\ell \\ \end{array} \right) D^\ell_\mu \left( \begin{array}{c} u^\ell \ \\ d^\ell \end{array} \right) = \left( \begin{array}{c} \bar{u}^\ell L_u^\dagger \ \\ \bar{d}^\ell L_d^\dagger \end{array} \right) D^\ell_\mu \left( \begin{array}{c} L_u u^\ell \\ L_d d^\ell \end{array} \right).$$

(251)

In most places, the unitary matrices just give unity, $L_u^\dagger L_u = I$ and $L_d^\dagger L_d = I$. The only changes are in the nonzero parts of $T^\pm$ which become

$$T_{CKM}^+ = \begin{pmatrix} 0 & L_u^\dagger L_d \\ 0 \\ \end{pmatrix} \quad \text{and} \quad T_{CKM}^- = \begin{pmatrix} 0 \\ L_d^\dagger L_u \\ 0 \end{pmatrix}. \quad (252)$$
The unitary matrix \( V = L_u^\dagger L_d \) is called the CKM matrix after Nicola Cabibbo, Makoto Kobayashi, and Toshihide Maskawa. The left-handed covariant derivative on the mass eigenfields then is

\[
D_\mu^\ell = i \partial_\mu + i \frac{e}{\sin \theta_w} (W_\mu^+ T_{\text{CKM}}^+ + W_\mu^- T_{\text{CKM}}^-) + i \frac{e}{\cos \theta_w} Z_\mu \left( \frac{T^3}{\sin \theta_w} - \sin \theta_w Q \right) + ie A_\mu Q.
\]

It has a second part that acts more or less like the right-handed covariant derivative, but the first part uses \( W_\mu^- T^- \) to map the up fields \( u, c, t \) into linear combinations of the down fields \( d, s, b \) and \( W_\mu^+ T^+ \) to map the down fields into linear combinations of the up fields. The \( W_\mu^\pm \) terms are sensitive to the CKM matrix

\[
V = L_u^\dagger L_d.
\]

We write them suggestively as

\[
(u \ c \ t \ d \ s \ b)^\dagger \begin{pmatrix} 0 & V W_\mu^- & V W_\mu^+ \\ V W_\mu^- & V W_\mu^- & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ c \\ t \\ d \\ s \\ b \end{pmatrix} = \begin{pmatrix} u \\ c \\ t \end{pmatrix} V \begin{pmatrix} d \\ s \\ b \end{pmatrix}^\dagger \begin{pmatrix} 0 & V W_\mu^- & V W_\mu^+ \\ V W_\mu^- & V W_\mu^- & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix} V \begin{pmatrix} d \\ s \\ b \end{pmatrix},
\]

(254)

By choosing the phases of the six quark fields, that is, \( u(x) \to e^{i \theta u} u(x) \ldots b(x) \to e^{i \theta b} b(x) \), one may make the CKM matrix \( L_u^\dagger L_d \) real apart from a single phase. The existence of that phase probably is the cause of most of the breakdown of \( CP \) invariance that Fitch and Cronin and others have observed since 1964. The magnitudes of the elements of the CKM matrix \( V \) are

\[
V = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.97427 & 0.22536 & 0.00355 \\ 0.22522 & 0.97343 & 0.0414 \\ 0.00886 & 0.0405 & 0.99914 \end{pmatrix}.
\]

(255)

Although there is only one phase \( \exp(i \delta) \) in the CKM matrix \( V \), the experimental constraints on this phase often are expressed in terms of the angles \( \alpha, \beta, \) and \( \gamma \) defined as

\[
\alpha = \arg \left[ -V_{td} V_{tb}^* / (V_{ud} V_{ub}^*) \right],
\]

(256)

\[
\beta = \arg \left[ -V_{cd} V_{cb}^* / (V_{td} V_{tb}^*) \right],
\]

(257)

\[
\gamma = \arg \left[ -V_{ud} V_{ub}^* / (V_{cd} V_{cb}^*) \right].
\]

(258)
If $V$ is unitary, then $\alpha + \beta + \gamma = 180^\circ$. From $B \to \pi\pi$, $\rho\pi$, and $\rho\rho$ decays, the limits on the angle $\alpha$ are roughly

\[ \alpha = (85.4 \pm 4)^\circ. \] (259)

From $B^\pm \to DK^\pm$ decays, the limits on the angle $\gamma$ are roughly

\[ \gamma = (68.0 \pm 8)^\circ. \] (260)

So the angle $\beta$ is about $26.6^\circ$.

One of the quark-Higgs interactions is

\[
-c_{dij} \overline{Q}_d^r H d_r^d = -\frac{\sqrt{2}}{v} Q_d^r M_d^r d_r^d H = -\frac{\sqrt{2}}{v} Q_d^r \Sigma_d^r R_d^r d_r^d H
\]

\[
= -\frac{\sqrt{2}}{v} Q_d^r \Sigma_d^r d_r^d H = -\frac{\sqrt{2}}{v} Q_d^r \left( \frac{0}{(v + h)/\sqrt{2}} \right) \Sigma_d^r d_r^d H
\]

\[
= -d_{d_i}^r \left( 1 + \frac{h}{v} \right) \Sigma_d^r d_r^d H = -d_{d_i}^r \left( 1 + \frac{h}{v} \right) d_r^d H.
\] (261)

A similar term describes the coupling of the up quarks to the Higgs

\[
-m_{u_i} \overline{u}_{d_i} \left( 1 + \frac{h}{v} \right) u_r^d,
\] (262)

Thus, the rate of quark-antiquark to Higgs is proportional to the mass of the quark in the standard model.

12 Lepton Masses

We can treat the leptons just like the quarks. The up leptons are the flavor neutrinos $\nu_e'$, $\nu_\mu'$, and $\nu_\tau'$, and the down leptons are the flavor charged leptons $e'$, $\mu'$, and $\tau'$. The action density

\[
\sum_{i,j=1}^3 -c_{eij} \overline{E}_{\ell_i}^d H e_{r_j}^d - c_{eij}^* \overline{e}_{r_j}^d H^\dagger E_{\ell_i}^d
\] (263)
gives for the $e'$, $\mu'$, and $\tau'$ the mixed mass terms

$$\frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} -c_{eij} \tau'_{ij} e'_{ij} - c_{eij}^{*} e'_{ij} \tau'_{ij}.$$  \hspace{1cm} (264)

The $3 \times 3$ mass matrix $M_e$ with entries

$$[M_e]_{ij} = \frac{v}{\sqrt{2}} c_{eij}^{*}$$  \hspace{1cm} (265)

has a singular value decomposition

$$M_e = L_e \Sigma_e R_e^{\dagger}$$  \hspace{1cm} (266)

in which $L_e$ and $R_e$ are $3 \times 3$ unitary matrices, and $\Sigma_e$ is a $3 \times 3$ diagonal matrix with nonincreasing positive singular values $m_{\tau}$, $m_{\mu}$, and $m_{e}$ on its main diagonal.

12.1 Before and After Symmetry Breaking

Before spontaneous symmetry breaking, all the fields of the standard model are massless, and the local symmetry under $SU(2)_{\ell} \otimes U(1)$ is exact. Under these gauge transformations, the left-handed electron and neutrino fields are rotated among themselves. If $e'_{ij}$ is a linear combination of itself and of $\nu'_{ij}$, then these two fields, $e'_{ij}$ and $\nu'_{ij}$, must be of the same kind. The left-handed electron field is a Dirac field. Thus, the left-handed neutrino field also must be a Dirac field. This makes sense because before symmetry breaking, all the fields are massless, and so there is no problem combining two Majorana fields of the same mass, namely zero, into one Dirac field. Thus, there are three flavor Dirac neutrino fields $\nu'_{\ell}$, $\nu'_{\mu}$, and $\nu'_{\tau}$.

A massless left-handed neutrino field $\nu'_{\ell}$ satisfies the two-component Dirac equation

$$(\partial_{0} I - \nabla \cdot \sigma) \nu'_{\ell}(x) = 0$$  \hspace{1cm} (267)

which in momentum space is

$$(E + \mathbf{p} \cdot \sigma) \nu'_{\ell}(p) = 0.$$  \hspace{1cm} (268)
Since the angular momentum is $\mathbf{J} = \sigma / 2$, and $E = |\mathbf{p}|$, we have

$$\hat{p} \cdot \mathbf{J} \nu'_\ell(p) = -\frac{1}{2} \nu'_\ell(p).$$  \hspace{1cm} (269)$$

The left-handed neutrino field $\nu'_\ell$ annihilates neutrinos of negative helicity and creates antineutrinos of positive helicity.

Since the neutrinos are massive, there may be right-handed neutrino fields. As for the $u$-$p$ quarks, we can use them to make an action density

$$\sum_{i,j=1}^{3} -c_{\nu ij} E'_{\ell i} \sigma_2 H^* \nu'_j - c^*_{\nu ij} \nu'_{r j} H^T \sigma_2 E'_{\ell i}$$

that is invariant under $SU(2)_\ell \otimes U(1)$ and that gives for the neutrinos the mixed mass terms

$$\sum_{i,j=1}^{3} \frac{iv}{\sqrt{2}} (c_{\nu ij} \nu^*_r \nu_{r j} - c^*_{\nu ij} \nu'_r \nu^{' r j}).$$  \hspace{1cm} (271)$$

The $3 \times 3$ mass matrix $M_\nu$ with entries

$$[M_\nu]_{ij} = \frac{iv}{\sqrt{2}} c_{\nu ij}$$

has a singular value decomposition

$$M_\nu = L_\nu \Sigma_\nu R_\nu^\dagger$$  \hspace{1cm} (273)$$

in which $L_\nu$ and $R_\nu$ are $3 \times 3$ unitary matrices, and $\Sigma_\nu$ is a $3 \times 3$ diagonal matrix with nonincreasing positive singular values $m_{\nu_\tau}$, $m_{\nu_\mu}$, and $m_{\nu_e}$ on its main diagonal (here, I have assumed that the neutrino masses mimic those of the charged leptons and quarks, rising with family number). The neutrino CKM matrix then would be $L^\dagger_\nu L_\nu$, but since we are accustomed to treating the charged leptons as flavor and mass eigenfields, we apply the neutrino CKM matrix to the neutrinos rather than to the charged leptons. Thus the neutrino CKM matrix is

$$V_\nu = L^\dagger_\nu L_\nu.$$  \hspace{1cm} (274)$$
By choosing the phases of the six lepton fields, we can make the neutrino CKM matrix real except for $CP$-breaking phases. If the neutrinos are Dirac fields, then there is one such phase; if not, there are three.

So far, I have assumed that the mass terms for the neutrinos are the usual Dirac mass terms. However, the right-handed Majorana neutrino fields $\nu'_{\ell}$ are not affected by the $SU(2)_{\ell} \otimes U(1)$.

**Note that a gauge transformation between $e$ and $\nu_e$ rotates the operators $a(p, s, e)$ and $a(p, s, \nu_e)$ into each other.** This rotation makes sense only when the two particles have the same mass. In the standard model, such a gauge transformation makes sense only before symmetry breaking when all the particles are massless. Moreover, only when the particles are massless can one say that they are left- or right-handed. While the particles are massless, the operator $a(p, -)$ annihilates a particle of negative helicity and occurs only in a left-handed field, while the operator $a(p, +)$ annihilates a particle of positive helicity and occurs only in a right-handed field. But when the particles are massive, the operator $a(p, \frac{1}{2})$ annihilates a particle that is spin up and occurs in both left-handed and right-handed fields. So a symmetry transformation that acted on the operator $a(p, \frac{1}{2})$ would change both left-handed and right-handed fields.

The left-handed fields of the neutrino and electron are

\[
\nu_{e,\ell}(x) = \int u(p, -) \frac{a_1(p, -, \nu_e) + ia_2(p, -, \nu_e)}{\sqrt{2}} e^{ipx} + v(p, +) \frac{a_1^\dagger(p, +, \nu_e) + ia_2^\dagger(p, +, \nu_e)}{\sqrt{2}} e^{-ipx} \frac{d^3p}{(2\pi)^{3/2}} \tag{275}
\]

\[
e_{\ell}(x) = \int u(p, -) \frac{a_1(p, -, e) + ia_2(p, -, e)}{\sqrt{2}} e^{ipx} + v(p, +) \frac{a_1^\dagger(p, +, e) + ia_2^\dagger(p, +, e)}{\sqrt{2}} e^{-ipx} \frac{d^3p}{(2\pi)^{3/2}} \tag{276}
\]

where $(p, -, \nu_e)$ means momentum $p$, spin down, and electron flavor, and $(p, +, \nu_e)$ means momentum $p$, spin up, and electron flavor. These fields satisfy equations like (267–269) apart from their interactions with other fields. Since a gauge transformation maps the fields $\nu_{e,\ell}(x)$ and $e_{\ell}(x)$ into each other, we know that when all the fields are massless, before symmetry breaking, there are (for each momentum) at least two neutrino and antineutrino states

\[
\frac{1}{\sqrt{2}} \left[ a_1^\dagger(p, -, \nu_e) - ia_2^\dagger(p, -, \nu_e) \right] |0\rangle \tag{277}
\]

\[
\frac{1}{\sqrt{2}} \left[ a_1^\dagger(p, +, \nu_e) + ia_2^\dagger(p, +, \nu_e) \right] |0\rangle \tag{278}
\]
for each of the three flavors, \( f = e, \mu, \tau \). So there are at least six neutrino (and antineutrino) states.

The right-handed electron field exists and interacts with gauge bosons and other fields. So there are 12 electron states \( a_i^\dagger(p, \pm, \epsilon_f)|0\rangle \) for \( i = 1 \) and 2 and for the three flavors, \( f = e, \mu, \tau \). We don’t know yet whether a right-handed neutrino field exists or interacts with other fields. So there may be only 6 neutrino states or as many as 12.

Neutrino oscillations tell us that neutrinos have masses. If there are 12 neutrino states, then there can be three massive Dirac neutrinos analagous to the \( e, \mu, \) and \( \tau \) or six massive Majorana neutrinos or some intermediate combination. If there are only 6, then there can be 3 massive Majorana neutrinos.

The Majorana mass terms for the right-handed neutrino fields are

\[
\sum_{ij=1}^{6} \frac{1}{2} \left[ im_{ij} \nu'^T_{ri} \sigma_2 \nu'_{rj} + (im_{ij} \nu'^T_{ri} \sigma_2 \nu'_{rj})^\dagger \right].
\]  

(279)

They are Lorentz invariant because under the Lorentz transformations (213)

\[
\nu''^{T_r} \sigma_2 \nu'' = \nu^{T_r} \exp(z^* \cdot \sigma^T) \sigma_2 \exp(z^* \cdot \sigma) \nu_r = \nu'^{T_r} \sigma_2 \nu' _r.
\]

(280)

The Majorana mass terms (279) are unrelated to the scale \( v \) of the Higgs field’s mean value. One can show that the complex matrix \( m_{ij} \) is symmetric. One then must combine the mass matrix in (279) with the mass matrix \( M_\nu \) in (272). The resulting mass matrix will have a singular-value decomposition with six singular values that would be the masses of the “physical” neutrinos. If these six masses are equal in pairs, then the three pairs would form three Dirac neutrinos.

Whether or not there are right-handed neutrinos, we can make Majorana mass terms like \( \nu^T \sigma_2 \nu \), which are Lorentz invariant but not invariant under \( SU_\ell(2) \) or \( U_Y(1) \). We can make them gauge invariant by using a triplet \( \vec{\phi} = \sigma_i \phi_i \) of Higgs fields that transforms as \( \vec{\phi}' \cdot \vec{\sigma} = g(\vec{\phi} \cdot \vec{\sigma}) g^\dagger \) for \( g \in SU_\ell(2) \) and that carries a value of \( Y = -1 \). Then if \( \sigma_2 \) has Lorentz indices and \( \sigma'_2 \) has \( SU_\ell(2) \) indices, the term

\[
E^\ell_\ell^T \sigma_2 \sigma'_2 (\vec{\phi} \cdot \vec{\sigma}) E_\ell
\]

(281)
is both Lorentz invariant and gauge invariant. If the potential $V(\tilde{\phi})$ has minima at $\tilde{\phi} \neq 0$, then this term violates lepton number and gives a Majorana mass to the neutrino. Another possibility is to say that at higher energies a theory with new fields of very high mass $\Lambda$ plays a role, and that when one path-integrates over these heavy fields, one is left with an effective, nonrenormalizable term in the action

$$-rac{(H^\dagger E_\ell)^2}{\Lambda}$$

(282)

which gives a Majorana mass to $\nu_\ell$.

Models with both right-handed and left-handed neutrinos are easier to think about, but only experiments can tell us whether right-handed neutrinos exist.

What is known experimentally is that there are at least three masses that satisfy

$$|\Delta m^2_{21}| \equiv |m_2^2 - m_1^2| = (7.53 \pm 0.18) \times 10^{-5} \text{eV}^2$$

$$|\Delta m^2_{32}| \equiv |m_3^2 - m_2^2| = (2.44 \pm 0.06) \times 10^{-3} \text{eV}^2 \quad \text{normal mass hierarchy}$$

$$|\Delta m^2_{32}| \equiv |m_3^2 - m_1^2| = (2.52 \pm 0.07) \times 10^{-3} \text{eV}^2 \quad \text{inverted mass hierarchy.}$$

If the neutrinos are Dirac particles, then they have a CKM matrix like that of the quarks with one $CP$-violating phase. But whereas one chooses to make the mass and flavor eigenfields the same for the up quarks $u, c, t$, for the leptons one makes the mass and flavor eigenfields the same for the down or charged leptons $e, \mu, \tau$. So the neutrino CKM matrix actually is $V = L^\dagger e L_\nu$. If they are three Majorana particles, then their CKM matrix has two extra $CP$-violating phases $\alpha_{12}$ and $\alpha_{31}$. A common convention for the neutrino CKM matrix is

$$V = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{23} & \sin \theta_{23} \\
0 & -\sin \theta_{23} & \cos \theta_{23}
\end{pmatrix}
\begin{pmatrix}
\cos \theta_{13} & 0 & \sin \theta_{13} e^{-i\delta} \\
0 & 1 & 0 \\
-\sin \theta_{13} e^{i\delta} & 0 & \cos \theta_{13}
\end{pmatrix}
\begin{pmatrix}
\cos \theta_{12} & \sin \theta_{12} & 0 \\
-\sin \theta_{12} & \cos \theta_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\alpha_{12}/2} & 0 \\
0 & 0 & e^{i\alpha_{31}/2}
\end{pmatrix}. $$

(284)

This convention without the last $3 \times 3$ matrix also is used for the quark CKM matrix. The current estimates
are

\[
\sin^2(2\theta_{12}) = 0.846 \pm 0.021 \tag{285}
\]
\[
\sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018} \text{ normal mass hierarchy} \tag{286}
\]
\[
\sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017} \text{ inverted mass hierarchy} \tag{287}
\]
\[
\sin^2(2\theta_{13}) = 0.093 \pm 0.008. \tag{288}
\]

Two of these are big angles: \(2\theta_{12} \approx 2\theta_{23} = \pi/2 \pm n\pi\). In the normal hierarchy, the lightest neutrino is about 2/3 electron, 1/6 muon, and 1/6 tau; the very slightly heavier neutrino is about 1/3 electron, 1/3 muon, and 1/3 tau; and the much heavier heavier neutrino is about 1/6 electron, 5/12 muon, and 5/12 tau.

### 12.2 The Seesaw Mechanism

Why are the neutrino masses so light? Suppose we wish to find the eigenvalues of the real, symmetric mass matrix

\[
\mathcal{M} = \begin{pmatrix}
0 & m \\
\frac{m}{M} & M
\end{pmatrix}
\]

in which \(m\) is an ordinary mass and \(M\) is a huge mass. The eigenvalues \(\mu\) of this hermitian mass matrix satisfy \(\det(\mathcal{M} - \mu I) = \mu(\mu - M) - m^2 = 0\) with solutions \(\mu_{\pm} = (M \pm \sqrt{M^2 + 4m^2})/2\). The larger mass \(\mu_+ \approx M + m^2/M\) is approximately the huge mass \(M\) and the smaller mass \(\mu_- \approx -m^2/M\) is tiny. The physical mass of a fermion is the absolute value of its mass parameter, here \(m^2/M\).

The product of the two eigenvalues is the constant \(\mu_+\mu_- = \det\mathcal{M} = -m^2\) so as \(\mu_-\) goes down, \(\mu_+\) must go up. In 1975, Gell-Mann, Ramond, Slansky, and Jerry Stephenson invented this “seesaw” mechanism as an explanation of why neutrinos have such small masses, less than 1 eV/c^2. If \(mc^2 = 10\) MeV, and \(\mu_-c^2 \approx 0.01\) eV, which is a plausible light-neutrino mass, then the rest energy of the huge mass would be \(Mc^2 = 10^7\) GeV. This huge mass would be one of the six neutrino masses and would point at new physics, beyond the standard model. Yet the small masses of the neutrinos may be related to the weakness of their interactions.
Before leaving the subject of fermion masses, let’s look more closely at Dirac and Majorana mass terms.

A Dirac field is a linear combination of two Majorana fields of the same mass

\[
\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} L + i\ell \\ R + ir \end{pmatrix}
\]

(290)

in which \( L \) and \( \ell \) are two-component left-handed spinors, and \( R \) and \( r \) are two-component right-handed spinors. The Dirac mass term

\[
m \bar{\psi} \psi = i \, m \, \psi^\dagger \gamma^0 \psi = m \psi^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \psi
\]

\[
= m \frac{1}{2} (L^\dagger - i\ell^\dagger, \ R^\dagger - ir^\dagger) \left( \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right) \left( \begin{pmatrix} L + i\ell \\ R + ir \end{pmatrix} \right)
\]

\[
= m \frac{1}{2} [(L^\dagger - i\ell^\dagger) (R + ir) + (R^\dagger - ir^\dagger) (L + i\ell)]
\]

\[
= m \frac{1}{2} (R^\dagger - ir^\dagger) (L + i\ell) + \text{h.c.,}
\]

(291)

in which h.c. means hermitian conjugate, gives mass \( m \) to the particle and antiparticle of the Dirac field \( \psi \).

We may set

\[
R = i\sigma_2 \, L^* \iff L = -i\sigma_2 \, R^*
\]

(292)

\[
r = i\sigma_2 \, \ell^* \iff \ell = -i\sigma_2 \, r^*
\]

(293)

(294)
or equivalently

\[
\begin{pmatrix}
R_1 \\
R_2
\end{pmatrix} = \begin{pmatrix} L_2^\dagger \\
-L_1^\dagger
\end{pmatrix} \iff \begin{pmatrix} L_1 \\
L_2
\end{pmatrix} = \begin{pmatrix} -R_2^\dagger \\
R_1^\dagger
\end{pmatrix}
\]

\[
\begin{pmatrix}
 r_1 \\
 r_2
\end{pmatrix} = \begin{pmatrix} \ell_2^\dagger \\
-\ell_1^\dagger
\end{pmatrix} \iff \begin{pmatrix} \ell_1 \\
\ell_2
\end{pmatrix} = \begin{pmatrix} -r_2^\dagger \\
r_1^\dagger
\end{pmatrix}
\]

which are the Majorana conditions. Since \( R^\dagger = -iL^T\sigma_2 \), we can write the Dirac mass term (291) in terms of left-handed fields as

\[
m\bar{\psi}\psi = \frac{1}{2} m \left(-i L^T - \ell^T\right) \sigma_2 (L + i\ell) + \text{h.c.}
\]

\[
= \frac{1}{2} m \left(L^T - i\ell^T\right) \left(-i \sigma_2\right) (L + i\ell) + \text{h.c.}
\]

\[
= \frac{1}{2} m \left(L_1 - i\ell_1, \quad L_2 - i\ell_2\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L_1 + i\ell_1 \\ L_2 + i\ell_2 \end{pmatrix} + \text{h.c.}
\]

\[
= \frac{1}{2} m \left(L_1 - i\ell_1\right) \left(-L_2 - i\ell_2\right) + \left(L_2 - i\ell_2\right) \left(L_1 + i\ell_1\right) + \text{h.c.}
\]

The fermion fields anticommute, so the Dirac mass term is

\[
m\bar{\psi}\psi = \frac{1}{2} m \left(-2L_1 L_2 - 2\ell_1\ell_2\right) + \text{h.c.} = -m \left(L_1 L_2 + \ell_1\ell_2\right) + \text{h.c.},
\]

and it says that the fields \( L \) and \( \ell \) have the same mass \( m \), as they must if they are to form a Dirac field.
Since $L^i = iR^T \sigma_2$, we also can write the Dirac mass term in terms of the right-handed fields as

$$m \bar{\psi} \psi = \frac{1}{2} m \left( R^T - i r^T \right) i \sigma_2 (R + i r) + \text{h.c.}$$

$$= \frac{1}{2} m \left( R_1 - i r_1, \quad R_2 - i r_2 \right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} R_1 + i r_1 \\ R_2 + i r_2 \end{pmatrix} + \text{h.c.}$$

$$= m (R_1 R_2 + r_1 r_2) + \text{h.c.}$$

So the fields $R$ and $r$ have the same mass $m$, as they must if they are to form a Dirac field. The Majorana mass term for a right-handed field $r$ of mass $m$ evidently is

$$m r_1 r_2 + \text{h.c.}$$

12.3 Neutrino Oscillations

The phase difference $\Delta \phi$ between two highly relativistic neutrinos of momentum $p$ going a distance $L$ in a time $t \approx L$ varies with their masses $m_1$ and $m_2$ as

$$\Delta \phi = t \Delta E = \frac{LE}{p} \Delta E = \frac{LE}{p} \left( \sqrt{p^2 + m_1^2} - \sqrt{p^2 + m_2^2} \right)$$

in natural units. We can approximate this phase by using the first two terms of the binomial expansion of the square roots with $y = 1$ and $x = m_2^2/p^2$

$$\Delta \phi \approx LE \left( \sqrt{1 + m_1^2/p^2} - \sqrt{1 + m_2^2/p^2} \right) \approx \frac{LE \Delta m^2}{p^2} \approx \frac{L \Delta m^2}{E}$$

or in ordinary units $\Delta \phi \approx L \Delta m^2 c^3/(\hbar E)$.

13 Grand Unification

The success of the electroweak unification of the standard model led physicists in the 1970s to propose what they called grand unification. Their goal was to unify the electroweak and the strong interactions.
Howard Georgi and Sheldon Glashow made the first attempt in 1974. They chose the group $SU(5)$ which with 24 generators is big enough to house $SU_c(3) \otimes SU_\ell(2) \otimes U_Y(1)$ and its 12 generators.

Compact internal-symmetry groups can’t rotate left-handed fields into right-handed fields. So the first problem they overcame was how to combine transformations $SU_\ell(2)$ that act only on left-handed fields with ones $SU_c(3)$ that act on both left- and right-handed fields. They solved that problem by writing all fields as left-handed fields. Recall for instance that if $u_r$ is a right-handed up-quark field, that is if it transforms like

$$u'_r = \exp(\vec{z} \cdot \vec{\sigma}) u_r \tag{310}$$

then

$$u_\ell^c = \sigma_2 u_r^* \tag{311}$$

is left-handed, that is, it transforms as

$$\begin{align*}
(u_\ell^c)' &= \sigma_2 (u'_r)^* = \sigma_2 [\exp(\vec{z} \cdot \vec{\sigma}) u_r]^*
\quad = \sigma_2 \exp(\vec{z}^* \cdot \vec{\sigma}^*) u_r^*
\quad = \exp(-\vec{z}^* \cdot \vec{\sigma}) u_\ell^c.
\end{align*} \tag{312}$$

because

$$\sigma_2 \vec{z}^* \cdot \vec{\sigma}^* = -\vec{z}^* \cdot \vec{\sigma} \sigma_2. \tag{315}$$

So they wrote all the fields as left-handed fields. They had 15 left-handed Fermi fields in each generation; at that time, only two generations were known. Since all 15 fields are left handed, we may drop the subscript $\ell$. They had $u_r$, $u_g$, $u_b$, $d_r$, $d_g$, $d_b$, $e$, $\nu = \nu_e$, $u_r^c$, $u_g^c$, $u_b^c$, $d_r^c$, $d_g^c$, $d_b^c$, and $e^c$. They left out $\nu^c$ which is a right-handed neutrino and took the neutrinos to be massless. (The physics community had not yet accepted Ray Davis’s late-1960s discovery of neutrino oscillations.)

Georgi and Glashow put the 15 left-handed quark and lepton fields into the $5^*$

$$5^* = \begin{pmatrix} d_r^c \\ d_g^c \\ d_b^c \\ e \\ \nu \end{pmatrix} \tag{316}$$
and a 10

$$10 = \begin{pmatrix}
0 & u_b^c & -u_g^c & -u_r & -d_r \\
-u_b^c & 0 & u_r^c & -u_g & -d_g \\
-u_g^c & -u_r^c & 0 & -u_b & -d_b \\
u_r & u_g & u_b & 0 & -e^c \\
d_r & d_g & u_b & e^c & 0
\end{pmatrix}$$

(317)

and introduced 13 new gauge bosons $Y_r^\mu$, $Y_g^\mu$, $Y_b^\mu$, $Y_r^{e\mu}$, $Y_g^{e\mu}$, $Y_b^{e\mu}$, $X_r^\mu$, $X_g^\mu$, $X_b^\mu$, $X_r^{e\mu}$, $X_g^{e\mu}$, and $X_b^{e\mu}$; and $A^\mu$.

All the generators of $SU(5)$ are traceless matrices. Thus the diagonalized charge operator $Q$ is traceless.

In the $5^*$ representation, the sum of its diagonal elements must vanish:

$$q(d_r^c) + q(d_g^c) + q(d_b^c) + q(e) + q(\nu) = 0.$$  

(318)

The neutrino is neutral, and the charges of the antidown quarks are color independent. Thus

$$3q(d_r^c) = - q(e)$$

(319)

or $q(d_r^c) = |e|/3$.

But gauge theories with quarks and antiquarks, leptons and antileptons in the same multiplet have gauge bosons that mediate changes of quark and lepton number. Putting quarks and antiquarks into the same multiplet means that nucleons are unstable. The proton is a colorless $s$-state of $u_r$, $u_g$, and $u_b$. The processes $u_g + d_b \rightarrow Y_r^c$ and $Y_r^c \rightarrow u_r^c + e^c$ lead to proton decay:

$$p = u_r + u_g + d_b \rightarrow u_r + Y_r^c$$

(320)

$$u_r + Y_r^c \rightarrow u_r + u_r^c + e^c$$

(321)

$$u_r + u_r^c + e^c \rightarrow \pi^0 + e^+.$$  

(322)

Other processes lead to other modes of proton decay and to the decay of neutrons in otherwise stable nuclei. The lifetime $\tau_p$ of the proton is proportional to the fourth power of the mass of the $Y$

$$\tau_p \propto \frac{M_Y^4}{\alpha^2 m_p^5}.$$  

(323)
in which $\alpha$ is the fine structure constant of $SU(5)$. The lower bound on the lifetime of the proton due to this decay mode is $8.3 \times 10^{33}$ years. Putting charge-conjugated right-handed fields into the same multiplet as left-handed fields changes the focus of physics from accessible energies to the GUT scale or $M_Y > 10^{16}$ GeV. This seems premature.

Harald Fritzsch, Peter Minkowski, Howard Georgi, and Edward Witten put the left-handed fields of a single family into a 16-dimensional multiplet of $SO(10)$, which to save paper I represent as a row vector

$$16^T = (\nu^c \ e^c \ u_r \ d_r \ u_g \ d_g \ u_b \ d_b \ u_r^c \ u_g^c \ u_b^c \ d_r^c \ d_g^c \ d_b^c \ \nu \ e)$$

This theory of grand unification is more symmetrical and has room for a right-handed neutrino, which appears as $\nu^c$. This theory also produces proton decay unless the masses of the gauge bosons exceed about $10^{16}$ GeV. Unfortunately, $SO(10)$ does not seem to explain the charges as directly as $SU(5)$ because the sum of the charges of the particles of the 16 vanishes no matter what they are as long as $q + q^c = 0$. This may be why Georgi and Glashow opted for $SU(5)$, which Georgi discovered only a few hours after figuring out $SO(10)$. But if one fits the 16 particles of the 16 into $SU(5)$ multiplets, then one recovers the $SU(5)$ version of charge quantization.

The gauge group of the standard model is $SU_c(3) \otimes SU_\ell(2) \otimes U_Y(1)$ with three coupling constants $g_s$, $g$, and $g'$ These coupling constants have nothing to do with each other. Grand unification puts these three groups into a simple group with a single coupling constant $T^a$ that are related to one another by the structure constants $f_{abc}$

$$[T^a, T^b] = i f_{abc} T^c$$

which are real and totally antisymmmteric. A simple group $G$ is one that has no nontrivial invariant subgroup $S$; that is, if

$$g^{-1} sg = s' \in S \text{ for all } s \in S \text{ and all } g \in G$$

then either $S = G$ or $S$ consists of the identity element of $G$. The group of the standard model $SU_c(3) \otimes SU_\ell(2) \otimes U_Y(1)$ is not simple (or semi-simple). Its structure constants don't relate the $SU_c(3)$ generators to the $SU_\ell(2)$ generators or to the $U_Y(1)$ generator.