The Photon Propagator in Coulomb's Gauge

In the radiation gauge, a.k.a. Coulomb's gauge, the field of the photon is

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2p^0}} \sum_{\substack{s=-1\\ \neq 0}}^{1} \left[e^{-ipx} e^\mu(p,s)\, a(p,s) + e^{ipx} e^{\mu *}(p,s)\, a^\dagger(p,s) \right]$$

where $\vec{p} \cdot \vec{e}(p,s) = 0$ enforces the Coulomb gauge choice

$$\vec{\nabla} \cdot \vec{A} = 0,$$

and $e^0(p,s) = 0$ implies

$$A^0(x) = 0$$

because $A^0(x)$ has been replaced in the theory by

$$A^0(t,\vec{x}) = \frac{1}{4\pi} \int \frac{\rho(\vec{y},t)\, d^3y}{|\vec{x} - \vec{y}|} = \frac{1}{4\pi} \int \frac{j^0(\vec{y},t)\, d^3y}{|\vec{x} - \vec{y}|}$$

in which $y^0 = t = x^0$.

We recall the spin sum

$$\sum_{\substack{s=-1\\ \neq 0}}^{1} e^i(\hat{p},s)\, e^{j*}(\hat{p},s) = \delta^{ij} - \hat{p}^i \hat{p}^j$$

$$= \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}$$

$$= \delta^{ij} - \frac{p^i p^j}{\vec{p}^2}.$$

The mean value in the vacuum of the time-ordered product of $A^\mu(x)$ with $A^\nu(y)$ is

$$\langle 0| T\left(A^\mu(x) A^\nu(y)\right)|0\rangle = \langle 0|\theta(x^0-y^0)A^\mu(x)A^\nu(y) + \theta(y^0-x^0)A^\nu(y)A^\mu(x)|0\rangle$$

$$= \langle 0|\theta(x^0-y^0)A^{\mu(+)}(x)A^{\nu(-)}(y) + \theta(y^0-x^0)A^{\nu(+)}(y)\,A^{\mu(-)}(x)\,|0\rangle$$

$$= \langle 0|\theta(x^0-y^0)[A^{\mu(+)}(x), A^{\nu(-)}(y)] + \theta(y^0-x^0)[A^{\nu(+)}(y), A^{\mu(-)}(x)]\,|0\rangle$$

$$= \theta(x^0-y^0)\,[A^{\mu(+)}(x), A^{\nu(-)}(y)] + \theta(y^0-x^0)\,[A^{\nu(+)}(y), A^{\mu(-)}(x)].$$

The first commutator is

$$[A^{\mu(+)}(x), A^{\nu(-)}(y)] = \int \frac{d^3p\, d^3q}{(2\pi)^6\sqrt{2p^0 2q^0}} \sum_{\substack{s,t=-1\\ \neq 0}}^{1} [a(p,s), a^\dagger(q,t)]\, e^{-ipx+iqy}\, e^\mu(p,s) e^{\nu *}(q,t)$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6\sqrt{2p^0 2q^0}} \sum_{\substack{s,t=-1\\ \neq 0}}^{1} (2\pi)^3 \delta_{st}\, \delta^{(3)}(\vec{p}-\vec{q})\, e^{-ipx+iqy}\, e^\mu(p,s) e^{\nu *}(q,t)$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0} \sum_{\substack{s=-1\\ \neq 0}}^{1} e^\mu(p,s) e^{\nu *}(p,s)\, e^{-ip(x-y)}.$$

So

$$[A^{i(+)}(x), A^{j(-)}(y)] = \int \frac{d^3p}{(2\pi)^3 2p^0}\left(\delta^{ij} - \frac{p^i p^j}{\vec{p}^{\,2}}\right) e^{-ip(x-y)}$$

and

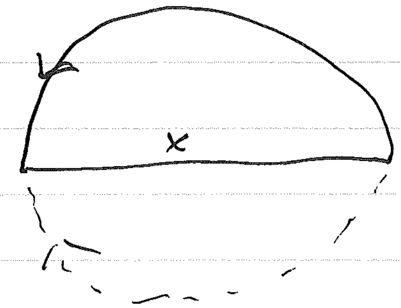
$$[A^{\mu(+)}(x), A^{0(-)}(y)] = [A^{0(+)}(x), A^{\nu(-)}(y)] = 0.$$

Similarly $x \leftrightarrow y$ and $i \leftrightarrow j$ give

$$[A^{j(+)}(y), A^{i(-)}(x)] = \int \frac{d^3p}{(2\pi)^3 2p^0} \left(\delta^{ij} - \frac{p^i p^j}{\vec{p}^{\,2}} \right) e^{ip(x-y)} .$$

Now we write

$$\theta(a) = \frac{1}{2\pi i} \int \frac{e^{isa}\, ds}{s - i\epsilon}$$



if $a > 0$

if $a < 0$

so the time part of $[A^{i(+)}(x), A^{i(-)}(y)]$ is

$$\frac{1}{2\pi i} \int ds \frac{e^{-ip^0(x^0-y^0) + is(x^0-y^0)}}{s - i\epsilon} = +\int \frac{dq^0}{2\pi i} \frac{e^{-iq^0(x^0-y^0)}}{p^0 - q^0 - i\epsilon}$$

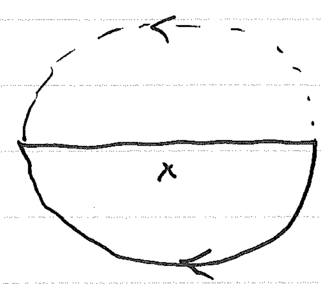
where

We set $q^0 = p^0 - s$ so $s = p^0 - q^0$. We now switch from $\vec{p}$ to $\vec{q}$ keeping $p^0 = E_p = E_q = \sqrt{\vec{q}^{\,2}}$ and get

$$\theta(x^0 - y^0)[A^{i(+)}(x), A^{j(-)}(y)] = -\frac{1}{i} \int \frac{d^4q}{(2\pi)^4} \frac{1}{2E_q} \left(\delta^{ij} - \frac{q^i q^j}{\vec{q}^{\,2}} \right) \frac{e^{-iq(x-y)}}{q^0 - E_q + i\epsilon} .$$

Similarly, we use

$$\theta(-a) = -\int \frac{ds}{2\pi i} \frac{e^{isa}}{s + i\epsilon}$$



$a > 0$

$a < 0$

to write the time integral in $[A^{j(+)}(y), A^{i(-)}(x)]$ as

$$-\int \frac{ds}{2\pi i} \frac{e^{ip^0(x^0-y^0)+is(x^0-y^0)}}{s+i\epsilon} = -\int \frac{dq^0}{2\pi i} \frac{e^{-iq^0(x^0-y^0)}}{-p^0-q^0+i\epsilon}$$

where we set $-q^0 = p^0 + s$ so that $s = -p^0 - q^0$. Keeping $p^0 = E_q = \sqrt{\vec{q}^{\,2}}$ and using $\vec{q}$ for $\vec{p}$, we get

$$\theta(y^0-x^0)\,[A^{j(+)}(y), A^{i(-)}(x)] = \frac{-1}{i}\int \frac{d^4q}{(2\pi)^4}\frac{1}{2E_q}\left(\delta^{ij} - \frac{q^iq^j}{\vec{q}^{\,2}}\right)\frac{e^{-iq^0(x^0-y^0)-i\vec{q}\cdot(\vec{x}-\vec{y})}}{-E_q-q^0+i\epsilon}.$$

Adding the two terms

$$\langle 0|T(A^i(x)A^j(y))|0\rangle = \frac{-1}{i}\int\frac{d^4q}{(2\pi)^4}\frac{1}{2E_q}\left(\delta^{ij} - \frac{q^iq^j}{\vec{q}^{\,2}}\right)$$

$$\times\left[\frac{e^{-iq(x-y)}}{q^0-E_q+i\epsilon} + \frac{e^{-iq^0(x^0-y^0)-i\vec{q}\cdot(\vec{x}-\vec{y})}}{-E_q-q^0+i\epsilon}\right].$$

In the second integral, all terms apart from $\vec{q}\cdot(\vec{x}-\vec{y})$ are invariant under $\vec{q}\to-\vec{q}$, so changing $\vec{q}$ to $-\vec{q}$, we get

$$\langle 0|T(A^i(x)A^j(y))|0\rangle = \frac{-1}{i}\int\frac{d^4q}{(2\pi)^4}\frac{1}{2E_q}\left(\delta^{ij} - \frac{q^iq^j}{\vec{q}^{\,2}}\right)$$

$$\times\, e^{-iq(x-y)}\left[\frac{1}{q^0-E_q+i\epsilon} + \frac{1}{-E_q-q^0+i\epsilon}\right]$$

The [ ] term is

$$\frac{1}{q^0 - E_q + i\epsilon} + \frac{1}{-q^0 - E_q + i\epsilon} = \frac{-E_q - q^0 + i\epsilon + q^0 - E_q + i\epsilon}{-q^{0\,2} + E_q^2 + i\epsilon(q^0 - q^0 - 2E_q)}$$

$$= \frac{-2E_q}{-q^{0\,2} + \vec{q}^{\,2} - 2E_q i\epsilon} = \frac{2E_q}{q^2 + i\epsilon'}$$

in which $\epsilon' = 2E_q\epsilon$ is a small positive number. We'll drop the prime since its actual magnitude is immaterial and infinitesimal. So

$$\langle 0|T(A^i(x)A^j(y))|0\rangle = \frac{-1}{i}\int \frac{d^4q}{(2\pi)^4}\left(\delta^{ij} - \frac{q^i q^j}{\vec{q}^{\,2}}\right)\frac{e^{-iq(x-y)}}{q^2 + i\epsilon}$$

$$= \int \frac{d^4q}{(2\pi)^4}\left(\delta^{ij} - \frac{q^i q^j}{\vec{q}^{\,2}}\right)\frac{i}{q^2 + i\epsilon}\, e^{-iq(x-y)}.$$

One may check that for $e^\mu = (1,0,0,0)$

$$-\eta^{\mu\nu} + \frac{q^0 q^\mu e^\nu + q^0 q^\nu e^\mu - q^\mu q^\nu - q^2 e^\mu e^\nu}{\vec{q}^{\,2}}$$

vanishes when μ or ν or both are zero and that

$$-\eta^{ij} + \frac{q^0 q^i e^j + q^0 q^j e^i - q^i q^j - q^2 e^i e^j}{\vec{q}^{\,2}} = \delta^{ij} - \frac{q^i q^j}{\vec{q}^{\,2}}.$$

So

$$\langle 0|T(A^\mu(x)A^\nu(y))|0\rangle =$$

$$\int \frac{d^4q}{(2\pi)^4}\left(-\eta^{\mu\nu} + \frac{q^0 q^\mu e^\nu + q^0 q^\nu e^\mu - q^\mu q^\nu - q^2 e^\mu e^\nu}{\vec{q}^{\,2}}\right)\frac{i}{q^2+i\epsilon}e^{-iq(x-y)}.$$

This formula holds whatever q^0 is. So we may set q^0 in ( ) to be the q^0 of d^4q. Then the q^μ and q^ν terms will couple to conserved currents in Feynman diagrams

$$\int d^4q\,d^4x\,d^4y\, e^{-iq(x-y)} q^\mu j_\mu(x) = \int d^4q\,d^4x\,d^4y\, i\partial^\mu e^{-iq(x-y)} j_\mu(x)$$

$$= -\int d^4q\,d^4x\,d^4y\, e^{iq(x-y)}\partial^\mu j_\mu(x) = 0.$$

So the terms $q^0 q^\mu e^\nu$, $q^0 q^\nu e^\mu$, and $-q^\mu q^\nu$ don't contribute when A^μ is coupled to a conserved current. The other term is

$$-\int \frac{d^4q}{(2\pi)^4}\frac{q^2\, i}{q^2+i\epsilon}\frac{1}{\vec{q}^{\,2}}e^{-iq(x-y)}$$

$$= -i\int \frac{d^4q}{(2\pi)^4}\frac{1}{\vec{q}^{\,2}}e^{-iq(x-y)}$$

$$= -i\int \frac{d^3q}{(2\pi)^3}\frac{1}{\vec{q}^{\,2}}e^{i\vec{q}\cdot(\vec{x}-\vec{y})}\delta(x^0-y^0)$$

But this is multiplied by $j^\mu j^\nu$ for $\mu=\nu=0$ so

$$-i\frac{1}{2}\int d^4x\int d^4y\,[-ij^0(x)][-ij^0(y)]\frac{-i}{(2\pi)^4}\int\frac{d^4q}{\vec{q}^{\,2}}e^{-iq(x-y)}$$

$$=-\frac{1}{2}\int d^3x\,d^3y\,\frac{j^0(x,t)\,j^0(y,t)}{4\pi|\vec{x}-\vec{y}|}$$

which cancels the Coulomb interaction

$$V_c(t)=\frac{1}{2}\int d^3x\,d^3y\,\frac{j^0(x,t)\,j^0(y,t)}{4\pi|x-y|}.$$

The upshot is that we can drop this $V_c(t)$ if we simplify the photon propagator to

$$\langle 0|T(A^\mu(x)A^\nu(y))|0\rangle=\int\frac{d^4q}{(2\pi)^4}\frac{-i\eta^{\mu\nu}}{q^2+i\epsilon}e^{-iq(x-y)}.$$