Vortices and Monopoles

Let's look for a kink in 2+1 dimensions. Its energy would be

$$M = \int d^2x \left[\partial_i \varphi^* \partial_i \varphi + \lambda (\varphi^*\varphi - v^2)^2 \right]. \qquad (1)$$

We need $\vec{\varphi}(\vec{x}) \to v e^{i\theta}$ as $|\vec{x}| \to \infty$.

Now this is okay if θ is a constant.

But if we want $\vec{\varphi}(\vec{x}) = v \hat{x}$,

then $|d\vec{\varphi}|\,|\vec{x}| = v$

so

$$|\nabla \varphi| \sim \frac{v}{r}$$

and

$$M \geq \int r\,dr\,2\pi\, |\nabla\varphi|^2 = \int 2\pi \frac{dr}{r} v^2 = \infty$$

has a logarithmic divergence as $r \to \infty$. (Also, as $r \to 0$, but we can fix that.).

The solution is to change the theory. We can make $D\varphi = \vec{\nabla}\varphi - ie\vec{A}\varphi$ small at spatial infinity.

We let

$$A_i \xrightarrow[r\to\infty]{} -\frac{i}{e}\frac{1}{|\phi|^2}\phi^* \partial_i \phi$$

So if $\phi \to v e^{i\theta}$

$$A_i \to -\frac{i}{e}\frac{1}{v^2} v^2 \, i \partial_i \theta = \frac{\partial_i \theta}{e} .$$

But then the flux is

$$\phi = \int d^2x \cdot B = \oint A_i dx_i = \frac{2\pi}{e}$$

This sort of vortex appears as a flux tube in type II superconductors, where

$$\Phi_0 = \frac{2\pi \hbar c}{e} = \frac{hc}{e}$$

is the elemental unit of flux.

Cosmic strings may be of this form and may exist. May exist.

But in 3+1 dimensions, an ungauged vortex can form a loop, and such a vortex can have finite energy. So cosmic strings can occur even in the ungauged theory (1).

Homotopy Groups.

Spatial infinity is topologically a unit circle S^1!

The field configuration $\phi = e^{i\theta} v$ also is a circle. So ϕ is a map from

$$\underset{\substack{\text{spatial}\\\text{infinity}}}{S^1} \rightarrow \underset{\substack{\text{circle}\\\text{of radius } v.}}{S^1}$$

Maps of the n sphere S^n into a manifold M are classified by the homotopy group

$$\pi_n(M).$$

For $n \geq 1$, $\pi_n(S^n) = \mathbb{Z}$ the group of integers. For instance the maps of

$$\pi_1(S^1) = \mathbb{Z}$$

are just $\phi_n(re^{i\theta}) = re^{in\theta}$.

The kink was $\pi_0(S^0) = \mathbb{Z}_2$.

Magnetic Monopoles Again

Go to 3+1 dimensions. Consider 3 real fields $\vec{\phi}(x)$.

The mass now is

$$M = \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + \lambda(\vec{\phi}^2 - v^2)^2 \right]$$

So $|\vec{\phi}(\vec{r})| \to v$ as $|\vec{r}| \to \infty$.

The kinetic energy of the field

$$\phi^a = v\frac{x^a}{r} \qquad \text{as } r \to \infty$$

will diverge unless we "gauge" the theory.

So we let $D_i\phi^a = \partial_i\phi^a + e\,\epsilon^{abc}A_i^b\phi^c$

The gauge field will make $|D_i\phi^a|^2$ small if

$$A_i^b \longrightarrow \frac{1}{e}\epsilon^{bij}\frac{x^j}{r^2} \qquad (j \equiv j).$$

The tensor

$$F_{\mu\nu} \equiv \frac{F_{\mu\nu}^a\phi^a}{|\phi|} - \frac{\epsilon^{abc}\phi^a(D_\mu\phi)^b(D_\nu\phi)^c}{e|\phi|^3}$$

is gauge invariant and plays the role of the electromagnetic field.

Now the mass of the field $\vec{\phi}$, A is

$$M = \int d^3x \left[\frac{1}{4} F_{ij}^2 + \frac{1}{2}(D_i\vec{\phi})^2 + V(\phi) \right].$$

One can show that

$$\frac{1}{4} F_{ij}^2 + \frac{1}{2}(D_i\phi)^2 = \frac{1}{4}(\vec{F}_{ij} \pm \epsilon_{ijk} D_k\vec{\phi})^2$$
$$\mp \frac{1}{2} \epsilon_{ijk} \vec{F}_{ij} \cdot D_k\vec{\phi}$$

So

$$M \geq \left| \int d^3x \left[\mp \frac{1}{2} \epsilon_{ijk} \vec{F}_{ij} \cdot D_k\vec{\phi} + V(\phi) \right] \right.$$

But

$$\int d^3x \, \frac{1}{2} \epsilon_{ijk} \vec{F}_{ij} \cdot D_k\vec{\phi}$$
$$= \int d^3x \, \frac{1}{2} \epsilon_{ijk} \partial_k (\vec{F}_{ij} \cdot \vec{\phi})$$
$$= v \int dS \cdot B = 4\pi v g$$

where g is the charge of the monopole.

If we really minimize $V(\varphi)$,
then we even can get

$$M \geq 4\pi v(g)$$

to be just $M = 4\pi v(g)$.