The Propagator

$$-i\Delta_{\ell m}(x,y) = \langle 0| T(\psi_\ell(x)\,\psi_m^\dagger(y))|0\rangle$$

$$= \theta(x^0-y^0)\,[\psi_\ell^{(+)}(x), \psi_m^{(+)\dagger}(y)]_\mp$$

$$\pm\, \theta(y^0-x^0)\,[\psi_m^{(-)\dagger}(y), \psi_\ell^{(-)}(x)]_\mp$$

$$= -i\int \frac{d^4q}{(2\pi)^4}\,\frac{P_{\ell m}(q)\,e^{iq(x-y)}}{q^2+m^2-i\epsilon}$$

where

$$P_{\ell m}(q) = 2\sqrt{\vec{q}^{\,2}+m^2}\,\sum_s u_\ell(\vec{q},s)\,u_m^*(\vec{q},s)$$

is the propagator we used to do perturbation theory.

Let

$$G(q_1,q_2) = \int d^4x_1\, d^4x_2\, e^{-iq_ix_i}\,\langle 0| T(A_\ell(x_1)\,A_m(x_2))|0\rangle$$

If there's a particle that couples $A_\ell(x)$ to the vacuum, then near $q_1^2 = -m^2$

$$G(q_1,q_2) \simeq \frac{-2i\sqrt{\vec{q}_1^{\,2}+m^2}\,(2\pi)^3}{q_1^2+m^2-i\epsilon}\sum_s \langle 0| A_\ell(0)|\vec{q}_1,s\rangle$$

$$\times \int d^4x_2\, e^{-iq_2x_2}\,\langle \vec{q}_1,s| A_m(x_2)|0\rangle .$$

If $A_\ell(x_1)$ is a field that belongs to an irreducible representation of the Lorentz group, then one may show that *

$$\langle 0 | A_\ell(0) | \vec{q}_1, s\rangle = \frac{N}{(2\pi)^{3/2}} u_\ell(\vec{q}_1, s)$$

and so

$$\int d^4x_2 \, e^{-iq_2x_2} \langle \vec{q}_1, s | A_m(x_2) | 0\rangle$$

$$= \int d^4x_2 \, e^{-iq_2x_2} \langle q_1, s | e^{-iPx_2} A_m(0) e^{iPx_2} | 0\rangle$$

$$= \int d^4x_2 \, e^{-iq_2x_2 - iq_1x_2} \langle q_1, s | A_m(0) | 0\rangle$$

$$= (2\pi)^4 \delta^4(q_1 + q_2) \langle q_1, s | A_m(0) | 0\rangle$$

$$= \frac{N^*}{(2\pi)^{3/2}} u_m^*(q_1, s) \, (2\pi)^4 \delta^4(q_1 + q_2).$$

So near $q_1^2 = -m^2$, G looks like

$$G(q_1, q_2) \simeq \frac{-2i\sqrt{\vec{q}_1^{\,2} + m^2}\, |N|^2}{q_1^2 + m^2 - i\epsilon} \sum_s u_\ell(q_1, s) u_m^*(q_1, s) \times (2\pi)^4 \delta^4(q_1 + q_2)$$

$$= \frac{-i P_{\ell m}(q_1)}{q_1^2 + m^2 - i\epsilon} |N|^2 (2\pi)^4 \delta^4(q_1 + q_2). \qquad (G)$$

* For a scalar field $u_\ell(q_1, s) = \dfrac{1}{\sqrt{2 q_1^0}}$, $q^0 = \sqrt{\vec{q}_1^{\,2} + m^2}$.

The renormalized field has $N = 1$.

$$\langle 0 | \psi_\ell(0) | \vec{q}, s \rangle = \frac{u_\ell(\vec{q}, s)}{(2\pi)^{3/2}}$$

Its propagator is

$$-i\Delta_{\ell m}(q_1) = \frac{-i\, P_{\ell m}(q_1)}{q_1^2 + m^2 - i\epsilon}$$

which is Eq. (6) with $N = 1$ (apart from the factor $(2\pi)^4\,\delta(q_1 + q_2)$).

So we want the renormalized propagator to have a pole at $q^2 = -m^2$ with unit residue — apart from $P_{\ell m}(q)$.

Scalar case

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_b\,\partial^\mu\phi_b - \frac{1}{2}m_b^2\,\phi_b^2 - V_b(\phi_b)$$

In general $\langle 0 | \phi_b(0) | q \rangle = \dfrac{N}{(2\pi)^{3/2}}\, u(q)$

$$= \frac{N}{\sqrt{(2\pi)^3\, 2q^0}} \equiv \frac{\sqrt{Z}}{\sqrt{(2\pi)^3\, 2q^0}}$$

So $\phi_b = \sqrt{Z}\,\phi$ makes the renormalized field satisfy $\langle 0 | \phi(0) | q \rangle = \dfrac{1}{\sqrt{(2\pi)^3\, 2q^0}}$.

And as before, we set

$$m_b^2 = m^2 - \delta m^2$$

so that

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\mathcal{L}_1 = -\frac{1}{2}(Z-1)\left[\partial_\mu\phi\,\partial^\mu\phi + m^2\phi^2\right] + \frac{1}{2}Z\,\delta m^2\phi^2 - V_b(\sqrt{Z}\,\phi).$$

The sum of all graphs that can't be disconnected by cutting a single line — one-particle-irreducible graphs — is $i(2\pi)^4\,\Pi^*(q^2)$, apart from two external-line propagators

 is one of the 1pi graphs.

If $\Delta = \dfrac{1}{q^2+m^2-i\epsilon}$, then

$$\Delta'(q) = \Delta + \Delta\Pi^*\Delta + \Delta(\Pi^*\Delta)^2 + \cdots$$

$$= \Delta\sum_{n=0}^{\infty}(\Pi^*\Delta)^n = \frac{\Delta}{1-\Pi^*\Delta}$$

That is,

$$\Delta'(q) = \frac{1}{q^2+m^2-i\epsilon} \quad \frac{1}{1-\dfrac{\Pi^*(q^2)}{q^2+m^2-i\epsilon}}$$

$$= \frac{1}{q^2+m^2-\Pi^*(q^2)-i\epsilon}\,.$$

So we want $\Pi^*(q^2) = \sum_{n=2}^{\infty} p_n (q^2)^n$

so the pole stays at $q^2=-m^2$ with unit residue.

Now first- and second-order perturbation theory gives

$$\Pi^*(q^2) = \underbrace{-(Z-1)(q^2+m^2)+Z\,\delta m^2}_{\text{first order}} + \underbrace{\Pi^*_{loop}(q^2)}_{\text{second order}}$$

We want $\Pi^*(-m^2) = 0 = -(Z-1)(0) + Z\,\delta m^2 + \Pi^*_{loop}(-m^2)$
so we set

$$Z\,\delta m^2 = -\Pi^*_{loop}(-m^2)$$

and we also want

$$\left.\frac{d\Pi^*}{dq^2}\right|_{q^2=-m^2} = 0$$

so we set $\quad Z = 1 + \left.\dfrac{d\Pi^*_{loop}(q^2)}{dq^2}\right|_{q^2=-m^2}$.

So both $Z-1$ and $Z\delta m^2$ are of second and higher orders in the coupling constant. In

$\lambda\phi^4$ theory, for example,

k  k $$\propto \lambda^2 \int \frac{d^4p\, d^4q}{(p^2+m^2-i\epsilon)(q^2+m^2-i\epsilon)}\,\frac{1}{((k-p-q)^2+m^2-i\epsilon)}$$

Note that external lines that represent particles on their mass shells $q^2+m^2=0$ have no radiative corrections.

Dirac case

$$\mathcal{L} = -\bar\psi_b(\not\partial + m_b)\psi_b - V_b(\bar\psi_b\psi_b)$$

$$\psi_b = \sqrt{Z_2}\,\psi$$

$$m_b = m - \delta m$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \qquad \mathcal{L}_0 = -\bar\psi(\not\partial + m)\psi$$

$$\mathcal{L}_1 = -(Z_2-1)\left(\bar\psi(\not\partial+m)\psi\right) + Z_2\delta m\,\bar\psi\psi - V_b(\bar\psi\psi Z_2)$$

$$\Sigma^*(\not k) = -(Z_2-1)(i\not k + m) + Z_2\delta m + \Sigma^*_{loop}(\not k)$$

renormalized propagator $S'(k) = \dfrac{1}{i\not k + m - \Sigma^*(\not k) - i\epsilon}$

1pi self-energy of fermion

$$\Sigma^*(im) = 0 = \frac{\partial \Sigma^*(k\!\!\!/)}{\partial k\!\!\!/}\bigg|_{k\!\!\!/=im} \quad \text{give}$$

$$Z_2 \delta m = - \Sigma^*_{loop}(im)$$

$$Z_2 = 1 - i \, \frac{\partial \Sigma^*_{loop}(k\!\!\!/)}{\partial k\!\!\!/}\bigg|_{k\!\!\!/=im}.$$

We've seen that $\partial_\mu J^\mu = 0$ or current conservation implies that

$$p_\mu M^{\mu\nu}(p,p') = \int d^4x \, d^4x' \, p_\mu \langle 0| T(J^\mu(x) J^\nu(x'))|0\rangle e^{-ixp-ix'p'}$$

$$= -i \int d^4x \, d^4x' \, e^{-ixp-ix'p'} \frac{\partial}{\partial x^\mu} \langle 0| T(J^\mu(x) J^\nu(x'))|0\rangle$$

$$= 0.$$

Thus terms proportional to p^μ or p^ν in the photon propagator $\Delta_{\mu\nu}$ don't matter.

The current $J^\mu = \frac{\partial \mathcal{L}_m}{\partial A_\mu}$ where

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu b} F_b^{\mu\nu} + \mathcal{L}_m(\psi, \gamma^\mu(\partial_\mu - i q_b A_{\mu b})\psi)$$

$A_b^\mu = \sqrt{Z_3}\, A^\mu$ so we set $q_b = q/\sqrt{Z_3}$ so that

$$J^\mu = +iq\, \bar\psi \gamma^\mu \psi.$$

Γ^μ is defined by

$$-i(2\pi)^4 q\, S'_{nn'}(k)\, \Gamma^\mu_{n'm'}(k,\ell)\, S'_{m'm}(\ell)\, \delta^4(p+k-\ell)$$

$$\equiv \int d^4x\, d^4y\, d^4z\, e^{-ipx-iky+i\ell z} \langle 0|T(J^\mu(x)\psi_n(y)\bar\psi_m(z))|0\rangle$$

where

$$-i(2\pi)^4 S'_{nm}(k)\,\delta^4(k-\ell) \equiv \int d^4y\, d^4z \langle 0|T(\psi_n(y)\bar\psi_m(z))|0\rangle \times e^{-iky+i\ell z}.$$

With no interactions at all

$$S'(k) \to \frac{1}{i\not{k}+m-i\epsilon} \quad \text{and} \quad \Gamma^\mu(k,\ell) \to \gamma^\mu.$$

We derived in class the Ward-Takahashi identity

$$(\ell-k)_\mu\, S'(k)\Gamma^\mu(k,\ell)S'(\ell) = iS'(\ell) - iS'(k)$$

or

$$(\ell-k)_\mu\, \Gamma^\mu(k,\ell) = iS'^{-1}(k) - iS'^{-1}(\ell).$$

As $\ell \to k$, we get

$$\Gamma^\mu(k,k) = -i\frac{\partial}{\partial k_\mu} S'^{-1}(k).$$

We've seen that

$$S'^{-1}(k) = i\not{k} + m - \Sigma^*(\not{k})$$

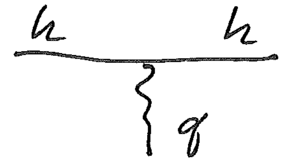
So $\Gamma^\mu(k,k) = \gamma^\mu + i \frac{\partial}{\partial k_\mu} \Sigma^*(\not{k})$

We've also seen that for $k^2 = -m^2$

$$\Sigma^*(im) = \frac{\partial \Sigma^*}{\partial \not{k}}\Big|_{\not{k}=im} = 0$$

So for a renormalized Dirac field on mass shell
$(i\not{k}+m)u_k = 0$, $(i\not{k}+m)u_{k'} = 0$

$$\bar{u}'_k \Gamma^\mu(k,k) u_k = \bar{u}'_k \gamma^\mu u_k .$$

Radiative corrections vanish 

when $k^2 = -m^2$ which means that $q \to 0$.