Vortices and Monopoles

Let's look for a kink in 2+1 dimensions. Its energy would be

\[ M = \int d^2 x \left[ \partial_i \phi^* \partial_i \phi + \lambda (\phi^* \phi - \nu^2)^2 \right]. \quad (a) \]

We need \( \phi(\vec{x}) \to \nu e^{i \vec{k} \cdot \vec{x}} \) as \( |\vec{x}| \to \infty \).

Now this is okay if \( \nu \) is a constant.

But if we want \( \phi(\vec{x}) = \nu \vec{x} \),

then \( \nabla \phi \propto 1/|\vec{x}| \mathbf{v} \)

so

\[ |\nabla \phi| \propto \frac{\nu}{r} \]

and

\[ M \geq \int d^2 r 2\pi |\nabla \phi|^{2} = \int 2\pi d\nu \nu^2 = \infty \]

has a logarithmic divergence as \( \nu \to \infty \).

(Also, as \( \nu \to 0 \), but we can fix that.)

The solution is to change the theory. We can make \( D \phi = \vec{E} \phi - \text{ie} \vec{A} \phi \)

small at spatial infinity.
We let

$$A_i \to -\frac{i}{e} \frac{1}{r} \phi \frac{d\phi}{r} \quad \text{as} \quad r \to 0$$

So if \( \phi \to \nu e^{i\theta} \)

$$A_i \to -\frac{1}{e} \frac{1}{\nu^2} \nu^2 \nu \frac{d\theta}{\nu^2} = \frac{2i\theta}{e}.$$

But then the flux is

$$\phi = \int A \cdot d\mathbf{x} = \int A_i d\mathbf{x}_i = \frac{2\pi}{e}$$

This sort of vortex appears as a flux tube in Type II superconductors, where

$$\Phi_0 = \frac{2\pi \hbar c}{e} = \frac{\hbar c}{e}$$

is the elemental unit of flux.

Cosmic strings may be of this form and may exist. May exist.

But in 3+1 dimensions, an ungauged vortex can form a loop, and such a vortex can have finite energy. So cosmic strings can occur even in the ungauged theory (1).
Homotopy Groups.

Spatial infinity is topologically a unit circle. The field configuration \( \Phi = e^{i\theta} \) also is a circle. So if \( \Phi \) is a map from

\[
S^1 \rightarrow S^1
\]

spatial circle

infinity quadrants.

Maps of the \( n \) sphere \( S^n \) into a manifold \( M \) are classified by the homotopy group

\[ \pi_n(M) \].

For \( n \geq 1 \), \( \pi_n(S^n) = \mathbb{Z} \) the group of integers. For instance the maps \( \Phi \)

\[ \pi_1(S^1) = \mathbb{Z} \]

are just \( \Phi_m(re^{i\theta}) = re^{im\theta} \).

The trick was \( \pi_0(S^0) = \mathbb{Z} \).
Magnetic Monopoles Again.

Go to 3+1 dimensions. Consider 3 real fields \( \Phi (x) \).

The mass now is

\[
M = \int d^3x \left[ -\frac{1}{2} (\nabla \Phi)^2 + \lambda (\Phi^2 - \nu^2)^2 \right]
\]

So \( |\Phi(x)| \rightarrow \nu \quad \text{as} \quad (r^2) \rightarrow \infty \).

The kinetic energy of the field

\[
\Phi^q = \nu \frac{x^q}{r} \quad \text{as} \quad r \rightarrow \infty
\]

will diverge unless we "gauge" the theory.

So we let \( D_i \Phi^q = \partial_i \Phi^q + e \varepsilon^{abc} A^b_i \Phi^c \)

The gauge field will make \( |D_i \Phi^q|^2 \) small if

\[
A^b_i \rightarrow \frac{1}{e} \varepsilon^{bij} \frac{x^j}{r^2} \quad (j \neq i).
\]

The tensor

\[
F_{\mu \nu} = \frac{\partial \Phi^q}{\partial x^\nu} - e \varepsilon^{abc} \frac{\Phi_{(\mu}^a}{(D_{\nu})^b} \frac{\Phi_{(\nu}^b}{(D_{\mu})^c}
\]

\[
|D \Phi|^2 \rightarrow \frac{1}{e} \|D \Phi\|^2
\]
is gauge invariant and plays the role of the electromagnetic field.

Now the mass of the field $\Phi, A$ is

$$M = \int d^3x \left[ \frac{1}{4} F_{ij}^2 + \frac{1}{2} (D_i \Phi)^2 + V(\Phi) \right].$$

One can show that

$$\frac{1}{4} F_{ij}^2 + \frac{1}{2} (D_i \Phi)^2 = \frac{1}{4} \left( \nabla_i \cdot \epsilon_{ijk} D_k \Phi \right)^2 + \frac{1}{2} \epsilon_{ijk} F_{ij} \cdot D_k \Phi$$

So

$$M > \int d^3x \left[ \frac{1}{4} \epsilon_{ijk} F_{ij} \cdot D_k \Phi + V(\Phi) \right].$$

But

$$\int d^3x \frac{1}{2} \epsilon_{ijk} F_{ij} \cdot D_k \Phi$$

$$= \int d^3x \frac{1}{2} \epsilon_{ijk} \delta_{ik} \left( \nabla_j \Phi \right)$$

$$= v \int d\vec{S} \cdot \vec{B} = 4\pi v \Phi$$

where $\Phi$ is the charge of the monopole.
If we really minimize $V(\theta)$, then we even can get

$$M \geq 4\pi\nu(\nu)$$

to be just $M = 4\pi\nu(\nu)$. 