The action of a plaquette (little square) is made from the trace of the path-ordered product along the links around the plaquette of group elements $\exp(-igaA)$ in which $A$ is a linear combination of the $n \times n$ generators $t_a$ of the gauge group multiplied by the fields $A_a^i$ in the direction of the link, $A_i = t_a A_a^i$. If the center of the plaquette is $x$, then the action for a plaquette in the 1-2 plane is

$$S_\Box = \beta \left\{ 1 - (1/n) \mathrm{Re} \mathrm{Tr} \left[ \exp(-igaA_1(x - aj/2)) \exp(-igaA_2(x + ai/2)) \times \exp(igaA_1(x + aj/2)) \exp(igaA_2(x - ai/2)) \right] \right\}$$  \hspace{1cm} (1)$$

where $ai/2$ adds $a/2$ to $x_1$, and similarly $aj/2$ adds $a/2$ to $x_2$.

Expand the exponentials to order $a^2$ and show that the product of the four of them to that order is

$$\exp \left( -iga^2 F_{12} \right)$$  \hspace{1cm} (2)$$

in which to order $a^2$

$$F_{12} = \frac{A_2(x + ai/2) - A_2(x - ai/2)}{a} - \frac{A_1(x + aj/2) - A_1(x - aj/2)}{a}$$

$$\approx \partial_1 A_2(x) - \partial_2 A_1(x) - ig[A_1(x), A_2(x)].$$  \hspace{1cm} (3)$$

Solution: One can expand the four exponentials to order $a^2$, but it is faster to use the formula

$$e^{aA} e^{aB} \approx e^{a(A+B) + (a^2/2)[A,B]}$$  \hspace{1cm} (4)$$

which holds to order $a^2$ and is exact when $[A, B]$ commutes with both $A$ and $B$. Thus to order $a^2$, the product of the first two exponentials is

$$\exp(-igaA_1(x - aj/2)) \exp(-igaA_2(x + ai/2)) \approx$$

$$\exp \left\{ -iga \left[ A_1(x - aj/2) + A_2(x + ai/2) - ig[A_1(x), A_2(x)]/2 \right] \right\}. \hspace{1cm} (5)$$

Similarly, the product of the second two exponentials to order $a^2$ is

$$\exp(igaA_1(x + aj/2)) \exp(igaA_2(x - ai/2)) \approx$$

$$\exp \left\{ ig \left[ A_1(x + aj/2) + A_2(x - ai/2) - ig[A_1(x), A_2(x)]/2 \right] \right\}. \hspace{1cm} (6)$$
Thus the product of all four exponentials is to order $a^2$

$$\exp\left\{ -iga^2 \left[ \frac{A_2(x + ai/2) - A_2(x - ai/2)}{a} - \frac{A_1(x + aj/2) - A_1(x - aj/2)}{a} \right] \\
- ig[A_1(x), A_2(x)] \right\} = \exp (-iga^2 F_{12}). \tag{7}$$

If the matrices of the representation have unit determinant, as they will for the special unitary groups $SU(N)$ and the special orthogonal groups $SO(N)$, then the generators $t_a$ are traceless, and so to order $a^4$

$$\text{Tr} \exp (-iga^2 F_{12}) = n - \frac{1}{2} g^2 a^4 \text{Tr} F_{12}^2. \tag{8}$$

Thus the action of the plaquette is

$$S_\Box = \frac{\beta g^2}{2n} a^4 \text{Tr} F_{12}^2. \tag{9}$$

So replacing $a^4$ by $d^4x$ and summing over all six plaquettes at each vertex of the lattice, we get in the $a \to 0$ limit

$$S = \frac{\beta g^2}{2n} \int \frac{1}{2} \text{Tr} F_{\mu\nu}^2 d^4x \tag{10}$$

in which the factor of two lets us sum over all $\mu\nu$ pairs.

In the above discussion, we used the definition

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)]. \tag{11}$$

Another convention is to set

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]. \tag{12}$$

This convention has the advantage that the plaquette action is

$$S_\Box = \beta \left\{ 1 - (1/n) \text{Re} \text{Tr} \left[ \exp(iga_1(x - aj/2)) \exp(iga_2(x + ai/2)) \times \exp(-iga_1(x + aj/2)) \exp(-iga_2(x - ai/2)) \right] \right\} \tag{13}$$

which is more natural.