The Renormalization Group in Continuum Field Theory

Let's recall that when we computed vacuum polarization

\[ g \rightarrow \frac{g}{\sqrt{m^2}} \]

in QED using dimensional regularization, we got

\[ \Pi(q^2) = \frac{e^2}{2\pi^2} \int_0^1 x(1-x) \ln \left( 1 + \frac{q^2 x(1-x)}{m^2} \right) \, dx \]

in which \( q^2 = q^2 - q_0^2 \) is the square of the photon's momentum.

Now \( \Pi(q^2) \) is dimensionless. So we'd expect from dimensional analysis that as \( q^2 \rightarrow \infty \)

\[ \Pi(q^2, m^2, e^2) = \Pi(1, m^2, e^2) \rightarrow \Pi(1, 0, e^2) \]

That is, \( \Pi \) should go to a constant as \( q^2 \rightarrow \infty \). This behavior is spoiled by the log term.

Such large logarithms are generic. They occur because we used a renormalization scheme at a fixed \( q^2 \), namely \( q^2 = 0 \).
More generally, a quantity $\Gamma$ of dimension $D$ should vary with $E$ as 

$$\Gamma(E, x, g, m) = E^D \Gamma(1, x, g, \frac{m}{E})$$

and so we'd expect 

$$\Gamma(E, x, g, m) \rightarrow E^D \Gamma(1, x, g, 0)$$

as $E \rightarrow \infty$. But large logs spoil this simple behavior if we renormalize at a fixed energy $E$.

So we try to renormalize at a sliding scale $\mu$. Our coupling constant now is $g_\mu = g(\mu)$. Now 

$$\Gamma(E, x, g_\mu, m, \mu) = E^D \Gamma(1, x, g_\mu, \frac{m}{E}, \frac{\mu}{E}).$$

And to compute $\Gamma$ at $E$, we use $m = E$, so 

$$\Gamma(E, x, g_\mu, m, \mu) = E^D \Gamma(1, x, g_\mu, \frac{m}{E}, 1).$$

The idea here is to have $g_E$ independent of $m$ for $E \gg m$. Then as $E \rightarrow \infty$ we may have 

$$\Gamma(E, x, g_\mu, m, m) \rightarrow E^D \Gamma(1, x, g_E, 0, 1),$$

apart from possible non-perturbative corrections.
We expect
\[ q_{m'} = G(q_m, m'/m, \frac{m}{m'}) \]

Then
\[ \frac{dq_{m'}}{dm'} = \frac{1}{m} \frac{dG(q_m, z, \frac{m}{m'})}{dz} \bigg|_{z = m'/m} \]

So
\[ m \frac{dq_{m'}}{dm'} = \frac{dG(q_m, z, \frac{m}{m'})}{dz} \bigg|_{z = m'/m} \]

And setting \( m' = m \), we have
\[ m \frac{dq_m}{dm} = \frac{dG(q_m, \frac{m}{m})}{d(\frac{m}{m})} = \frac{dG(q_m, z, \frac{m}{m})}{dz} \bigg|_{z = 1}. \]

Thus for \( m >> m \), we find
\[ m \frac{dq_m}{dm} = \beta(g_m, 0) \equiv \beta(g_m) \]

which is the Callan-Symanzik equation.
Integrating
\[ \int_0^E \frac{dq_g}{\beta(q)} = \int_0^E \frac{dm}{m} = \ln(E/m). \]
We need to choose $M$ so that for $\mu > M$ we can neglect $m/M$ and so that $\ln (M/m)$ is not so big as to prevent us from using perturbation theory to compute $g_M$ from $g_0$, the conventionally renormalized coupling constant.

Example: In the scalar theory with

$$\mathcal{L} = -\frac{1}{2} \partial_\tau \phi \partial_\tau \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{24} f \phi^4$$

the diagrams give

$$A = g - \frac{\theta^2}{32\pi^2} \int_0^1 dx \left\{ \ln \frac{\Lambda^2}{m^2 - s x (1-x)} + \ln \left( \frac{\Lambda^2}{m^2 - t x (1-x)} \right) - 3 \right\}$$

where $\Lambda$ is a UV cut-off and

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_2)^2$$

and $s + t + u = 4m^2$ when all $p_i$ are on mass shell $p_i^2 = -m^2$. 
In the conventional approach, we replace the bare $g$ with $g_R$ defined at some fixed scale, e.g.,
\[ g_R \equiv A(s=t=a=0) = g - \frac{3g^2}{32\pi^2} \left\{ \ln \frac{\Lambda^2}{m^2} - 1 \right\}. \]

Setting $t = \frac{3}{32\pi^2} \left\{ \ln \frac{\Lambda^2}{m^2} - 1 \right\}$, we have
\[ -t \frac{g^2}{4} + g - g_R = 0 \quad \text{or} \quad g = \frac{1}{2t} \pm \sqrt{\frac{1}{4t^2} - \frac{g_R}{t}}. \]

Anyway, $A$ then is
\[ A = g_R + \frac{g_R^2}{32\pi^2} \int_0^{\infty} dx \left\{ \ln \left( 1 - \frac{tx(1-x)}{m^2} \right) + \ln \left( 1 - \frac{tx(1-x)}{m^2} \right) + \ln \left( 1 - \frac{nu(1-nu)}{m^2} \right) + \ldots \right\}, \]

and for large $s,t,$ and $a$
\[ A \sim g_R + \frac{g_R^2}{32\pi^2} \left\{ \ln \left( -\frac{s}{m^2} \right) + \ln \left( -\frac{t}{m^2} \right) + \ln \left( -\frac{u}{m^2} \right) + b \right\}, \]
which has big logs as $s, -t, a, -u \to \infty$.

Instead, we'll define
\[ g_m \equiv A(s=t=a=U = -m^2). \]
That is,

\[ g_m = A \left( s^2 + t^2 = -\mu^2 \right) \]

\[ = g - \frac{3g^2}{32\pi^2} \int_0^1 dx \left\{ \ln \left( \frac{\Lambda^2}{m^2 + \mu^2 x(1-x)} \right) - \frac{1}{3} \right\} + O(g^3) \]

In terms of \( g_R = g - \frac{3g^2}{32\pi^2} \left\{ \ln \left( \frac{\Lambda^2}{m^2} \right) - \frac{1}{3} \right\} \), this is

\[ g_R = g_R + \frac{3g^2}{32\pi^2} \int_0^1 dx \ln \left( 1 + \frac{m^2 x(1-x)}{m^2} \right) + O(g^3) \]

but this works only if \( |g_R \ln (\Lambda/m)| \ll 1 \).

But in terms of \( g_m \), \( g_m' \) is better behaved:

\[ g_m = g - \frac{3g^2}{32\pi^2} \int_0^1 dx \left\{ \ln \Lambda^2 - \ln m^2 + \mu^2 x(1-x) - \frac{1}{3} \right\} \]

\[ g_m = g - \frac{3g^2}{32\pi^2} \int_0^1 dx \left\{ \ln \Lambda^2 - \ln m^2 + m^2 x(1-x) \right\} \]

so

\[ g_m' = g_m - \frac{3g^2}{32\pi^2} \int_0^1 dx \ln \left( \frac{m^2 + m^2 x(1-x)}{m^2 + \mu^2 x(1-x)} \right) \]

Here we replace \( g^2 \) with \( g_m^2 \) or \( g_m'^2 \) because the difference is of order \( g^3 \).
Now
\[ \beta(g_m, \frac{m}{m'}) = \frac{d g_m}{d m'} \bigg|_{m'=m} \]
\[ = \frac{3 g_m^2}{16 \pi^2} \int_0^1 dx \frac{m^2 x (1-x)}{m^2 + m'^2 x (1-x)} + O(g_m^3) \]

For \( m > m' \), this is
\[ \beta(g_m) = \frac{3 g_m^2}{16 \pi^2} + O(g_m^3) \]

To two-loop order, the beta function is
\[ \beta(g_m) = g_m \left( \frac{3 g_m}{16 \pi^2} - \frac{17}{3} \left( \frac{g_m}{16 \pi^2} \right)^2 \right) + \cdots \]

So to one-loop order
\[ \frac{\ln E}{E} = \int \frac{dx}{x} = \int \frac{g_m^2}{g_m} \frac{dg}{\beta} = \sqrt{\frac{\int \frac{d g}{g_m}}{\frac{3 g_m}{16 \pi^2}}} = \frac{16 \pi^2}{3} \left[ \frac{1}{g_m} - \frac{1}{g_E} \right] \]

which gives us after setting \( E = \mu \)
\[ g_E = g_m = \frac{g_m}{1 - \frac{3}{16 \pi^2} g_m \ln \left( \frac{M}{m} \right)} \]

(\( m \to \) big legs!)
We may relate $g_m$ to $g_R$ by using (6), at values of $m > m^2$, where

$$g_m \approx g_R + \frac{3g_R^3}{32\pi^2} \int_0^1 dx \ln \frac{m^2}{m_2^2} x(1-x)$$

$$\approx g_R + \frac{3g_R^3}{32\pi^2} \int_0^1 dx \left[ 2 \ln \frac{m}{m_2} + \ln x(1-x) \right]$$

$$\approx g_R + \frac{3g_R^3}{16\pi^2} \ln \frac{\rho}{m}.$$ 

Although $m > m_2$, we want $|g_R \ln \frac{m}{m_2}| < < 1$.

So, for such a $m$, say $m = M$, we have

$$g_m = g_R \quad \text{whence for } m = E$$

$$g_m = \frac{g_R}{1 - \frac{3}{16\pi^2} g_R \ln \frac{M}{M} \rho}.$$ 

So, the $\phi^4$ theory exhibits asymptotic slavery.

$$g_E = \frac{g_R}{1 - \frac{3}{16\pi^2} g_R \ln \frac{E}{M} \rho}.$$