Massless particles and fields:

We begin with the definition

$$|p,s\rangle \equiv U(L(p))\,|k,s\rangle$$

where $k = (k^0, 0, 0, k) = k(1,0,0,1)$ is a massless fiducial momentum $k^2 = k^{0\,2} - \vec{k}^{\,2} = 0$.

Now $L(p)$ is a boost in the z-direction followed by a rotation that takes $\hat{z}$ to $\hat{p}$

$$L(p) = R(\hat{p})\,B(p) = e^{-i\phi J_3}\, e^{-i\theta J_2}\, B(p).$$

As with massive particles

$$U(\Lambda)\,|p,s\rangle = U(L(\Lambda p))\,U(L^{-1}(\Lambda p)\,\Lambda L(p))\,|k,s\rangle$$

where $L^{-1}(\Lambda p)\,\Lambda\, L(p)\, k \equiv Wk = k$ but now the group of such transformations — the little group — is no longer the rotation group, because it must leave the vector $k = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ invariant.

If $W = I + \omega$, then $\omega k = 0$.

The most general combination of B's and R's that send k to 0 is of the form

$$\omega\, k \equiv \begin{pmatrix} 0 & a & b & 0 \\ a & 0 & -\theta & -a \\ b & \theta & 0 & -b \\ 0 & a & b & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

That is, ω is of the form

$$\omega = a B_1 + b B_2 - a R_2 + b R_1 + \theta R_3$$

$$= -i\left(a i B_1 + b i B_2 - a i R_2 + b i R_1 + \theta i R_3\right)$$

$$= -i\left(a K_1 + b K_2 - a J_2 + b J_1 + \theta J_3\right)$$

$$= -i\left[a(K_1 - J_2) + b(K_2 + J_1) + \theta J_3\right].$$

We set $T_1 = J_2 - K_1$ and $T_2 = -J_1 - K_2$ and find that these generators commute

$$[T_1, T_2] = 0 = [J_2 - K_1, -J_1 - K_2] = +[J_1, J_2] + [K_1, K_2]$$
$$= i J_3 - i J_3 = 0$$

and rotate into each other under J_3

$$[J_3, T_1] = i T_2 = [J_3, J_2 - K_1] = -i J_1 - i K_2$$

$$[J_3, T_2] = -i T_1 = [J_3, -J_1 - K_2] = -i J_2 + i K_1,$$

This little group is $ISO(2)$ which includes the translations T_1 & T_2 and the rotations. The translations form an invariant abelian subalgebra $\left([J_3, aT_1 + bT_2] = cT_1 + dT_2 \right)$, and so the group $ISO(2)$ is neither simple nor semi-simple.

Any $W \in ISO(2)$ is a Lorentz transformation of the form

$$W = L^{-1}(\Lambda p) \Lambda L(p) \equiv SR \equiv e^{-ia(K_1 - J_2) - ib(K_2 - J_2)} e^{-i\theta J_3}.$$

As far as we know, the states $|p,s\rangle$ of massless particles are eigenstates of T_1 & T_2 with eigenvalue zero. So

$$U(W)|k,s\rangle = e^{-ia(K_1 - J_2) - ib(K_2 - J_2)} e^{-is\theta} |k,s\rangle$$

$$= e^{-is\theta} |k,s\rangle. \qquad \text{So}$$

$$U(\Lambda)|p,s\rangle = U(L(\Lambda p)) e^{-is\theta} |k,s\rangle = e^{-is\theta} |\Lambda p, s\rangle$$

where the angle $\theta = \theta(\Lambda, p)$. (Weinberg uses $-\theta$ instead of θ.)

So

$$U(\Lambda) a(p,s) U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}}\, e^{+is\theta(p,\Lambda)} a(\Lambda p, s)$$

$$U(\Lambda) a^{\dagger}(p,s) U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}}\, e^{-is\theta(p,\Lambda)} a^{\dagger}(\Lambda p, s).$$

As for massive particles

$$A^{\mu(+)}(x) = \sum_s \int d^3p \; u^\mu(p,s)\, a(p,s)\, e^{-ipx} \quad \text{and}$$

$$U(\Lambda) A^{\mu(+)}(x) U^{-1}(\Lambda) = \sum_s \int d^3p \; u^\mu(p,s) \sqrt{\frac{(\Lambda p)^0}{p^0}}\, e^{+is\theta(p,\Lambda)}\, a(\Lambda p,s)\, e^{-i\Lambda p \Lambda x} .$$

We want

$$U(\Lambda) A^{\mu(\pm)}(x) U^{-1}(\Lambda) = D^\mu{}_\nu(\Lambda^{-1}) A^{\nu(\pm)}(\Lambda x) + R^\mu$$

where R^μ is a remainder. So instead of (3) and (4) of the notes on Wigner rotations, we get

(3′) $$u^\mu(\Lambda p,s) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\nu D^\mu{}_\nu(\Lambda)\, e^{+is\theta(p,\Lambda)}\, u^\nu(p,s) + r_u^\mu \quad \text{and}$$

(4′) $$v^\mu(\Lambda p,s) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\nu D^\mu{}_\nu(\Lambda)\, e^{-is\theta(p,\Lambda)}\, v^\nu(p,s) + r_v^\mu ,$$

which are SW's (5.9.6 & 7) with remainders, and $-\theta$ for θ. Set $p = k = (k,0,0,k)$ and $\Lambda = L(q)$. Then $\theta = 0$ and we get

$$u^\mu(q,s) = \sqrt{\frac{k^0}{q^0}}\, D^\mu{}_\nu(L(q))\, u^\nu(k,s) + r_u^\mu$$

$$v^\mu(q,s) = \sqrt{\frac{k^0}{q^0}}\, D^\mu{}_\nu(L(q))\, v^\nu(k,s) + r_v^\mu .$$

Here $L(q)k = R(\hat{q})B(q^0)k = R(\hat{q})\begin{pmatrix} q^0 \\ 0 \\ 0 \\ q^0 \end{pmatrix} = q$.
So here $r^\mu = 0$.

But Now set $\Lambda = W = SR$ and $p = k$ in (3′-4′):
Then like SW's (5.9.10-11), we get

$$(3'')\quad u^\mu(k,s)\, e^{-is\theta(k,w)} = D^\mu{}_\nu(SR)\, u_\nu(k,s) + \tilde{r}_u{}^\mu$$

$$= (SR)^\mu{}_\nu\, u_\nu(k,s) + \tilde{r}_u{}^\mu \quad \text{and}$$

$$(4'')\quad v^\mu(k,s)\, e^{+is\theta(k,w)} = (SR)^\mu{}_\nu\, v^\nu(k,s) + \tilde{r}_v{}^\mu.$$

Now we saw that in the Coulomb gauge

$$u^\mu(p,s) = \frac{1}{\sqrt{2p^0}}\, e^\mu(p,s)$$

But $Be = e$

$$e^\mu(p,s) = R(\hat{p})^\mu{}_\nu\, B^\nu(p)_\sigma\, e^\sigma(k,s)$$

so

$$= R(\hat{p})^\mu{}_\nu\, e^\nu(k,s)$$

where

$$e(k,s) = e(k,\pm 1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \quad \text{for } \mu = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases}$$

In (3″) R is

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $c = \cos\theta$ and $s = \sin\theta$. so

$$R\, e(k,\pm 1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ e^{\mp i\theta} \\ \pm i e^{\mp i\theta} \\ 0 \end{pmatrix} = e^{\mp i\theta}\, e(k,\pm 1)$$

And S is the matrix

$$S = \begin{pmatrix} 1 & a & b & 0 \\ a & 1 & 0 & -a \\ b & 0 & 1 & -b \\ 0 & a & b & 1 \end{pmatrix}$$

so

$$Se = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & a & b & 0 \\ a & 1 & 0 & -a \\ b & 0 & 1 & -b \\ 0 & a & b & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} a \pm ib \\ 1 \\ \pm i \\ a \pm ib \end{pmatrix} = e + \frac{a \pm ib}{\sqrt{2}} \hat{k}$$

where $\hat{k} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is the unit fiducial vector. So

$$(SR)^{\mu}{}_{\nu}\, e^{\nu}(k, \pm 1) = e^{\mp i\theta} \left[e^{\mu}(k, \pm 1) + \frac{a \pm ib}{\sqrt{2}} \frac{k^{\mu}}{k^{0}} \right],$$

which is a gauge transformation.

More generally, equations (3') and (4') tell us that if $\Lambda = S$, then

$$u^\mu(Sp, s) = \sqrt{\frac{p^0}{(Sp)^0}}\; S^\mu{}_\nu\, e^{is\theta}\, u^\nu(p,s) + r_u{}^\mu$$

$S^\mu{}_\nu u^\nu$ is simplest if $p = (p^0, 0, 0, p^0)$. Then

$$S^\mu{}_\nu u^\nu = \frac{1}{\sqrt{2p^0}}\, S^\mu{}_\nu e^\nu$$

$$= \frac{1}{\sqrt{2p^0}}\left[e^\mu + \frac{a \pm ib}{\sqrt{2}}\frac{h^\mu}{h^0}\right]$$

$$= u^\mu(p,s) + \frac{a \pm ib}{\sqrt{2}}\frac{h^\mu}{h^0\sqrt{2p^0}}$$

$$= u^\mu(p,s) + \frac{a \pm ib}{\sqrt{2}}\frac{p^\mu}{p^0\sqrt{2p^0}},$$

which again looks like a gauge transformation since

$$\partial_\mu \theta(x) = \partial_\mu \int e^{ixp}\,\theta(p)\, d^3p$$

$$= \int p^\mu e^{ixp}\,\theta(p)\, d^3p.$$